Marking Scheme

Module 1 (Calculus and Statistics)

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

5. In the marking scheme, 'r.t.' stands for 'accepting answers which can be rounded off to' and 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles. All fractional answers must be simplified.

6. Unless otherwise specified in the question, numerical answers should either be exact or given to 4 decimal places. Answers not accurate up to the required degree of accuracy should not be accepted.

Solution	Marks	Remarks
1. (a) $(1-4p)+ap+p=1$	1M	
$(a-3)p=0$		
$a-3=0$ (by $p>0$)		
$a=3$	1A	
(b) $\mathrm{E}(X)$		
$=0(1-4p)+1(ap)+2(p)$	1M	
$=5p$		
$\mathrm{E}(X^2)$		
$=(0^2)(1-4p)+(1^2)(ap)+(2^2)(p)$		
$=7p$		
$\mathrm{Var}(X)$		
$=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$		
$=7p-25p^2$	1M	
$\mathrm{Var}(2X+a^2)=8\mathrm{E}(aX-1)$		
$4\mathrm{Var}(X)=8a\mathrm{E}(X)-8$	1M	for either one
$4(7p-25p^2)=(8)(3)(5p)-8$		
$25p^2+23p-2=0$		
$p=\frac{2}{25}$ or $p=-1$ (rejected)		
Thus, we have $p=\frac{2}{25}$.	1A	
	----------(6)	
2. (a) The required probability		
$=1-\left(1-\frac{1}{5}\right)^6$	1M	
$=\frac{11\,529}{15\,625}$	1A	r.t. 0.7379
(b) (i) The required probability		
$=1-\left(1-\frac{1}{5}\right)^6-C_1^6\left(1-\frac{1}{5}\right)^5\left(\frac{1}{5}\right)-C_2^6\left(1-\frac{1}{5}\right)^4\left(\frac{1}{5}\right)^2$	1M	
$=\frac{309}{3\,125}$	1A	r.t. 0.0989
(ii) The expected number of photocopies		
$=\frac{1}{\frac{309}{3\,125}}-1$	1M	
$=\frac{2\,816}{309}$	1A	r.t. 9.1133
	----------(6)	

Solution	Marks	Remarks
3. (a) $P(A \cap B)$		
$= P(B \mid A)P(A)$		
$= \frac{1}{2}P(A)$		
$P(B) = P(A \cap B) + P(A' \cap B)$		
$P(B) = \frac{1}{2}P(A) + kP(A)$	1M	
$P(B) = \left(\frac{1}{2} + k\right)P(A)$		
Assume that $k = \frac{1}{2}$.		
Then, we have $P(B) = P(A)$.		
Since $P(B) = \frac{1}{3} + P(A)$, we have $0 = \frac{1}{3}$.		
This is impossible.		
Thus, we have $k \neq \frac{1}{2}$.	1	
Since $P(B) = \left(\frac{1}{2} + k\right)P(A)$, we have $P(B) = \left(\frac{1}{2} + k\right)\left(P(B) - \frac{1}{3}\right)$.		
Solving, we have $P(B) = \frac{2k+1}{3(2k-1)}$.	1A	
(b) $P(A \cap B)$		
$= \frac{1}{2}\left(P(B) - \frac{1}{3}\right)$		
$= \frac{1}{2}\left(\frac{2k+1}{3(2k-1)} - \frac{1}{3}\right)$	1M	for using the result of (a)
$= \frac{1}{3(2k-1)}$		
$\neq 0$		
Thus, A and B are not mutually exclusive.	1A	f.t.
(c) $P(B \mid A) = \frac{1}{2}$		
$P(B) = \frac{1}{2}$	1M	
$\frac{2k+1}{3(2k-1)} = \frac{1}{2}$		
$k = \frac{5}{2}$	1A	
	----------(7)	

Solution	Marks	Remarks
4. (a) An approximate 95% confidence interval for p		
$=\left(\frac{441}{841}-1.96\left(\sqrt{\frac{\left(\frac{441}{841}\right)\left(1-\frac{441}{841}\right)}{841}}\right), \frac{441}{841}+1.96\left(\sqrt{\frac{\left(\frac{441}{841}\right)\left(1-\frac{441}{841}\right)}{841}}\right)\right)$	1M+1A	
$=\left(\frac{59\,829}{121\,945}, \frac{68\,061}{121\,945}\right)$		
$\approx(0.490622821, 0.558128664)$		
$\approx(0.4906, 0.5581)$	1A	r.t. $(0.4906, 0.5581)$
(b) $2z\sqrt{\frac{\left(\frac{441}{841}\right)\left(1-\frac{441}{841}\right)}{841}}=0.088$	1M	
$z\approx 2.555038095$		
The confidence level		
$=100(2)(0.4948)\%$	1M	
$=98.96\%$		
Thus, we have $\beta=99$ (correct to the nearest integer).	1A	
	----------(6)	
5. (a) $(1+ke^x)^3$		
$=1+3ke^x+3k^2e^{2x}+k^3e^{3x}$	1M	
$=1+3k\left(1+x+\frac{x^2}{2}+\cdots\right)+3k^2\left(1+2x+\frac{4x^2}{2}+\cdots\right)+k^3\left(1+3x+\frac{9x^2}{2}+\cdots\right)$	1M	for expanding e^x, e^{2x} or e^{3x}
$=1+3k+3k^2+k^3+(3k+6k^2+3k^3)x+\left(\frac{3k+12k^2+9k^3}{2}\right)x^2+\cdots$		
Thus, the constant term and the coefficient of x^2 are $1+3k+3k^2+k^3$	1A	
and $\frac{3k+12k^2+9k^3}{2}$ respectively.	1A	
(b) $1+3k+3k^2+k^3=27$	1M	
$(1+k)^3=27$		
$1+k=3$		
$k=2$		
The coefficient of x^2		
$=\frac{3(2)+12(2)^2+9(2)^3}{2}$		
$=63$	1A	
	----------(6)	

Solution	Marks	Remarks
6. (a) $g'(x)$		
$=1-\frac{5}{x^2}+\frac{1}{x^4}(4x^3)$	1M	
$=1+\frac{4}{x}-\frac{5}{x^2}$	1A	
(b) $g'(x)=0$		
$1+\frac{4}{x}-\frac{5}{x^2}=0$		
$x^2+4x-5=0$		
$x=-5$ or $x=1$	1M	

x	$(-\infty,-5)$	-5	$(-5,0)$	$(0,1)$	1	$(1,\infty)$
$g'(x)$	+	0	−	−	0	+
$g(x)$	↗	$4\ln 5-6$	↘	↘	6	↗

Solution	Marks	Remarks
(table above)	1M	for testing
So, the maximum value and the minimum value of $g(x)$ are $4\ln 5-6$ and 6 respectively.		
Since $4\ln 5-6<6$, the claim is agreed.	1A	f.t.

Solution	Marks	Remarks
$g'(x)=0$		
$1+\frac{4}{x}-\frac{5}{x^2}=0$		
$x^2+4x-5=0$		
$x=-5$ or $x=1$	1M	
$g''(x)=\frac{-4}{x^2}+\frac{10}{x^3}$		
$g''(-5)=\frac{-6}{25}<0$ and $g(-5)=4\ln 5-6$	1M	for testing
$g''(1)=6>0$ and $g(1)=6$		
So, the maximum value and the minimum value of $g(x)$ are $4\ln 5-6$ and 6 respectively.		
Since $4\ln 5-6<6$, the claim is agreed.	1A	f.t.

Solution	Marks	Remarks
(c) The equations of the two horizontal tangents to the graph of $y=g(x)$ are $y=4\ln 5-6$ and $y=6$.	1A ----------(6)	for both correct

Solution	Marks	Remarks
7. (a) Let h cm be the height of the circular cylinder.		
$2\pi r^2 + 2\pi rh = 486\pi$	1M	
$h = \dfrac{243 - r^2}{r}$		
V $= \pi r^2 h$		
$= \pi r^2 \left(\dfrac{243 - r^2}{r}\right)$	1M	
$= 243\pi r - \pi r^3$		
Thus, we have $\dfrac{dV}{dr} = 243\pi - 3\pi r^2$.	1A	
(b) $\dfrac{dV}{dr} = 0$		
$243\pi - 3\pi r^2 = 0$		
$r = 9$ or $r = -9$ (rejected)	1M	
$\dfrac{d^2V}{dr^2} = -6\pi r$		
$\left.\dfrac{d^2V}{dr^2}\right\|_{r=9} = -54\pi < 0$	1M	for testing
Since there is only one extreme value, V attains its greatest value when $r = 9$.		
The greatest value of V $= 243\pi(9) - \pi(9)^3$ $\approx 4\,580.442089$ $< 5\,000$		
Thus, the volume of the circular cylinder cannot exceed $5\,000\text{ cm}^3$.	1A	f.t.
$\dfrac{dV}{dr} = 0$		
$243\pi - 3\pi r^2 = 0$		
$r = 9$ or $r = -9$ (rejected)	1M	
(table below)	1M	for testing
So, V attains its greatest value when $r = 9$.		
The greatest value of V $= 243\pi(9) - \pi(9)^3$ $\approx 4\,580.442089$ $< 5\,000$		
Thus, the volume of the circular cylinder cannot exceed $5\,000\text{ cm}^3$.	1A	f.t.
	----------(6)	

r	$0 \le r < 9$	$r = 9$	$9 < r < \sqrt{243}$
$\dfrac{dV}{dr}$	+	0	−
V	↗	$1\,458\pi$	↘

Solution	Marks	Remarks
8. (a) $\frac{d}{dx}(xe^{mx})$		
$= mxe^{mx} + e^{mx}$	1A	
So, we have $xe^{mx} = \frac{1}{m}\left(\frac{d}{dx}(xe^{mx}) - e^{mx}\right)$.		
$\int xe^{mx}\,dx$		
$= \frac{1}{m}\left(xe^{mx} - \int e^{mx}\,dx\right)$	1M	
$= \frac{xe^{mx}}{m} - \frac{e^{mx}}{m^2} + \text{constant}$	1A	
(b) Note that the x-intercept of the curve $y = xe^{mx}$ is 0.		
$\int_0^1 xe^{mx}\,dx = \frac{1}{m}$	1M	
$\left[\frac{xe^{mx}}{m} - \frac{e^{mx}}{m^2}\right]_0^1 = \frac{1}{m}$	1M	for using the result of (a)
$\frac{e^m}{m} - \frac{e^m}{m^2} + \frac{1}{m^2} = \frac{1}{m}$		
$me^m - e^m - m + 1 = 0$		
$(m-1)(e^m - 1) = 0$	1M	
$m = 1$ or $m = 0$ (rejected)		
Thus, we have $m = 1$.	1A	
	----------(7)	

Solution	Marks	Remarks
9. (a) The required probability		
$= P\left(\frac{0-15}{2} < Z < \frac{13-15}{2}\right)$	1M	
$= P(-7.5 < Z < -1)$		
$= 0.1587$	1A	
	----------(2)	
(b) The required probability		
$= P\left(\frac{13-15}{2} < Z < \frac{20-15}{2}\right)$		
$= P(-1 < Z < 2.5)$		
$= 0.8351$	1A	
	----------(1)	
(c) (i) The probability that Mary greets Tom on a certain morning		
$= (0.1587)(0.3015) + (0.8351)(0.6328)$	1M	
$= 0.57629933$		
≈ 0.5763		
The required probability		
$= C_1^3(0.57629933)(1-0.57629933)^2(0.57629933)$	1M	
≈ 0.178869291		
≈ 0.1789	1A	r.t. 0.1789
(ii) The required probability		
$= \frac{((0.8351)(0.6328))^2}{(0.57629933)^2}$	1M+1M	
≈ 0.84084061		
≈ 0.8408	1A	r.t. 0.8408
(iii) The required probability		
$= 1 - \frac{((0.8351)(0.6328))^4}{(0.57629933)^4}$	1M	
≈ 0.292987067		
≈ 0.2930	1A	r.t. 0.2930
(iv) Assume that Tom leaves home x minutes before 7:23.		
$P\left(\frac{0-15}{2} < Z < \frac{x-15}{2}\right) > 0.3015$	1M	
$\frac{x-15}{2} > -0.52$		
$x > 13.96$		
Thus, Tom should leave home at 7:09 the latest.	1A	
	---------(10)	

Solution	Marks	Remarks
10. (a) The required probability		
$=\left(\frac{1}{6}\right)^4$		
$=\frac{1}{1\,296}$	1A	r.t. 0.0008
	----------(1)	
(b) The required probability		
$=\frac{1}{1\,296}+C_1^4\left(\frac{1}{6}\right)^3\left(\frac{2}{6}\right)+C_1^4\left(\frac{1}{6}\right)^3\left(\frac{3}{6}\right)+C_2^4\left(\frac{1}{6}\right)^2\left(\frac{2}{6}\right)^2$	1M	
$=\frac{5}{144}$	1A	r.t. 0.0347
	----------(2)	
(c) (i) The probability of getting an *Excellent*		
$=\frac{e^{-5}5^5}{5!}$	1M	for Poisson probability (covers both the *Excellent* and *Good* probabilities)
$=\frac{625e^{-5}}{24}$		
The probability of getting a *Good*		
$=\frac{e^{-5}5^1}{1!}+\frac{e^{-5}5^2}{2!}+\frac{e^{-5}5^3}{3!}+\frac{e^{-5}5^4}{4!}$		
$=\frac{515e^{-5}}{8}$		
The required probability		
$=3\left(\frac{625e^{-5}}{24}\right)^2\left(\frac{515e^{-5}}{8}\right)$	1M	
$=\frac{201\,171\,875e^{-15}}{1\,536}$	1A	r.t. 0.0401
(ii) The required probability		
$=\frac{\left(\frac{625e^{-5}}{24}\right)^3}{3\left(\frac{625e^{-5}}{24}\right)\left(\frac{515e^{-5}}{8}\right)^2+\frac{201\,171\,875e^{-15}}{1\,536}+\left(\frac{625e^{-5}}{24}\right)^3}$	1M+1M	
$=\frac{15\,625}{417\,943}$	1A	r.t. 0.0374
(iii) The required probability		
$=\frac{\left(1-\frac{5}{144}\right)(0.01)}{\left(1-\frac{5}{144}\right)(0.01)+\left(\frac{5}{144}\right)3\left(\frac{625e^{-5}}{24}\right)\left(\frac{515e^{-5}}{8}\right)^2}$	1M+1M	
≈ 0.737323792		
≈ 0.7373	1A	r.t. 0.7373
	----------(9)	

Solution	Marks	Remarks
11. (a) (i) α_1		
$= \frac{1}{2}\left(\frac{2-0}{5}\right)(\mathrm{A}(0)+\mathrm{A}(2)+2(\mathrm{A}(0.4)+\mathrm{A}(0.8)+\mathrm{A}(1.2)+\mathrm{A}(1.6)))$	1M	
$= \ln 20 + \ln 8 + 2(\ln 16.96 + \ln 14.24 + \ln 11.84 + \ln 9.76)$		
≈ 25.54855095		
≈ 25.5486	1A	r.t. 25.5486
(ii) $\mathrm{A}'(t)$		
$= 5\left(\frac{1}{t^2-8t+20}\right)(2t-8)$	1M	
$= \frac{10t-40}{t^2-8t+20}$		
$\mathrm{A}''(t)$		
$= \frac{(t^2-8t+20)(10)-(10t-40)(2t-8)}{(t^2-8t+20)^2}$	1M	
$= \frac{-10t^2+80t-120}{(t^2-8t+20)^2}$		
$= \frac{-10(t-2)(t-6)}{(t^2-8t+20)^2}$		
So, we have $\mathrm{A}''(t) < 0$ for $0 \le t < 2$.		
Thus, α_1 is an under-estimate.	1A	f.t.
	----------(5)	
(b) (i) Let $u = 1 + 3^{2t}$.	1M	
So, we have $\frac{du}{dt} = 2(\ln 3)(3^{2t})$.		
The required number		
$= \int_0^2 \mathrm{B}(t)\,dt$		
$= \frac{9}{2\ln 3}\int_2^{82} \frac{1}{u}\,du$	1M	
$= \frac{9}{2\ln 3}[\ln u]_2^{82}$	1M	
$= \frac{9}{2\ln 3}(\ln 82 - \ln 2)$		
$= \frac{9\ln 41}{2\ln 3}$ thousand	1A	r.t. 15.2111 thousand

Solution	Marks	Remarks
(ii) By (a)(ii), α_1 is an under-estimate of α .		
So, we have $\alpha > \alpha_1$.		
$\left(\alpha - \int_0^2 \mathrm{B}(t)\,\mathrm{d}t\right) - 0.4\alpha$	1M	
$= 0.6\alpha - \int_0^2 \mathrm{B}(t)\,\mathrm{d}t$		
$> 0.6\alpha_1 - \int_0^2 \mathrm{B}(t)\,\mathrm{d}t$	1M	
$\approx (0.6)(25.54855095) - 15.21107535$		
≈ 0.118055221		
> 0		
Therefore, we have $\alpha - \int_0^2 \mathrm{B}(t)\,\mathrm{d}t > 0.4\alpha$.		
Thus, the claim is agreed.	1A ----------(7)	f.t.

Solution	Marks	Remarks
12. (a) $P=\dfrac{32}{a^{5+bt}+8}$		
$\dfrac{32}{P}-8=a^{5+bt}$	1M	
$\ln\left(\dfrac{32}{P}-8\right)=(b\ln a)t+5\ln a$	1A	
	----------(2)	
(b) (i) $5\ln a=\ln 32$ $a=2$	1A	
$\ln 2=(b\ln 2)+5\ln 2$ $b=-4$	1A	
(ii) $\dfrac{dP}{dt}$		
$=\dfrac{-32(\ln 2)(2^{5-4t})(-4)}{(2^{5-4t}+8)^2}$	1M	for $\dfrac{d}{dt}2^{5-4t}$
$=\dfrac{128(\ln 2)(2^{5-4t})}{(2^{5-4t}+8)^2}$	1A	
$\dfrac{d^2P}{dt^2}$		
$=\dfrac{128(\ln 2)((2^{5-4t}+8)^2(2^{5-4t})(\ln 2)(-4)-(2^{5-4t})(2)(2^{5-4t}+8)(2^{5-4t})(\ln 2)(-4))}{(2^{5-4t}+8)^4}$	1M	for quotient rule
$=\dfrac{512(\ln 2)^2 2^{5-4t}(2^{5-4t}-8)}{(2^{5-4t}+8)^3}$	1A	
(iii) The estimated number of ducks $=\lim\limits_{t\to\infty}P$ $=\lim\limits_{t\to\infty}\left(\dfrac{32}{2^{5-4t}+8}\right)$ $=4$ thousand	1A	
As $\dfrac{dP}{dt}>0$ for all $t\geq 0$, P is increasing. Thus, the number of ducks in the farm does not exceed 4 thousand since the start of the study.	1	

Solution	Marks	Remarks
(iv) By (b)(ii), we have $\frac{d}{dt}\left(\frac{dP}{dt}\right)=0$ when $2^{5-4t}=8$.		
Hence, we have $\frac{d}{dt}\left(\frac{dP}{dt}\right)=0$ when $t=0.5$.	1M	
(sign table below)	1M	for testing
Therefore, $\frac{dP}{dt}$ attains its greatest value when $t=0.5$.		
The required number of ducks $=\frac{32}{2^{5-4(0.5)}+8}$ = 2 thousand	1A	
	---------(11)	

t	$[0, 0.5)$	0.5	$(0.5, \infty)$
$\frac{d}{dt}\left(\frac{dP}{dt}\right)$	+	0	−