SECTION A (50 marks)

1. Let A and B be two events. Suppose that $P(A)=0.8$, $P(B|A)=0.45$ and $P(B|A')=0.6$, where A' is the complementary event of A. Find

(a) $P(B)$,

(b) $P(A|B)$,

(c) $P(A \cup B)$.

(5 marks)

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2. In an estate, Peter wants to study the proportion p of households who keep pets. He conducts a survey of a random sample of 64 households and finds that an approximate $\beta\%$ confidence interval for p is $(0.0915, 0.3085)$.

(a) Find

(i) the sample proportion of households who keep pets,

(ii) β.

(b) Using the sample proportion obtained in (a)(i), find the least number of households such that the probability of at least 1 of these households who keeps pets is greater than 0.999.

(6 marks)

Answers written in the margins will not be marked.

3. A factory manufactures a batch of marbles. The diameters of the marbles follow a normal distribution with a mean of 9 mm and a standard deviation of 0.125 mm . A marble is classified as *oversized* if its diameter is more than 9.16 mm .

(a) Find the probability that a randomly selected marble from the batch is *oversized*.

(b) The diameters of the marbles are measured one by one. Let X be the random variable representing the number of measurements taken when the first *oversized* marble is found. Find

(i) $P(X \leq 3)$,

(ii) $E(X)$.

(6 marks)

Answers written in the margins will not be marked.

Please stick the barcode label here.

4. The table below shows the probability distribution of a discrete random variable Y, where m and p are constants:

y	-2	2	m
$P(Y = y)$	p	0.25	0.5

(a) Prove that $\text{Var}(Y) = 0.25m^2 + 2$.

(b) If $\text{Var}(2Y - 1) = 8\text{E}(2Y - 1)$, find m.

(7 marks)

5. Let $f(x)$ be a continuous function such that $f'(x) = \dfrac{12x - 48}{(3x^2 - 24x + 49)^2}$ for all real numbers x.

(a) If $f(x)$ attains its minimum value at $x = \alpha$, find α.

(b) It is given that the extreme value of $f(x)$ is 5. Find

(i) $f(x)$,

(ii) $\lim\limits_{x \to \infty} f(x)$.

(6 marks)

Answers written in the margins will not be marked.

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Let k be a constant.

(a) Expand $e^{kx}+e^{2x}$ in ascending powers of x as far as the term in x^2.

(b) If the coefficient of x and the coefficient of x^2 in the expansion of $(1-3x)^8(e^{kx}+e^{2x}-1)$ are equal, find k.

(6 marks)

Answers written in the margins will not be marked.

7. Let h be a constant. Consider the curve $C: y = x^2\sqrt{h-x}$, where $0 < x < h$. It is given that $\frac{dy}{dx} = 30$ when $x = 4$.

(a) Prove that $h = 20$.

(b) Find the maximum point(s) of C.

(c) Write down the equation(s) of the horizontal tangent(s) to C.

(7 marks)

Answers written in the margins will not be

8. (a) By considering $\frac{d}{dx}(x \ln x)$, find $\int \ln x \, dx$.

(b) Find $\int \frac{\ln x}{x} dx$.

(c) Let C be the curve $y = \frac{(x-1)(\ln x - 1)}{x}$, where $x > 0$. Express, in terms of e, the area of the region bounded by C and the x-axis.

(7 marks)

Answers written in the margins will not be marked.

SECTION B (50 marks)

9. A fruit wholesaler, John, grades a batch of apples according to their weights. The following table shows the classification of the apples, where a is a constant.

Weight of an apple (W g)	$W \leq a$	$a < W \leq 260$	$W > 260$
Classification	*small*	*medium*	*large*

The weights of the apples follow a normal distribution with a mean of μ g and a standard deviation of 16 g. It is known that 10.56% and 73.57% of the apples are *large* and *medium* respectively. Every 8 of the apples are packed in a box. A box of apples is regarded as *regular* if there are at least 6 *medium* apples in the box.

(a) Find μ and a. (3 marks)

(b) Find the probability that a randomly chosen box of apples is *regular*. (2 marks)

(c) John randomly chooses 3 boxes of apples.

(i) Find the probability that these 3 boxes of apples are *regular* and there are totally 21 *medium* apples and 3 *small* apples.

(ii) Given that these 3 boxes of apples are *regular*, find the probability that there are totally 21 *medium* apples and 3 *small* apples.

(iii) Given that there are totally 21 *medium* apples and 3 *small* apples in these 3 boxes of apples, find the probability that these 3 boxes of apples are *regular*.

(7 marks)

Answers written in the margins will not be marked.

18. A company records the numbers of lateness of its staff monthly. The performance of a staff member in a month is regarded as *good* if the staff member is late for fewer than 2 times in that month. Albert is a staff member of the company. The number of lateness of Albert in a month follows a Poisson distribution with a mean of 1.8.

(a) Find the probability that Albert's performance in a certain month is *good*. (2 marks)

(b) To improve the performance of the staff, the company launches a bonus scheme on staff performance in the coming four months. Two suggestions for the bonus scheme are listed below:

Suggestion I

Number of month with *good* performance	4	3	2	1	0
Bonus	$5000	$2500	$1500	$600	$0

Suggestion II

Total number of lateness in these four months	Fewer than 5	Otherwise
Bonus	$8000	$0

Which one of the above suggestions is more favourable to Albert? Explain your answer.

(6 marks)

(c) The company also records the numbers of early leaves of its staff monthly. The number of early leaves of Albert in a month follows a Poisson distribution with a mean of λ. It is assumed that whether Albert is late and whether he leaves early are independent events.

(i) Express, in terms of e and λ, the probability that Albert is late for 2 times and does not leave early in a certain month.

(ii) Given that the sum of the number of lateness and the number of early leaves of Albert in a certain month is 2, the probability that Albert is late for 2 times and does not leave early in that month is 0.36. Find λ.

(5 marks)

Answers written in the margins will not be marked.

11. In a research, the rate of change of the distance (in cm/s) travelled by a particle is given by

$$A(t) = 60(1+10t)e^{-2t},$$

where t is the number of seconds elapsed since the start of the research. Let D cm be the distance travelled by the particle from $t = 0.1$ to $t = 0.5$. Denote the estimate of D by using the trapezoidal rule with 4 sub-intervals by D_1.

(a) (i) Find D_1.

(ii) Is D_1 an over-estimate or an under-estimate? Explain your answer.

(6 marks)

(b) In order to estimate D, a researcher, Mary, models the rate of change of the distance travelled by the particle by

$$B(t) = \frac{50(1+10t)}{1+2t},$$

where t is the number of seconds elapsed since the start of the research. Let D_2 cm be the distance travelled by the particle from $t = 0.1$ to $t = 0.5$ under this model.

(i) Find D_2.

(ii) Mary claims that in order to estimate D, D_2 is more accurate than D_1. Do you agree? Explain your answer.

(6 marks)

12. In an experiment, the number of certain bacteria in a room under controlled conditions is recorded. The temperature Q (in °C) in the room can be modelled by the following linear function

$$Q = \ln r + (s \ln 3)t ,$$

where r and s are constants and $t\ (0 \le t \le 20)$ is the number of hours elapsed since the start of the experiment. It is given that the slope and the intercept on the vertical axis of the graph of this linear function of t are $-0.1 \ln 9$ and $\ln 9$ respectively.

(a) Find r and s. (2 marks)

(b) It is given that

$$Q = \ln\left(\frac{120 - 3N}{N}\right),$$

where N is the number in millions of bacteria.

(i) Prove that $N = \dfrac{40}{3^{1-0.2t} + 1}$.

(ii) Is it possible that there are 4 million bacteria in the room during the experiment? Explain your answer.

(iii) Find $\dfrac{dN}{dt}$ and $\dfrac{d^2N}{dt^2}$.

(iv) Describe how $\dfrac{dN}{dt}$ varies during the experiment. Explain your answer.

(11 marks)