Marking Scheme

Module 1 (Calculus and Statistics)

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

5. In the marking scheme, 'r.t.' stands for 'accepting answers which can be rounded off to' and 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles. All fractional answers must be simplified.

6. Unless otherwise specified in the question, numerical answers should either be exact or given to 4 decimal places. Answers not accurate up to the required degree of accuracy should not be accepted.

Solution	Marks	Remarks
1. (a) $P(Y \mid X)$ $= 0.5$ $\neq 0.7$ $= P(Y)$	1M	
Thus, X and Y are not independent.	1A	f.t.
$P(X)P(Y)$ $= (0.4)(0.7)$ $= 0.28$		
$P(X \cap Y)$ $= P(Y \mid X)P(X)$ $= (0.5)(0.4)$ $= 0.2$		
$P(X \cap Y) \neq P(X)P(Y)$	1M	
Thus, X and Y are not independent.	1A	f.t.
(b) $P(X \cap Y)$ $= P(Y \mid X)P(X)$ $= (0.5)(0.4)$	1M	
$= 0.2$		
$P(X \cup Y)$ $= P(X) + P(Y) - P(X \cap Y)$ $= 0.4 + 0.7 - 0.2$	1M	
$= 0.9$	1A	
	----------(5)	
2. (a) The required probability $= \dfrac{\left(\frac{1}{6}\right)^3 (3)}{\left(\frac{1}{6}\right)^3 (3 + 3! + 3 + 3)}$	1M+1M+1M	1M for $\left(\frac{1}{6}\right)^3$ + 1M for numerator + 1M for denominator
$= \dfrac{1}{5}$	1A	
(b) The required probability $= \dfrac{\left(\frac{1}{5}\right)^3 (3)}{\left(\frac{1}{5}\right)^3 (3 + 3! + 3 + 3)}$	1M	
$= \dfrac{1}{5}$		
Thus, the required probability will not change.	1A	f.t.
	----------(6)	

Solution	Marks	Remarks
3. (a) The variance of the number of visitors entering the museum in a minute is 1.8 .	1A	
(b) The required probability		
$=\frac{e^{-3.6}3.6^3}{3!}$	1M+1M	1M for Poisson probability + 1M using mean 3.6
$=\frac{7.776}{e^{3.6}}$	1A	r.t. 0.2125
≈ 0.212469265		
≈ 0.2125		
The required probability		
$=2\left(\frac{e^{-1.8}1.8^0}{0!}\right)\left(\frac{e^{-1.8}1.8^3}{3!}\right)+2\left(\frac{e^{-1.8}1.8^1}{1!}\right)\left(\frac{e^{-1.8}1.8^2}{2!}\right)$	1M + 1M	1M for 4 cases + 1M for Poisson probability using mean 1.8
$=\frac{7.776}{e^{3.6}}$	1A	r.t. 0.2125
≈ 0.212469265		
≈ 0.2125		
(c) P(at most 3 visitors in a minute)		
$=\frac{e^{-1.8}1.8^0}{0!}+\frac{e^{-1.8}1.8^1}{1!}+\frac{e^{-1.8}1.8^2}{2!}+\frac{e^{-1.8}1.8^3}{3!}$	1M	
≈ 0.891291605		
≈ 0.8913		
The required probability		
$\approx (0.891291605)(1-0.891291605)^2$	1M	
≈ 0.010532851		
≈ 0.0105	1A ----------(7)	r.t. 0.0105
4. (a) The point estimate of p is $\frac{64}{100}=0.64$.	1A	
An approximate 95% confidence interval for p		
$=\left(\frac{64}{100}-1.96\sqrt{\frac{(0.64)(0.36)}{100}},\frac{64}{100}+1.96\sqrt{\frac{(0.64)(0.36)}{100}}\right)$	1M+1A	1A for 1.96
$=(0.54592, 0.73408)$	1A	
$\approx(0.5459, 0.7341)$		
(b) Let n be the number of packs in the sample.		
The width of a 90% confidence interval for p is $(2)(1.645)\left(\frac{0.05}{\sqrt{n}}\right)$.		
$(2)(1.645)\left(\frac{0.05}{\sqrt{n}}\right)<0.04$	1M+1A	
$\sqrt{n}>4.1125$		
$n>16.91265625$		
Thus, the least sample size is 17 .	1A ----------(7)	

Solution	Marks	Remarks
5. (a) e^{kx}		
$=1+kx+\frac{(kx)^2}{2!}+\cdots$		
$=1+kx+\frac{k^2x^2}{2}+\cdots$	1A	
(b) $(1+2x)^7e^{kx}$		
$=\left(1+C_1^7(2x)+C_2^7(2x)^2+\cdots+(2x)^7\right)\left(1+kx+\frac{k^2x^2}{2}+\cdots\right)$	1M	
$=\left(1+14x+84x^2+\cdots+(2x)^7\right)\left(1+kx+\frac{k^2x^2}{2}+\cdots\right)$		
$\therefore\ 14+k=8$	1M	
$k=-6$		
The coefficient of x^2		
$=(1)\left(\frac{(-6)^2}{2}\right)+14(-6)+(84)(1)$	1M	
$=18$	1A	
	----------(5)	
6. (a) $\int \mathrm{f}(x)\,\mathrm{d}x$		
$=\int (3^{2x}-10(3^x)+9)\,\mathrm{d}x$		
$=\frac{3^{2x}}{2\ln 3}-\frac{10(3^x)}{\ln 3}+9x+\text{constant}$	1M+1A	1M for $\int a^x\mathrm{d}x=\frac{a^x}{\ln a}+\text{constant}$
(b) (i) $3^{2x}-10(3^x)+9=0$		
$(3^x)^2-10(3^x)+9=0$	1M	
$3^x=1$ or $3^x=9$		
$x=0$ or $x=2$	1A	for both
Thus, the x-intercepts are 0 and 2 .		
(ii) The area of the region bounded by C and the x-axis		
$=-\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$	1M	
$=-\left[\frac{3^{2x}}{2\ln 3}-\frac{10(3^x)}{\ln 3}+9x\right]_0^2$ (by (a))		
$=-\left(\frac{81}{2\ln 3}-\frac{90}{\ln 3}+18\right)+\left(\frac{1}{2\ln 3}-\frac{10}{\ln 3}\right)$		
$=\frac{40}{\ln 3}-18$	1A	
	----------(6)	

Solution	Marks	Remarks
7. (a) $\frac{dy}{dx}$		
$=\left(\frac{3}{2}\right)(2x+8)^{\frac{1}{2}}(2)+6x$	1M	for chain rule
$=3\sqrt{2x+8}+6x$	1A	
(b) Note that the slope of the straight line $6x+y+4=0$ is -6 . So, the slope of the tangent is -6 .		
$3\sqrt{2x+8}+6x=-6$	1M+1A	1M for using (a)
$\sqrt{2x+8}=-2(x+1)$		
$2x+8=4(x+1)^2$		
$2x^2+3x-2=0$	1M	for $ax^2+bx+c=0$
$x=-2$ or $x=\frac{1}{2}$ (rejected)	1A	for ‘ $x=-2$ or $x=\frac{1}{2}$ ’
Hence, there is only one tangent to C parallel to the straight line $6x+y+4=0$. Thus, the claim is disagreed.	1A	f.t.
	----------(7)	
8. (a) $f'(x)$		
$=\frac{x\left(2(\ln x)\frac{1}{x}\right)-(\ln x)^2}{x^2}$	1M	for quotient rule
$=\frac{2\ln x-(\ln x)^2}{x^2}$		
$=\frac{(2-\ln x)(\ln x)}{x^2}$		
$f'(x)=0$		
$\ln x=2$ or $\ln x=0$		
$x=e^2$ or $x=1$		
$\alpha=e^2$ and $\beta=1$	1A+1A	
(b) Let $u=\ln x$.	1M	
Then, we have $\frac{du}{dx}=\frac{1}{x}$.		
$\int_{\beta}^{\alpha} f(x)\,dx$		
$=\int_1^{e^2}\frac{(\ln x)^2}{x}dx$		
$=\int_0^2 u^2\,du$	1A	
$=\left[\frac{u^3}{3}\right]_0^2$	1M	
$=\frac{8}{3}$	1A	r.t. 2.6667
≈ 2.666666667		
≈ 2.6667		
	----------(7)	

Solution	Marks	Remarks
9. Let J minutes and K minutes be the random variables representing the daily reading times of the students in schools X and Y respectively.		
(a) Let μ_1 minutes and σ_1 minutes be the mean and the standard deviation of the daily reading times of the students in school X respectively, while μ_2 minutes and σ_2 minutes be the mean and the standard deviation of the daily reading times of the students in schools Y respectively.		
$\begin{cases} \frac{40-\mu_1}{\sigma_1}=-2.51 \\ \frac{70-\mu_1}{\sigma_1}=2.17 \end{cases}$ $\begin{cases} \frac{48-\mu_2}{\sigma_2}=-2.17 \\ \frac{72-\mu_2}{\sigma_2}=2.12 \end{cases}$	1M+1A	either one
Solving, we have $\mu_1=\frac{4375}{78}, \sigma_1=\frac{250}{39}$	1A	for both
$\mu_1 \approx 56.08974359, \sigma_1 \approx 6.41025641$ $\mu_1 \approx 56.0897, \sigma_1 \approx 6.4103$		r.t. $\mu_1 \approx 56.0897, \sigma_1 \approx 6.4103$
$\mu_2=\frac{8600}{143}, \sigma_2=\frac{800}{143}$	1A	for both
$\mu_2 \approx 60.13986014, \sigma_2 \approx 5.594405594$ $\mu_2 \approx 60.1399, \sigma_2 \approx 5.5944$		r.t. $\mu_2 \approx 60.1399, \sigma_2 \approx 5.5944$
P(students reading more than 60 minutes daily in school X) $=\mathrm{P}(J>60)$ $=\mathrm{P}\left(Z>\frac{60-\frac{4375}{78}}{\frac{250}{39}}\right)$ $=\mathrm{P}(Z>0.61)$ $=0.2709$	1M	either one
P(students reading more than 60 minutes daily in school Y) $=\mathrm{P}(K>60)$ $=\mathrm{P}\left(Z>\frac{60-\frac{8600}{143}}{\frac{800}{143}}\right)$ $=\mathrm{P}(Z>\frac{-1}{40})$ $>\mathrm{P}(Z>0)$ $=0.5$ >0.2709		
Thus, there are less students reading more than 60 minutes daily in school X.	1A ----------(6)	f.t.

Solution	Marks	Remarks
(b) The required probability		
$= C_1^3(0.2709)(1-0.2709)^2(0.2709)$	1M	
≈ 0.11703438		
≈ 0.1170	1A	r.t. 0.1170
	----------(2)	
(c) For school X,		
$\mathrm{P}(J \geq T) \leq 0.1$		
$\dfrac{T - \frac{4375}{78}}{\frac{250}{39}} \geq 1.29$	1M+1A	1A for 1.29
$T \geq \frac{2510}{39}$		either one
$T \geq 64.35897436$		
$T \geq 65$	1A	accept $T = 65$
For school Y,		
$\mathrm{P}(K \geq T) \leq 0.1$		
$\dfrac{T - \frac{8600}{143}}{\frac{800}{143}} \geq 1.29$		
$T \geq \frac{9632}{143}$		either one
$T \geq 67.35664336$		
$T \geq 68$		
Thus, the least value of T should be 68 .	1A	f.t.
	----------(4)	

Solution	Marks	Remarks
10. (a) The required probability		
$= 0.9+(1-0.9)(0.9)$	1M	
$= 0.99$	1A	
	----------(2)	
(b) The required probability		
$= (0.9)(0.1)+(1-0.9)(0.9)(0.4)+(1-0.9)^2(1)$	1M	
$= 0.136$	1A	
	----------(2)	
(c) The required probability		
$= C^6_2(1-0.136)^4(0.136)^2$	1M	
≈ 0.154605181		
≈ 0.1546	1A	r.t. 0.1546
	----------(2)	
(d) (i) The required probability		
$= (0.136)(0.7)^6+(1-0.136)(0.3)^6$	1M	
≈ 0.01663012		
≈ 0.0166	1A	r.t. 0.0166
(ii) The required probability		
$= (0.136)C^6_3(0.7)^3(0.3)^3+(1-0.136)C^6_2(0.7)^4(0.3)^2$	1M+1M	
≈ 0.30524256		
≈ 0.3052	1A	r.t. 0.3052
(iii) The required probability		
$= \dfrac{(0.136)C^6_3(0.7)^3(0.3)^3}{(0.136)C^6_3(0.7)^3(0.3)^3+(1-0.136)C^6_2(0.7)^4(0.3)^2}$	1M	1M for denominator using (d)(ii)
≈ 0.082524271		
≈ 0.0825	1A	r.t. 0.0825
	----------(7)	

Solution	Marks	Remarks
11. (a) (i) P_1		
$= \int_{0}^{12} \mathrm{A}(t)\,\mathrm{d}t$		
$\approx \frac{1}{2}\left(\frac{12-0}{4}\right)(\mathrm{A}(0)+\mathrm{A}(12)+2(\mathrm{A}(3)+\mathrm{A}(6)+\mathrm{A}(9)))$	1M	
≈ 54.61085671		
≈ 54.6109	1A	r.t. 54.6109
(ii) $\frac{\mathrm{dA}(t)}{\mathrm{d}t}$		
$= \frac{2t-8}{t^2-8t+95}$	1A	
$\frac{\mathrm{d}^2\mathrm{A}(t)}{\mathrm{d}t^2}$		
$= \frac{2(t^2-8t+95)-(2t-8)^2}{(t^2-8t+95)^2}$		
$= \frac{-2t^2+16t+126}{(t^2-8t+95)^2}$	1A	
$= \frac{-2(t^2-8t-63)}{(t^2-8t+95)^2}$		
	----------(4)	
(b) (i) Let $u = t+3$.	1M	
Then, we have $\frac{\mathrm{d}u}{\mathrm{d}t} = 1$.		
P_2		
$= \int_{0}^{12} \mathrm{B}(t)\,\mathrm{d}t$	1M	
$= \int_{0}^{12} \frac{t+8}{\sqrt{t+3}}\,\mathrm{d}t$		
$= \int_{3}^{15} \frac{u-3+8}{\sqrt{u}}\,\mathrm{d}u$	1A	
$= \int_{3}^{15} (u^{\frac{1}{2}}+5u^{\frac{-1}{2}})\,\mathrm{d}u$		
$= \left[\frac{2}{3}u^{\frac{3}{2}}+10u^{\frac{1}{2}}\right]_{3}^{15}$	1M	
$= 20\sqrt{15}-12\sqrt{3}$	1A	r.t. 56.6751
≈ 56.67505723		
≈ 56.6751		

Solution	Marks	Remarks
(b) (ii) $\frac{d^2A(t)}{dt^2}=\frac{-2[t-(4-\sqrt{79})][t-(4+\sqrt{79})]}{(t^2-8t+95)^2}$	1M	1M for considering $\frac{d^2A(t)}{dt^2}$
Note that $4-\sqrt{79}<0$ and $4+\sqrt{79}>12$.		
Therefore, we have $\frac{\left(t-(4-\sqrt{79})\right)\left(t-(4+\sqrt{79})\right)}{(t^2-8t+95)^2}<0$ for $0\le t\le 12$.		
Hence, we have $\frac{d^2A(t)}{dt^2}>0$ for $0\le t\le 12$.	1A	f.t.
So, the estimate of P_1 is an over-estimate. $P_1<54.61085671$.		
P_2-P_1 $=20\sqrt{15}-12\sqrt{3}-P_1$		
$>20\sqrt{15}-12\sqrt{3}-54.61085671$	1M	
≈ 2.064200523 >2		
Thus, the claim is disagreed.	1A ----------(9)	f.t.

Solution	Marks	Remarks
12. (a) $N=\dfrac{27}{2+\alpha te^{\beta t}}$		
$\dfrac{27-2N}{Nt}=\alpha e^{\beta t}$	1M	
$\ln\left(\dfrac{27-2N}{Nt}\right)=\ln\alpha+\beta t$	1A	
	----------(2)	
(b) (i) $\beta=-0.1$	1A	
$0=-0.1(10\ln 0.03)+\ln\alpha$		
$\ln\alpha=\ln 0.03$		
$\alpha=0.03$	1A	
(ii) $\dfrac{dN}{dt}$		
$=-27(2+0.03te^{-0.1t})^{-2}(0.03)(e^{-0.1t}-0.1te^{-0.1t})$	1M	for $\dfrac{d}{dt}e^{\beta t}=\beta e^{\beta t}$
$=\dfrac{0.081(t-10)e^{-0.1t}}{(2+0.03te^{-0.1t})^2}$	1A	
For $\dfrac{dN}{dt}=0$, we have $t=10$.		

t	$0\le t<10$	$t=10$	$t>10$
$\dfrac{dN}{dt}$	$-$	0	+

Solution	Marks	Remarks
	1M	
So, N attains its least value when $t=10$.	1A	
The least value of $N=\dfrac{27}{2+0.3e^{-1}}\approx 12.79400243>12$.		
Thus, the number of chickens will not be less than 12 thousand on a certain day after the start of the spread of the bird flu.	1A	f.t.
(iii) $\dfrac{d^2N}{dt^2}$		
$=\dfrac{d}{dt}\left(\dfrac{dN}{dt}\right)$		
$=\dfrac{0.081(2+0.03te^{-0.1t})^2(e^{-0.1t}-0.1(t-10)e^{-0.1t})}{(2+0.03te^{-0.1t})^4}$		
$-\dfrac{0.081(t-10)e^{-0.1t}(2)(2+0.03te^{-0.1t})(0.03)(e^{-0.1t}-0.1te^{-0.1t})}{(2+0.03te^{-0.1t})^4}$	1M	for quotient rule
$=0.0081\left(\dfrac{(2+0.03te^{-0.1t})(20-t)e^{-0.1t}+0.06(t-10)^2e^{-0.2t}}{(2+0.03te^{-0.1t})^3}\right)$	1A	
Hence, we have $\dfrac{d^2N}{dt^2}>0$ for $0\le t\le 20$.		
So, $\dfrac{dN}{dt}$ increases for $0\le t\le 20$.	1A	f.t.
Thus, the rate of change of the number of chickens increases.	---------(10)	