Marking Scheme

Module 1 (Calculus and Statistics)

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

5. In the marking scheme, 'r.t.' stands for 'accepting answers which can be rounded off to' and 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles. All fractional answers must be simplified.

6. Unless otherwise specified in the question, numerical answers should either be exact or given to 4 decimal places. Answers not accurate up to the required degree of accuracy should not be accepted.

Solution	Marks	Remarks
1. (a) $0.08+0.15+a+0.45+b=1$	1M	either one
$2(0.08)+3(0.15)+5a+7(0.45)+9b=5.64$		
Solving, we have $a=0.25$ and $b=0.07$.	1A	for both
(b) $\mathrm{E}((6-5X)^2)$		
$=\mathrm{E}(36-60X+25X^2)$		
$=36-60\mathrm{E}(X)+25\mathrm{E}(X^2)$	1M	
$=36-60(5.64)+25(35.64)$		
$=588.6$	1A	
$\mathrm{Var}(6-5X)$		
$=\mathrm{E}((6-5X)^2)-(\mathrm{E}(6-5X))^2$		
$=\mathrm{E}((6-5X)^2)-(6-5\mathrm{E}(X))^2$	1M	accept $(-5)^2\mathrm{Var}(X)$
$=588.6-(6-5(5.64))^2$		
$=95.76$	1A	
	----------(6)	
2. (a) $\mathrm{P}(A'\cap B')$		
$=\mathrm{P}(B'\mid A')\mathrm{P}(A')$		
$=0.6(1-0.3)$	1M	
$=0.42$	1A	
$\mathrm{P}(A'\cap B)$		
$=\mathrm{P}(A')-\mathrm{P}(A'\cap B')$		
$=1-0.3-0.42$	1M	
$=0.28$	1A	
(b) Note that $\mathrm{P}(B)=\mathrm{P}(A\cap B)+\mathrm{P}(A'\cap B)$.		
Since $\mathrm{P}(B)=\mathrm{P}(A'\cap B)=0.28$, we have $\mathrm{P}(A\cap B)=0$.	1M	
Thus, A and B are mutually exclusive.	1A	f.t.
	----------(6)	

Solution	Marks	Remarks
3. (a) The required probability		
$=\frac{2}{7}p+\frac{5}{7}(1-p)$	1M	for $rs+(1-r)(1-s)$
$=\frac{5-3p}{7}$	1A	
(b) (i) $\frac{5-3p}{7}=\frac{50}{91}$	1M	for using (a)
$p=\frac{5}{13}$	1A	
$p\approx 0.384615384$		r.t. 0.3846
$p\approx 0.3846$		
(ii) The required probability		
$=\frac{\frac{2}{7}\left(1-\frac{5}{13}\right)}{1-\frac{50}{91}}$	1M	for numerator using (b)(i)
$=\frac{16}{41}$	1A	
≈ 0.3902439024		r.t. 0.3902
≈ 0.3902		
	----------(6)	

Solution	Marks	Remarks
4. (a) The required probability		
$=(0.75)^3(1-0.75)$	1M	for $p^3(1-p)$, $0<p<1$
$=\frac{27}{256}$	1A	
$=0.10546875$		r.t. 0.1055
≈ 0.1055		
(b) The required probability		
$=1-\left((0.75)^4+4\left(\frac{27}{256}\right)\right)$	1M+1M	1M for $1-p$ + 1M for using (a)
$=\frac{67}{256}$	1A	
$=0.26171875$		r.t. 0.2617
≈ 0.2617		
The required probability		
$=(1-0.75)^4+C_1^4(1-0.75)^3(0.75)+C_2^4(1-0.75)^2(0.75)^2$	1M+1M	1M for the 3 cases + 1M for binomial probability
$=\frac{67}{256}$	1A	
$=0.26171875$		r.t. 0.2617
≈ 0.2617		
(c) The required probability		
$=\frac{\left(1-(0.75)^3\right)(1-0.75)}{0.26171875}$	1M	for denominator using (b)
$=\frac{37}{67}$	1A	
≈ 0.552238806		r.t. 0.5522
≈ 0.5522		
The required probability		
$=\frac{\left((1-0.75)^3+C_1^3(1-0.75)^2(0.75)+C_2^3(1-0.75)(0.75)^2\right)(1-0.75)}{0.26171875}$	1M	for denominator using (b)
$=\frac{37}{67}$	1A	
≈ 0.552238806		r.t. 0.5522
≈ 0.5522		
	----------(7)	

Solution	Marks	Remarks
5. (a) e^{-4x}		
$=1+(-4x)+\frac{(-4x)^2}{2!}+\cdots$	1M	
$=1-4x+8x^2-\cdots$	1A	
(b) $(2+x)^5$		
$=2^5+C_1^5(2^4)x+C_2^5(2^3)x^2+\cdots+x^5$	1M	
$=32+80x+80x^2+\cdots+x^5$		
The required coefficient		
$=(1)(80)+(-4)(80)+(8)(32)$	1M	
$=16$	1A	
	----------(5)	
6. (a) $e^{2x}+e^4=e^{x+3}+e^{x+1}$		
$(e^x)^2-(e^3+e)e^x+e^4=0$	1M	
$(e^x-e)(e^x-e^3)=0$		
$e^x=e$ or $e^x=e^3$		
$x=1$ or $x=3$	1A	
Thus, the x-coordinates are 1 and 3 .		
(b) The area of the region bounded by C_1 and C_2		
$=\int_1^3(e^{x+3}+e^{x+1}-(e^{2x}+e^4))\,dx$	1M+1A	
$=\left[e^{x+3}+e^{x+1}-\frac{e^{2x}}{2}-e^4x\right]_1^3$	1M	
$=\frac{e^6}{2}-2e^4-\frac{e^2}{2}$	1A	
	----------(6)	

Solution	Marks	Remarks
7. (a) $y = x\sqrt{2x^2+1}$		
$\frac{dy}{dx} = \sqrt{2x^2+1} + x\left(\frac{1}{2}\right)(2x^2+1)^{\frac{-1}{2}}(4x)$	1M	for chain rule
$\frac{dy}{dx} = \frac{4x^2+1}{\sqrt{2x^2+1}}$	1A	
(b) Note that the slope of the straight line is $\frac{-3}{17}$.		
So, the slope of each tangent is $\frac{17}{3}$.		
$\frac{4x^2+1}{\sqrt{2x^2+1}} = \frac{17}{3}$	1M+1A	1M for using (a)
$3(4x^2+1) = 17\sqrt{2x^2+1}$		
$9(4x^2+1)^2 = 289(2x^2+1)$		
$72x^4 - 253x^2 - 140 = 0$	1M	for $ax^4 + bx^2 + c = 0$
$x = 2$ or $x = -2$		
For $x = 2$, we have $y = 6$.		
The equation of the tangent to C at the point $(2, 6)$ is		
$y - 6 = \frac{17}{3}(x-2)$	1M	either one
$17x - 3y - 16 = 0$		
For $x = -2$, we have $y = -6$.		
The equation of the tangent to C at the point $(-2, -6)$ is		
$y + 6 = \frac{17}{3}(x+2)$		
$17x - 3y + 16 = 0$	1A	for both
	----------(7)	
8. (a) $\frac{d}{dx}\left((x^6+1)\ln(x^2+1)\right)$		
$= (x^6+1)\frac{2x}{x^2+1} + 6x^5\ln(x^2+1)$	1M+1A	1M for product rule
$= (x^2+1)(x^4-x^2+1)\frac{2x}{x^2+1} + 6x^5\ln(x^2+1)$	1M	
$= 2x^5 - 2x^3 + 2x + 6x^5\ln(x^2+1)$	1A	
(b) $(x^6+1)\ln(x^2+1) = 2\int(x^5 - x^3 + x)\,dx + 6\int x^5\ln(x^2+1)\,dx$	1M	
Note that $\int(x^5 - x^3 + x)\,dx = \frac{x^6}{6} - \frac{x^4}{4} + \frac{x^2}{2} + \text{constant}$.	1A	
Thus, we have		
$\int x^5\ln(x^2+1)\,dx = \frac{1}{6}(x^6+1)\ln(x^2+1) - \frac{x^6}{18} + \frac{x^4}{12} - \frac{x^2}{6} + \text{constant}$.	1A	
	----------(7)	

Solution	Marks	Remarks
9. (a) (i) The sample mean		
$= 68.64$ km/h	1A	
A 95% confidence interval for μ		
$= \left(68.64 - 1.96\left(\frac{16}{\sqrt{25}}\right), 68.64 + 1.96\left(\frac{16}{\sqrt{25}}\right)\right)$	1M+1A	1A for 1.96
$= (62.368, 74.912)$	1A	
(ii) Let n be the sample size.		
$2(2.24)\left(\frac{16}{\sqrt{n}}\right) < 9$	1M+1A	1A for 2.24
$n > 63.43237531$		
Thus, the least sample size is 64 .	1A	
	----------(7)	
(b) (i) The required probability		
$= \mathrm{P}\left(Z > \frac{90-66}{16}\right)$	1M	
$= \mathrm{P}(Z > 1.5)$		
$= 0.5 - 0.4332$		
$= 0.0668$	1A	
(ii) The required probability		
$= 1-(1-0.0668)^{12} - C_1^{12}(1-0.0668)^{11}(0.0668) - C_2^{12}(1-0.0668)^{10}(0.0668)^2$	1M+1M	1M for using (b)(i) + 1M for binomial probability
≈ 0.041574551		
≈ 0.0416	1A	r.t. 0.0416
	----------(5)	

Solution	Marks	Remarks
10. (a) The required probability		
$=\frac{3.2^0 e^{-3.2}}{0!}+\frac{3.2^1 e^{-3.2}}{1!}+\frac{3.2^2 e^{-3.2}}{2!}+\frac{3.2^3 e^{-3.2}}{3!}$	1M+1M	1M for the 4 cases + 1M for Poisson probability
≈ 0.602519724		
≈ 0.6025	1A	r.t. 0.6025
	----------(3)	
(b) The required probability		
$=C_2^7(0.7)^2(1-0.7)^5(0.7)$	1M	for binomial probability
≈ 0.01750329		
≈ 0.0175	1A	r.t. 0.0175
	----------(2)	
(c) The required probability		
$=\frac{3.2^3 e^{-3.2}}{3!}(0.7)^3$	1M	
≈ 0.076357282		
≈ 0.0764	1A	r.t. 0.0764
	----------(2)	
(d) The required probability		
$\approx 0.076357282+\frac{3.2^3 e^{-3.2}}{3!}\left(3(0.12)^2(0.04)+3!(0.12)(0.7)(0.08)\right)$	1M+1A	1M for using (c) + 1A for any one correct
≈ 0.085717839		
≈ 0.0857	1A	r.t. 0.0857
	----------(3)	
(e) The required probability		
$\approx \frac{\left(\frac{3.2e^{-3.2}}{1!}\right)(0.02)+\left(\frac{3.2^2 e^{-3.2}}{2!}\right)\left(2(0.12)(0.03)+2(0.7)(0.04)+(0.08)^2\right)+0.085717839}{0.602519724}$	1M+1M	1M for numerator using (d) +1M for denominator using (a)
≈ 0.170703644		
≈ 0.1707	1A	r.t. 0.1707
	----------(3)	

Solution	Marks	Remarks
11. (a) The total amount of oil produced by oil company X		
$=\int_2^{12} \mathrm{f}(t)\,\mathrm{d}t$	1M	
$\approx \frac{1}{2}\left(\frac{12-2}{5}\right)(\mathrm{f}(2)+\mathrm{f}(12)+2(\mathrm{f}(4)+\mathrm{f}(6)+\mathrm{f}(8)+\mathrm{f}(10)))$	1M	
≈ 69.49587529		
≈ 69.4959 hundred barrels	1A	r.t. 69.4959
	----------(3)	
(b) $\frac{\mathrm{d}\mathrm{f}(t)}{\mathrm{d}t}$		
$=\frac{e^t-1}{e^t-t}$	1A	
$\frac{\mathrm{d}^2\mathrm{f}(t)}{\mathrm{d}t^2}$		
$=\frac{(e^t-t)e^t-(e^t-1)(e^t-1)}{(e^t-t)^2}$	1A	
$=\frac{e^t(2-t)-1}{(e^t-t)^2}$		
<0 (since $2\le t\le 12$)		
Thus, the estimate in (a) is an under-estimate.	1A	f.t.
	----------(3)	
(c) Let $u=1+t$. Then, we have $\frac{\mathrm{d}u}{\mathrm{d}t}=1$.	1M	
$\int \frac{t}{1+t}\mathrm{d}t$		
$=\int \frac{u-1}{u}\mathrm{d}u$		
$=\int \left(1-\frac{1}{u}\right)\mathrm{d}u$	1A	
$=u-\ln u+\text{constant}$		
$=t-\ln(1+t)+\text{constant}$	1A	
Note that $\frac{t}{1+t}=1-\frac{1}{1+t}$.	1A	
$\int \frac{t}{1+t}\mathrm{d}t$		
$=\int \left(1-\frac{1}{1+t}\right)\mathrm{d}t$	1M	
$=t-\ln(1+t)+\text{constant}$	1A	
	----------(3)	
(d) The total amount of oil produced by oil company Y		
$=8\int_2^{12}\frac{t}{1+t}\mathrm{d}t$		
$=8\left[t-\ln(1+t)\right]_2^{12}$ (by (c))	1M	for using the result of (c)
≈ 68.26930345	1A	
<69.49587529		
By (b), the claim is disagreed.	1A	f.t.
	----------(3)	

Solution	Marks	Remarks
12. (a) $S=\dfrac{200}{1+a2^{bt}}$		
$\dfrac{200}{S}-1=a2^{bt}$	1M	
$\ln\left(\dfrac{200}{S}-1\right)=(b\ln 2)t+\ln a$	1A	
	----------(2)	
(b) (i) $\ln a=\ln 4$ $a=4$	1A	
$b\ln 2=\dfrac{0-\ln 4}{4-0}$ $b=-0.5$	1A	
(ii) $\dfrac{\mathrm{d}S}{\mathrm{d}t}$		
$=\dfrac{-200(4)2^{-0.5t}(-0.5)\ln 2}{(1+4(2^{-0.5t}))^2}$	1M	for $\dfrac{\mathrm{d}}{\mathrm{d}t}2^{bt}$
$=\dfrac{(400\ln 2)2^{-0.5t}}{(1+4(2^{-0.5t}))^2}$	1A	
$\dfrac{\mathrm{d}^2S}{\mathrm{d}t^2}$		
$=\dfrac{-200(\ln 2)^2(1+4(2^{-0.5t}))^2 2^{-0.5t}+1600(\ln 2)^2(1+4(2^{-0.5t}))2^{-t}}{(1+4(2^{-0.5t}))^4}$	1M	for quotient rule
$=\dfrac{-200(\ln 2)^2 2^{-0.5t}(1-4(2^{-0.5t}))}{(1+4(2^{-0.5t}))^3}$	1A	
(iii) Note that $\dfrac{\mathrm{d}S}{\mathrm{d}t}>0$ for $0\le t\le 48$.	1M	
Therefore, S increases for $0\le t\le 48$.	1A	f.t.
For $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}S}{\mathrm{d}t}\right)=0$, we have $4(2^{-0.5t})=1$.		
Hence, we have $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}S}{\mathrm{d}t}\right)=0$ when $t=4$.		
(table below)	1M+1A	
Thus, $\dfrac{\mathrm{d}S}{\mathrm{d}t}$ increases for $0\le t\le 4$ and		
$\dfrac{\mathrm{d}S}{\mathrm{d}t}$ decreases for $4\le t\le 48$.	1A	f.t.
	--------(11)	

t	$0\le t<4$	$t=4$	$4<t\le 48$
$\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}S}{\mathrm{d}t}\right)$	+	0	–