Module 1 (Calculus and Statistics) Marking Scheme

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct alternative solution merits all the marks allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept / technique had been used.

3. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

4. Unless the form of the answer is specified in the question, alternative simplified forms of answers different from those in the marking scheme should be accepted if they are correct.

5. In the marking scheme, marks are classified into the following three categories:

'M' marks	–	awarded for applying correct methods
'A' marks	–	awarded for the accuracy of the answers
Marks without 'M' or 'A'	–	awarded for correctly completing a proof or arriving at an answer given in the question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. (I.e. Markers should follow through candidates' work in awarding 'M' marks.) However, 'A' marks for the corresponding answers should NOT be awarded, unless otherwise specified.

6. In the marking scheme, steps which can be skipped are enclosed by dotted rectangles, whereas alternative answers are enclosed by solid rectangles.

7. Unless otherwise specified in the question, numerical answers should either be exact or given to 4 decimal places. Answers not accurate up to the required degree of accuracy should not be accepted.

Solution	Marks	Remarks
1. Let V and r be the volume and radius of the spherical balloon respectively.		
$V=\frac{4}{3}\pi r^3$		
$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$	1M	
$\therefore\ -100=4\pi\cdot 10^2\frac{dr}{dt}$	1M	
$\frac{dr}{dt}=\frac{-1}{4\pi}$	1A	OR -0.0796
Hence the rate of change of the radius is $\frac{-1}{4\pi}$ cm s^{-1}		
	(3)	
2. (a) $f(x)=\frac{x^x}{(2x+13)^6}$		
$\ln f(x)=x\ln x-6\ln(2x+13)$	1A	
$\frac{f'(x)}{f(x)}=\ln x+x\cdot\frac{1}{x}-6\cdot\frac{2}{2x+13}$	1M+1A	
$f'(x)=\left(\ln x+1-\frac{12}{2x+13}\right)f(x)$		
$=\left(\ln x+\frac{2x+1}{2x+13}\right)\frac{x^x}{(2x+13)^6}$	1A	Accept $\left(\ln x+\frac{2x+1}{2x+13}\right)f(x)$
(b) For $x>1$, we have $\ln x>0$, $\frac{2x+1}{2x+13}>0$ and $\frac{x^x}{(2x+13)^6}>0$.	1M	
$\therefore\ f'(x)>0$		OR $f'(x)\geq 0$
Hence $f(x)$ is an increasing function.	1	
	(6)	
3. (a) $\left.\frac{dy}{dx}\right\vert_{(1,5)}=\left(2\cdot 1-\frac{1}{1}\right)^3$		
$=1$	1A	
Hence the equation of tangent is $y-5=1(x-1)$.		
i.e. $x-y+4=0$	1A	
(b) (i) $\left(2x-\frac{1}{x}\right)^3=(2x)^3-3(2x)^2\left(\frac{1}{x}\right)+3(2x)\left(\frac{1}{x}\right)^2-\left(\frac{1}{x}\right)^3$	1M	
$=8x^3-12x+\frac{6}{x}-\frac{1}{x^3}$	1A	
(ii) $y=\int\left(2x-\frac{1}{x}\right)^3dx$		
$=\int\left(8x^3-12x+\frac{6}{x}-\frac{1}{x^3}\right)dx$ by (i)		
$=2x^4-6x^2+6\ln\vert x\vert+\frac{1}{2x^2}+C$	1M	

Solution	Marks	Remarks
Since $P(1,5)$ lies on S, $5=2(1)^4-6(1)^2+6\ln\|1\|+\frac{1}{2(1)^2}+C$.	1M	
i.e. $C=\frac{17}{2}$		
Hence the equation of S is $y=2x^4-6x^2+6\ln x+\frac{1}{2x^2}+\frac{17}{2}$ for $x>0$.	1A	
	(7)	
4. (a) Let $u=t^2+4t+11$. $\mathrm{d}u=(2t+4)\mathrm{d}t$	1A	
When $t=1, u=16$; when $t=3, u=32$.		
$\int_1^3 \frac{t+2}{t^2+4t+11}\,\mathrm{d}t=\int_{16}^{32}\frac{1}{u}\frac{\mathrm{d}u}{2}$	1M	OR $\frac{1}{2}\int_{t=1}^{3}\frac{\mathrm{d}(t^2+4t+11)}{t^2+4t+11}$
$=\frac{1}{2}\left[\ln\|u\|\right]_{16}^{32}$	1A	
$=\frac{\ln 32-\ln 16}{2}$	1A	
$=\frac{\ln 2}{2}$		OR 0.3466
(b) $\int_1^3 \frac{t^2+3t+9}{t^2+4t+11}\,\mathrm{d}t=\int_1^3\left(1-\frac{t+2}{t^2+4t+11}\right)\mathrm{d}t$	1M	
$=[t]_1^3-\int_1^3\frac{t+2}{t^2+4t+11}\,\mathrm{d}t$		
$=2-\frac{\ln 2}{2}$	1A	OR 1.6534
	(6)	
5. $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{t\sqrt{9-t^2}}{3}$		
Let $u=9-t^2$. $\mathrm{d}u=-2t\mathrm{d}t$	1M	
$x=\int\frac{t\sqrt{9-t^2}}{3}\,\mathrm{d}t$		
$=\int\frac{u^{\frac{1}{2}}}{3}\frac{\mathrm{d}u}{-2}$	1M	OR $\int\frac{(9-t^2)^{\frac{1}{2}}}{3}\frac{\mathrm{d}(9-t^2)}{-2}$
$=\frac{-1}{6}\cdot\frac{2u^{\frac{3}{2}}}{3}+C$	1M	
$=\frac{-1}{9}(9-t^2)^{\frac{3}{2}}+C$		
When $t=0$, $x=8$.		
$\therefore\ 8=\frac{-1}{9}(9-0)^{\frac{3}{2}}+C$	1M	
$C=11$		
i.e. $x=\frac{-1}{9}(9-t^2)^{\frac{3}{2}}+11$	1A	
	(5)	

Solution	Marks	Remarks
6. (a) $0.1k+0.2(0)+0.3(4)+0.4(6)=3.4$	1M	
$k=-2$	1A	
(b) $\text{Var}(3-4X)=16\text{Var}(X)$	1M	
$=16[\text{E}(X^2)-\text{E}(X)^2]$		
$=16[0.1(-2)^2+0.2(0)^2+0.3(4)^2+0.4(6)^2-3.4^2]$		

Alternative Solution

x	-2	0	4	6
$3-4x$	11	3	-13	-21
$\text{P}(X=x)$	0.1	0.2	0.3	0.4

(1M)

$\text{E}(3-4X)=0.1(11)+0.2(3)+0.3(-13)+0.4(-21)$ (OR $3-4(3.4)$)

$=-10.6$

$\text{Var}(3-4X)=0.1(11+10.6)^2+0.2(3+10.6)^2+0.3(-13+10.6)^2+0.4(-21+10.6)^2$

Solution	Marks	Remarks
$=128.64$	1A	
(c) $\text{P}(G\cap H)=\text{P}(-1\le X<4)$		
$=\text{P}(X=0)$		
$=0.2$	1A	
	(5)	
7. (a) $\text{P}(A\cap B)=\text{P}(B)\text{P}(A\mid B)$		
$=(1-0.75)\times 0.4$	1M	
$=0.1$	1A	
$\text{P}(A\cup B)=\text{P}(A)+\text{P}(B)-\text{P}(A\cap B)$		
$0.45=\text{P}(A)+(1-0.75)-0.1$	1M	
$\text{P}(A)=0.3$	1A	
(b) $\text{P}(A\mid B)=0.4$		
$\neq \text{P}(A)$	1M	

Alternative Solution

$\text{P}(A)\text{P}(B)=0.3\times 0.25$

$=0.075$

$\neq \text{P}(A\cap B)$ (1M)

Solution	Marks	Remarks
Hence A and B are not independent events.	1	
	(6)	

Solution	Marks	Remarks
8. (a) P(the selected microwave oven is produced by line B and functions properly)		
$= (1-0.04)\times\frac{2}{3}$		
$= 0.64$	1A	
(b) $P(A)$ P(functions properly | A) = P(functions properly) P(A | functions properly)		
$P(A)\,(1-0.02) = (1-0.04)\left(1-\frac{2}{3}\right)$	1M	
$P(A) = \frac{16}{49}$	1A	OR 0.3265
(c) P(functions properly | B) $= \frac{0.64}{1-\frac{16}{49}}$	1M	
$= \frac{784}{825}$	1A	OR 0.9503
	(5)	
9. (a) An estimate of $p = \frac{80}{200}$		
$= 0.4$	1A	
An approximate 95% confidence interval for p		
$= \left(0.4-1.96\sqrt{\frac{0.4\times0.6}{200}},\ 0.4+1.96\sqrt{\frac{0.4\times0.6}{200}}\right)$	1M	
$\approx (0.3321, 0.4679)$	1A	
(b) $X \sim N\left(0.85, \frac{0.85\,(1-0.85)}{n}\right)$		
$P\left(X > \frac{100}{n}\right) < 0.05$	1A	
$P\left(Z > \frac{\frac{100}{n}-0.85}{\sqrt{\frac{0.85(0.15)}{n}}}\right) < 0.05$	1M	
$\frac{100-0.85n}{n}\sqrt{\frac{n}{0.1275}} > 1.645$	1M	
$0.85n + 1.645\sqrt{0.1275n} - 100 < 0$		
$-11.19754391 < \sqrt{n} < 10.50650569$		
$0 < n < 110.3866618$		
Hence the maximum number of n is 110 .	1A	
	(7)	

Solution	Marks	Remarks
10. (a) (i) $\frac{d}{dv}(ve^{-v}) = e^{-v} - ve^{-v}$	1A	
(ii) $ve^{-v} = e^{-v} - \frac{d}{dv}ve^{-v}$		
$\int ve^{-v}dv = \int e^{-v}dv - ve^{-v}$	1M	
$= -e^{-v} - ve^{-v} + C$		
$= -e^{-v}(1+v) + C$	1	
	(3)	
(b) The area of the shaded region $= \int_1^2 \frac{\ln x}{x^2}\,dx$	1A	
Let $x = e^u$. $dx = e^u du$ When $x = 1$, $u = 0$; when $x = 2$, $u = \ln 2$	1A	OR $u = \ln x$
$\therefore$ the area $= \int_0^{\ln 2} \frac{u}{e^{2u}} \cdot e^u du$	1M	
$= \int_0^{\ln 2} ue^{-u}du$		
$= [-e^{-u}(1+u)]_0^{\ln 2}$ by (a)	1M	
$= \frac{-1}{2}(1+\ln 2) + 1$		
$= \frac{1-\ln 2}{2}$	1	
	(5)	
(c) (i) $\frac{d}{dx}\left(\frac{\ln x}{x^2}\right) = \frac{x^2 \cdot \frac{1}{x} - \ln x \cdot 2x}{(x^2)^2}$	1M	
$= \frac{1-2\ln x}{x^3}$		
$\frac{d^2}{dx^2}\left(\frac{\ln x}{x^2}\right) = \frac{x^3 \cdot \frac{-2}{x} - (1-2\ln x)3x^2}{x^6}$		
$= \frac{6\ln x - 5}{x^4}$	1A	
(ii) $\frac{d^2}{dx^2}\left(\frac{\ln x}{x^2}\right) < 0$ when $x < e^{\frac{5}{6}} \approx 2.30098$	1A	OR when $1 \le x \le 2$
Hence the trapezoidal rule will underestimate $\int_1^2 \frac{\ln x}{x^2}\,dx$.	1A	
Consider the trapezoidal rule with 10 intervals.		
$\therefore \frac{1}{2} \cdot \frac{1}{10}\left[\frac{\ln 1}{1^2} + 2\left(\frac{\ln 1.1}{1.1^2} + \frac{\ln 1.2}{1.2^2} + \cdots + \frac{\ln 1.9}{1.9^2}\right) + \frac{\ln 2}{2^2}\right] < \frac{1-\ln 2}{2}$	1M	For L.H.S.
$0 + 2\left(\frac{\ln 1.1}{1.1^2} + \frac{\ln 1.2}{1.2^2} + \cdots + \frac{\ln 1.9}{1.9^2}\right) + \frac{\ln 2}{4} < 10 - 10\ln 2$		
$\frac{\ln 1.1}{1.1^2} + \frac{\ln 1.2}{1.2^2} + \frac{\ln 1.3}{1.3^2} + \cdots + \frac{\ln 1.9}{1.9^2} < 5 - \frac{41}{8}\ln 2$	1	
	(6)	

Solution	Marks	Remarks
1. (a) $\lim_{t\to\infty}\frac{340}{2+e^{-t}-2e^{-2t}}=\frac{340}{2+0-2\cdot 0}$	1M	
$=170$		
Hence the value of y will not exceed 171 in the long run.	1A	
	(2)	
(b) $\frac{\mathrm{d}y}{\mathrm{d}t}=340[-(2+e^{-t}-2e^{-2t})^{-2}](-e^{-t}+4e^{-2t})$	1A	
$=\frac{340(e^{-t}-4e^{-2t})}{(2+e^{-t}-2e^{-2t})^2}$		
$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}t}=0$ when $e^{-t}-4e^{-2t}=0$	1M	For $\frac{\mathrm{d}y}{\mathrm{d}t}=0$
i.e. $t=\ln 4$	1A	

t	$0\le t<\ln 4$	$t=\ln 4$	$t>\ln 4$
$\frac{\mathrm{d}y}{\mathrm{d}t}$	-ve	0	+ve

Solution	Marks	Remarks
(sign table above)	1M	
Hence y is minimum when $t=\ln 4$.		
The minimum value of $y=\frac{340}{2+e^{-\ln 4}-2e^{-2\ln 4}}$		
$=160$	1A	
When $t=0$, $y=\frac{340}{2+e^0-2e^0}$		
$=340$		
As the graph of y is continuous, and by (a), the greatest value of y is 340 and the least value of y is 160.	1A	
	(6)	
(c) (i) $y=\frac{340}{2+e^{-t}-2e^{-2t}}$		
$2y+ye^{-t}-2ye^{-2t}=340$		
$2y(e^{-t})^2-ye^{-t}+340-2y=0$	1A	
(ii) Since $e^{-\alpha}$ and $e^{\alpha-3}$ are roots of the equation in (i),		
$\frac{340-2y}{2y}=e^{-\alpha}e^{\alpha-3}$	1M	
$340-2y=2ye^{-3}$		
Hence the equation becomes $2y(e^{-t})^2-ye^{-t}+2ye^{-3}=0$		
i.e. $2(e^{-t})^2-e^{-t}+2e^{-3}=0$	1A	
$\therefore\ e^{-\alpha}=\frac{1+\sqrt{1-16e^{-3}}}{4}$ or $\frac{1-\sqrt{1-16e^{-3}}}{4}$ (rejected as $e^{-\alpha}$ is the greater root)		
i.e. $\alpha=-\ln\frac{1+\sqrt{1-16e^{-3}}}{4}$	1A	OR $\ln\frac{1-\sqrt{1-16e^{-3}}}{4}+3$ OR 1.0140
	(4)	

Solution	Marks	Remarks
12. (a) $P(\mu-3.5 \le X \le \mu+3.5)=0.5160$	1A	
$P\left(0 \le Z \le \frac{3.5}{\sigma}\right)=0.2580$		
$\frac{3.5}{\sigma}=0.7$		
$\sigma=5$	1A	
$P(X>25)=0.2743$	1A	
$P\left(0<Z<\frac{25-\mu}{\sigma}\right)=0.2257$		
$\frac{25-\mu}{5}=0.6$		
$\mu=22$	1A	
	(4)	
(b) $P(X>k) \le \frac{5}{200}$	1A	
$P\left(0<Z<\frac{k-22}{5}\right) \ge 0.475$		
$\frac{k-22}{5} \ge 1.96$	1M	
$k \ge 31.8$		
Hence the minimum integral value of k is 32 .	1A	
	(3)	
(c) (i) Sample mean $=\frac{22+15+\cdots+24}{12}$		
$=21$	1A	
A 90% confidence interval		
$\approx\left(21-1.645\times\frac{4.7}{\sqrt{12}},21+1.645\times\frac{4.7}{\sqrt{12}}\right)$	1M	
$\approx(18.7681, 23.2319)$	1A	
(ii) Let $\overline{Y}$ be the mean delivery time of the n orders.		Alternative Solution
$P(\theta-3 \le \overline{Y} \le \theta+3)>0.99$		$\left(\overline{Y}-2.575\times\frac{4.7}{\sqrt{n}},\overline{Y}+2.575\times\frac{4.7}{\sqrt{n}}\right)$
$P\left(\frac{-3}{\frac{4.7}{\sqrt{n}}} \le Z \le \frac{3}{\frac{4.7}{\sqrt{n}}}\right)>0.99$	1M	$\subsetneqq(\overline{Y}-3,\overline{Y}-3)$
$\frac{3}{\frac{4.7}{\sqrt{n}}}>2.575$	1M	$\therefore 2.575\times\frac{4.7}{\sqrt{n}}<3$
$n>16.27450069$		
Hence the minimum value of n is 17 .	1A	
	(6)	

Solution	Marks	Remarks

13. (a) P(not more than 3 delays in a day)

$= e^{-4.8}\left(1 + 4.8 + \frac{4.8^2}{2!} + \frac{4.8^3}{3!}\right)$ — 1M

≈ 0.294229916

≈ 0.2942 — 1A

(2)

(b) P(at most 2 days with not more than 3 delays in a day in 3 consecutive days)

$\approx 1 - 0.294229916^3$ — 1M

≈ 0.9745 — 1A

(2)

Remarks: OR $\sum_{r=0}^{2} C_r^3 p^r (1-p)^{3-r}$, where $p \approx 0.294229916$

(c) Denote P(*bad day*) by k.

(i) $k = 1 - e^{-4.8}\left(1 + 4.8 + \frac{4.8^2}{2!} + \frac{4.8^3}{3!} + \frac{4.8^4}{4!} + \frac{4.8^5}{5!}\right)$

≈ 0.348993562 — 1A

$\therefore$ the mean number of *good days* between today and next *bad day*

$= \frac{1}{k} - 1$ — 1M

≈ 1.8654 — 1A

(ii) P(the last day in a week is the third *bad day* in that week)

$= C_2^6 k^2 (1-k)^4 k$ — 1M

≈ 0.1145 — 1A

(iii) P(there are at least 4 consecutive bad days in a week)

$= k^4 \cdot 1^3 + (1-k)k^4 \cdot 1^2 + 1(1-k)k^4 \cdot 1 + 1^2(1-k)k^4$ — 1M

Alternative Solution 1

$= [2k^4(1-k) + 2k^4(1-k)^2] + [2k^5(1-k) + k^5(1-k)^2] + 2k^6(1-k) + k^7$ — 1M

$= 2(k^4 - k^5 + k^4 - 2k^5 + k^6) + 2k^5 - 2k^6 + k^5 - 2k^6 + k^7 + 2k^6 - 2k^7 + k^7$

Alternative Solution 2

$= 4k^4(1-k)^3 + 9k^5(1-k)^2 + 6k^6(1-k) + k^7$ — 1M

$= 4(k^4 - 3k^5 + 3k^6 - k^7) + 9(k^5 - 2k^6 + k^7) + 6k^6 - 6k^7 + k^7$

$= 4k^4 - 3k^5$

≈ 0.0438 — 1A

(7)

Remarks:

(c)(iii)

S	M	T	W	T	F	S
B	B	B	B	/	/	/
G	B	B	B	B	/	/
/	G	B	B	B	B	/
/	/	G	B	B	B	B

(c)(iii) Alt Sol 1

S	M	T	W	T	F	S
B	B	B	B	G	/	/
G	B	B	B	B	G	/
/	G	B	B	B	B	G
/	/	G	B	B	B	B
B	B	B	B	B	G	/
G	B	B	B	B	B	G
/	G	B	B	B	B	B
B	B	B	B	B	B	G
G	B	B	B	B	B	B
B	B	B	B	B	B	B

(c)(iii) Alt Sol 2

S	M	T	W	T	F	S
B	B	B	B	G	G	G
G	B	B	B	B	G	G
G	G	B	B	B	B	G
G	G	G	B	B	B	B
B	B	B	B	B	G	G
G	B	B	B	B	B	G
G	G	B	B	B	B	B
B	B	B	B	G	B	G
B	B	B	B	G	G	B
G	B	B	B	B	G	B
B	G	B	B	B	B	G
B	G	G	B	B	B	B
G	B	G	B	B	B	B
B	B	B	B	B	B	G
B	B	B	B	B	G	B
B	B	B	B	G	B	B
B	B	G	B	B	B	B
B	G	B	B	B	B	B
G	B	B	B	B	B	B
B	B	B	B	B	B	B