Module 1 (Calculus and Statistics) Marking Scheme

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct alternative solution merits all the marks allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept / technique had been used.

3. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

4. Unless the form of the answer is specified in the question, alternative simplified forms of answers different from those in the marking scheme should be accepted if they are correct.

5. In the marking scheme, marks are classified into the following three categories:

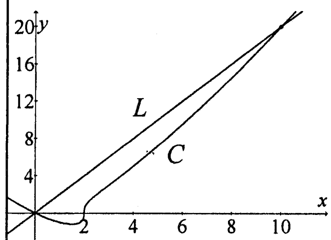
'M' marks	–	awarded for applying correct methods
'A' marks	–	awarded for the accuracy of the answers
Marks without 'M' or 'A'	–	awarded for correctly completing a proof or arriving at an answer given in the question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. (I.e. Markers should follow through candidates' work in awarding 'M' marks.) However, 'A' marks for the corresponding answers should NOT be awarded, unless otherwise specified.

6. In the marking scheme, steps which can be skipped are enclosed by dotted rectangles, whereas alternative answers are enclosed by solid rectangles.

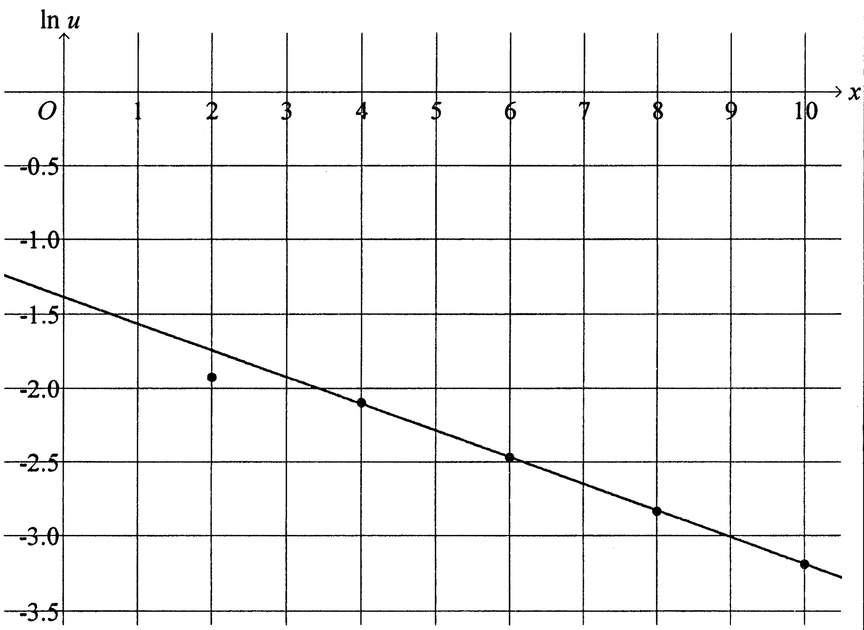
7. Unless otherwise specified in the question, numerical answers should either be exact or given to 4 decimal places. Answers not accurate up to the required degree of accuracy should not be accepted.

Solution	Marks	Remarks
1. (a) $\left(u+\frac{1}{u}\right)^4=u^4+4u^3\left(\frac{1}{u}\right)+6u^2\left(\frac{1}{u}\right)^2+4u\left(\frac{1}{u}\right)^3+\left(\frac{1}{u}\right)^4$		
$=u^4+4u^2+6+\frac{4}{u^2}+\frac{1}{u^4}$	1A	
(b) $(e^{ax}+e^{-ax})^4$		
$=e^{4ax}+4e^{2ax}+6+4e^{-2ax}+e^{-4ax}$ by (a)	1M	
$=\left[1+\frac{4ax}{1!}+\frac{(4ax)^2}{2!}+\cdots\right]+4\left[1+\frac{2ax}{1!}+\frac{(2ax)^2}{2!}+\cdots\right]+6$		
$+4\left[1+\frac{-2ax}{1!}+\frac{(-2ax)^2}{2!}+\cdots\right]+\left[1+\frac{-4ax}{1!}+\frac{(-4ax)^2}{2!}+\cdots\right]$	1M	
$=1+4ax+8a^2x^2+4+8ax+8a^2x^2+6+4-8ax+8a^2x^2+1-4ax+8a^2x^2+\cdots$		
$=16+32a^2x^2+\cdots$	1A	
(c) $32a^2=2$		
$a^2=\frac{1}{16}$		
$a=\pm\frac{1}{4}$	1A	
	(5)	
2. $p=8-\frac{2.1}{\sqrt{t+4}}$		
$\frac{dp}{dt}=\frac{2.1}{2(t+4)^{\frac{3}{2}}}$	1A	
$C=2^p$		
$\frac{dC}{dp}=2^p\ln 2$	1A	
$\frac{dC}{dt}=\frac{dC}{dp}\cdot\frac{dp}{dt}$		
$=2^p\ln 2\cdot\frac{2.1}{2(t+4)^{\frac{3}{2}}}$	1M	
When $t=5$, $p=7.3$ and hence		
$\frac{dC}{dt}=2^{7.3}\ln 2\cdot\frac{2.1}{2(5+4)^{\frac{3}{2}}}$		
≈ 4.2479	1A	
i.e. the rate of change of the concentration of carbon dioxide ≈ 4.2479 units/year.		
	(4)	

Solution	Marks	Remarks
3. (a) $y = x(x-2)^{\frac{1}{3}}$		
$\frac{dy}{dx} = (x-2)^{\frac{1}{3}} + \frac{1}{3}(x-2)^{\frac{-2}{3}} x$	1M	For product rule
When $x = 3$, $\frac{dy}{dx} = 2$.		
Hence the equation of L is $y = 2x$.	1A	
(b) Solving C and L :		
$x(x-2)^{\frac{1}{3}} = 2x$	1M	
$x\left[(x-2)^{\frac{1}{3}} - 2\right] = 0$		
$x = 0$ or 10	1A	
(c) The area bounded by L and C		
$= \int_0^{10} \left[2x - x(x-2)^{\frac{1}{3}}\right] dx$	1M	
$= \int_0^{10} 2x\,dx - \int_0^{10} x(x-2)^{\frac{1}{3}} dx$		
Let $u = x - 2$ and so $du = dx$.	1M	
When $x = 0$, $u = -2$; when $x = 10$, $u = 8$.		
$\therefore$ the area bounded by L and C		
$= \int_0^{10} 2x\,dx - \int_{-2}^{8} (u+2)u^{\frac{1}{3}} du$		
$= [x^2]_0^{10} - \int_{-2}^{8} \left(u^{\frac{4}{3}} + 2u^{\frac{1}{3}}\right) du$		
$= 100 - \left[\frac{3}{7}u^{\frac{7}{3}} + \frac{3}{2}u^{\frac{4}{3}}\right]_{-2}^{8}$	1M	For the primitive function
$= \frac{148 + 9\sqrt[3]{2}}{7}$	1A	OR 22.7628
	(8)	

Solution	Marks	Remarks
4. (a) (i) $u = ae^{-bx}$		
$\ln u = \ln a - bx$	1A	
(ii) $y = \dfrac{8(1-ae^{-bx})}{1+ae^{-bx}}$		
$= \dfrac{8-8u}{1+u}$		
$u = \dfrac{8-y}{8+y}$	1A	
(b) (i) By (a), $\ln\dfrac{8-y}{8+y} = \ln a - bx$		
(table below)	1A	For any two pairs of values
(graph below)	1A	
From the graph, we see that the value $y = 5.97$ is incorrect.	1A	
(ii) The y-intercept $= \ln a \approx -1.4$	1M	For either one
$\therefore\ a \approx 0.25$		
The slope $= -b \approx \dfrac{-3.19-(-2.10)}{10-4}$		
$\therefore\ b \approx 0.18$	1A	For both a and b
	(7)	

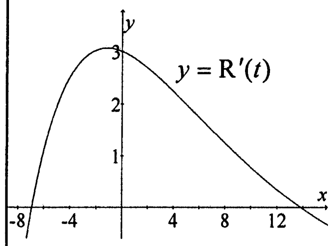
x	2	4	6	8	10
$\ln\dfrac{8-y}{8+y}$	−1.93	−2.10	−2.47	−2.83	−3.19



Solution	Marks	Remarks
5. (a) $\frac{d}{dx}(x\ln x)=(1)\ln x+x\left(\frac{1}{x}\right)$		
$=\ln x+1$	1A	
(b) $\ln x=\frac{d}{dx}(x\ln x)-1$		
$\int_1^e \ln x\,dx=[x\ln x]_1^e-\int_1^e 1\,dx$	1M	
$=e\ln e-\ln 1-[x]_1^e$	1A	For x
$=1$	1A	
	(4)	
6. An estimate for p is $\frac{75}{120}=0.625$.	1A	OR $\frac{5}{8}$
An approximate 90% confidence interval for p		
$\approx\left(0.625-1.645\sqrt{\frac{0.625(1-0.625)}{120}},\ 0.625+1.645\sqrt{\frac{0.625(1-0.625)}{120}}\right)$	1M+1M	
$\approx(0.5523, 0.6977)$	1A	
	(4)	
7. (a) $\text{E}(Y)=1\times0.4+2\times0.3+4\times0.2+m\times0.1=2.4$		
$\therefore\ m=6$	1A	
(b) (i) $\text{P}(A)=\text{P}(X=0,Y=1)+\text{P}(X=0,Y=2)+\text{P}(X=1,Y=1)$		
$=0.2\times0.4+0.2\times0.3+0.3\times0.4$	1M	
$=0.26$	1A	
(ii) $\text{P}(A\cap B)=\text{P}(X=0,Y=1)+\text{P}(X=0,Y=2)$		
$=0.2\times0.4+0.2\times0.3$		
$=0.14$	1A	
$\text{P}(A)\text{P}(B)=0.26\times0.2$		
$=0.052$		
$\neq\text{P}(A\cap B)$		
Alternative Solution		
$\text{P}(A\mid B)=\text{P}(Y=1)+\text{P}(Y=2)$		
$=0.4+0.3$		
$=0.7$	1A	
$\neq\text{P}(A)$ by (i)		
Thus, A and B are not independent.	1A	Follow through
	(5)	

Solution	Marks	Remarks
8. (a) P(get a prize | Mabel) $= 0.6 + 0.4\times 0.6 + 0.4^2 \times 0.6$	1M	OR $1-(1-0.6)^3$
$= 0.936$	1A	OR $(0.6)^3 + C_1^3(0.6)^2(0.4) + C_1^3(0.6)(0.4)^2$
(b) P(get a prize | Owen) $= 0.5 + 0.5\times 0.5 + 0.5^2 \times 0.5$ $= 0.875$		OR $1-(1-0.5)^3$
$\therefore$ P(win) $= 0.7\times 0.936 + 0.3\times 0.875$	1M	
$= 0.9177$	1A	
(c) P(Owen | does not get a prize) $= \dfrac{0.3\times(1-0.875)}{1-0.9177}$	1M	OR $\dfrac{0.3\times(1-0.5)^3}{1-0.9177}$
$= \dfrac{375}{823}$	1A	OR 0.4557
	(6)	
9. (a) P (lifetime of a bulb < 39000) $= 0.9641$		
$P\left(Z < \dfrac{39000-\mu}{5000}\right) = 0.9641$	1M	
$\dfrac{39000-\mu}{5000} \approx 1.8$ $\mu \approx 30000$	1A	
(b) P (30200 < sample mean < 30800)		Can use '$\le$' sign
$= P\left(\dfrac{30200-30000}{\frac{5000}{\sqrt{100}}} < Z < \dfrac{30800-30000}{\frac{5000}{\sqrt{100}}}\right)$	1M	
$= P(0.4 < Z < 1.6)$ $\approx 0.4452 - 0.1554$ $= 0.2898$	1A	
(c) P (sample mean > 28500) ≥ 0.985		
$P\left(Z > \dfrac{28500-30000}{\frac{5000}{\sqrt{n}}}\right) \ge 0.985$	1M	
$-0.3\sqrt{n} \le -2.17$	1A	
$n \ge 52.32111111$ Thus, the least value of n is 53 .	1A	
	(7)	

Solution	Marks	Remarks
10. (a) (i) $\ln(x^2+16)-\ln(3x+20)<0$		
$\ln(x^2+16)<\ln(3x+20)$		
$x^2+16<3x+20$	1A	
$x^2-3x-4<0$		
$-1<x<4$	1A	
(ii) (1) $I=\int_0^4[\ln(x^2+16)-\ln(3x+20)]\,dx$		
$\approx\frac{1}{2}[-0.223143551+0+2(-0.302280871-0.262364264-0.148420005)]$	1M	OR $\frac{1}{2}\{(\ln 16-\ln 20)+0+2[(\ln 17-\ln 23)+(\ln 20-\ln 26)+(\ln 25-\ln 29)]\}$
≈ -0.824636917		
≈ -0.8246	1A	
(2) $f(x)=\ln(x^2+16)-\ln(3x+20)$		
$f'(x)=\frac{2x}{x^2+16}-\frac{3}{3x+20}$	1M+1A	OR $\frac{3x^2+40x-48}{(x^2+16)(3x+20)}$
$f''(x)=2\cdot\frac{(x^2+16)-x(2x)}{(x^2+16)^2}-3\cdot\frac{(-1)\cdot 3}{(3x+20)^2}$	1M	
$=\frac{2(4+x)(4-x)}{(x^2+16)^2}+\frac{9}{(3x+20)^2}$		
>0 for $0\le x\le 4$		
Hence the estimate in (1) is an over-estimate.	1A	Follow through
	(8)	
(b) (i) $N'(t)=10\ln(t^2+16)-10\ln(3t+20)$	1A	
(ii) Assume that Jane's claim is true: the species will not die out until $t=4$, i.e. $N(t)>0$ for $0\le t\le 4$.		
$N(4)-N(0)=\int_0^4[10\ln(t^2+16)-10\ln(3t+20)]\,dt$	1A	
$N(4)-8<-8.24636917$ (since the estimate is an over-estimate)	1M	
$N(4)<0$		
Hence Jane's claim is false and cannot be agreed with.	1A	Follow through
	(4)	
11. (a) $R'(t)=0$		
$P'(t)-C'(t)=0$		
$4(4-e^{\frac{-t}{5}})-9(2-e^{\frac{-t}{10}})=0$	1A	
$-4\left(e^{\frac{-t}{10}}\right)^2+9e^{\frac{-t}{10}}-2=0$	1M	For $e^{\frac{-t}{5}}=\left(e^{\frac{-t}{10}}\right)^2$
$e^{\frac{-t}{10}}=0.25$ or 2		
$t=20\ln 2$ or $-10\ln 2$ (rejected as $t\ge 0$)	1A	OR $t\approx 13.8629$
	(3)	

Solution	Marks	Remarks
(b) $R'(t)=-4e^{\frac{-t}{5}}+9e^{\frac{-t}{10}}-2$		
$R''(t)=\frac{4}{5}e^{\frac{-t}{5}}-\frac{9}{10}e^{\frac{-t}{10}}$	1A	
$=\frac{1}{10}e^{\frac{-t}{10}}\left(8e^{\frac{-t}{10}}-9\right)$	1M	
<0 for $t\geq 0$ (since $e^{\frac{-t}{10}}\leq 1$ for $t\geq 0$)		
Therefore $R'(t)$ decreases with t.	1	
	(3)	
(c) By (a) and (b), $R'(t)>0$ when $0\leq t<20\ln 2$.		
The total redundant electric energy generated during the period when $R'(t)>0$		
$=\int_0^{20\ln 2}\left(-4e^{\frac{-t}{5}}+9e^{\frac{-t}{10}}-2\right)dt$	1M	For lower and upper limits
$=\left[20e^{\frac{-t}{5}}-90e^{\frac{-t}{10}}-2t\right]_0^{20\ln 2}$	1A	For primitive function
$=48.75-40\ln 2$	1A	OR 21.0241
	(3)	
(d) Consider $\int_5^8\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}dt$		
Let $u=\ln(t^2+2t+3)$.	1M	
$du=\frac{2t+2}{t^2+2t+3}dt$	1A	
When $t=5$, $u=\ln 38$; when $t=8$, $u=\ln 83$.		
$\therefore \int_5^8\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}dt=\int_{\ln 38}^{\ln 83}u^3\frac{du}{2}$	1A	For $\frac{u^3}{2}$
$=\frac{1}{8}\left[u^4\right]_{\ln 38}^{\ln 83}$		
$=\frac{1}{8}[(\ln 83)^4-(\ln 38)^4]$		
Hence the total electric energy produced for the first 3 years after the improvement		
$=\int_5^8\left[\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}+9\right]dt$	1A	
$=\int_5^8\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}dt+\int_5^8 9\,dt$		
$=\frac{1}{8}[(\ln 83)^4-(\ln 38)^4]+[9t]_5^8$		
$=\frac{1}{8}[(\ln 83)^4-(\ln 38)^4]+27$	1A	OR 52.7730
	(5)	

Solution	Marks	Remarks
12. (a) (i) $2\times 1.96\times\frac{\sigma}{\sqrt{49}}=5.044-4.596$	1M	
$\sigma=0.8$	1A	
(ii) The mean of the sample $=\frac{4.596+5.044}{2}$		
$=4.82$	1A	
	(3)	
(b) The combined sample mean $=\frac{4.82\times 49+3.6+3.8+\cdots+6.4}{49+15}$	1M	
$=4.83875$	1A	OR 4.8388
A 99% confidence interval for μ		
$\approx\left(4.83875-2.575\times\frac{0.8}{\sqrt{64}},\ 4.83875+2.575\times\frac{0.8}{\sqrt{64}}\right)$	1M	
$=(4.58125,5.09625)$	1A	OR $(4.5813, 5.0963)$
	(4)	
(c) Let X be the cholesterol level of a randomly selected adult.		
(i) $\text{P(low)}=\text{P}(X\le 5.2)$		
$=\text{P}\left(Z\le\frac{5.2-4.8}{0.8}\right)$		
$=\text{P}(Z\le 0.5)$		
≈ 0.6915	1A	
(ii) $\text{P(high)}=\text{P}(X\ge 6.2)$		
$=\text{P}\left(Z\ge\frac{6.2-4.8}{0.8}\right)$		
$=\text{P}(Z\ge 1.75)$		
≈ 0.0401	1A	
$\text{P(medium)}\approx 1-0.6915-0.0401$		
$=0.2684$	1A	
P(more than 17 adults with low level and at least 1 adult with medium level)		
$\approx C_{18}^{20}(0.6915)^{18}[C_1^2(0.2684)(0.0401)+(0.2684)^2]+C_{19}^{20}(0.6915)^{19}(0.2684)$	1M	
≈ 0.0281	1A	
	(5)	

Solution	Marks	Remarks
13. (a) P(the regular maintenance service of a lift in a certain month in the estate is unacceptable)		
$=1-e^{-1.9}\left(1+\frac{1.9^1}{1!}+\frac{1.9^2}{2!}\right)$	1M	
≈ 0.296279646		
≈ 0.2963	1A	
	(2)	
(b) P(the maintenance service of a lift in June of 2014 is the 3rd month unacceptable)		
$\approx C_2^5(0.296279646)^2(1-0.296279646)^3\cdot(0.296279646)$	1M	
≈ 0.0906	1A	
	(2)	
(c) The expected total number of unacceptable maintenance services of all lifts for one year		
$\approx 15\times 12\times 0.296279646$	1M	
≈ 53.3303	1A	
	(2)	
(d) (i) P(a warning letter will be issued for a lift on or before 30th April 2015)		
$\approx(0.296279646)^3+(1-0.296279646)\cdot(0.296279646)^3$	1M+1M	
≈ 0.044310205		
≈ 0.0443	1A	
(ii) P(3 or more warning letters will be issued on or before 30th April 2015)		
$\approx 1-(1-0.044310205)^{15}-C_1^{15}(0.044310205)(1-0.044310205)^{14}$ $-C_2^{15}(0.044310205)^2(1-0.044310205)^{13}$	1M+1M	
≈ 0.0265	1A	
	(6)	