5. Indefinite Integrals

Learning Unit	Learning Objective
Calculus Area	
Integration with Its Applications	
7. Indefinite integrals and their applications	7.1 recognise the concept of indefinite integration 7.2 understand the basic properties of indefinite integrals and basic integration formulae 7.3 use basic integration formulae to find the indefinite integrals of algebraic functions and exponential functions 7.4 use integration by substitution to find indefinite integrals 7.5 use indefinite integration to solve problems

Section A

1. Let $f(x)$ be a continuous function such that $f'(x)=\frac{12x-48}{(3x^2-24x+49)^2}$ for all real numbers x.

   (a) If $f(x)$ attains its minimum value at $x=\alpha$, find α.

   (b) It is given that the extreme value of $f(x)$ is 5 . Find

   (i) $f(x)$,

   (ii) $\lim_{x\to\infty} f(x)$.

(6 marks) (2018 DSE-MATH-M1 Q5)

2. (a) Express $\frac{d}{dx}\left((x^6+1)\ln(x^2+1)\right)$ in the form $f(x)+g(x)\ln(x^2+1)$, where $f(x)$ and $g(x)$ are polynomials.

   (b) Find $\int x^5\ln(x^2+1)dx$.

(7 marks) (2015 DSE-MATH-M1 Q8)

3. The slope of the tangent to a curve S at any point (x, y) on S is given by $\frac{dy}{dx} = \left(2x - \frac{1}{x}\right)^3$, where $x > 0$.
A point $P(1, 5)$ lies on S.
(a) Find the equation of the tangent to S at P.
(b) (i) Expand $\left(2x - \frac{1}{x}\right)^3$.
(ii) Find the equation of S for $x > 0$.
(7 marks) (2014 DSE-MATH-M1 Q3)

4. The government of a country is going to announce a new immigration policy which will last for 3 years. At the moment of the announcement, the population of the country is 8 million, After the announcement, the rate of change of the population can be modelled by

$$\frac{dx}{dt} = \frac{t\sqrt{9 - t^2}}{3} \qquad (0 \le t \le 3),$$

where x is the population (in million) of the country and t is the time (in years) which has elapsed since the announcement. Find x in terms of t.
(5 marks) (2014 DSE-MATH-M1 Q5)

5. The rate of change of the value V (in million dollars) of a flat is given by $\frac{dV}{dt} = \frac{t}{\sqrt{4t + 1}}$, where t is the number of years since the beginning of 2012. The value of the flat is 3 million dollars at the beginning of 2012. Find the percentage change in the value of the flat from the beginning of 2012 to the beginning of 2014.
(5 marks) (2012 DSE-MATH-M1 Q2)

6. An advertising company starts a media advertisement to recruit new members for a club. Past experience shows that the rate of change of the number of members N (in thousand) is given by

$$\frac{dN}{dt} = \frac{0.3e^{-0.2t}}{\left(1 + e^{-0.2t}\right)^2},$$

where $t\ (\ge 0)$ is the number of weeks elapsed after the launch of the advertisement. The club has 500 members before the launch of the advertisement.
(a) Using the substitution $u = 1 + e^{-0.2t}$, express N in terms of t.
(b) Find the increase in the number of members of the club 4 weeks after the launch of the advertisement. Correct your answer to the nearest integer.
(c) Will the number of members of the club ever reach 1300 after the launch of the advertisement? Explain your answer.
(7 marks) (2012 ASL-M&S Q3)

7. A company launches a promotion plan to raise revenue. The total amount of money X (in million dollars) invested in the plan can be modelled by

$$\frac{dX}{dt} = 6\left(\frac{t}{0.2t^3 + 1}\right)^2, \quad t \ge 0,$$

where t is the number of months elapsed since the launch of the plan.
Initially, 4 million dollars are invested in the plan.
(a) Using the substitution $u = 0.2t^3 + 1$, or otherwise, express X in terms of t.
(b) Find the number of months elapsed since the launch of the plan if a total amount of 13 million dollars are invested in the plan.
(c) If the company has a budget of 14.5 million dollars only, can the plan be run for a long time? Explain your answer.
(7 marks) (2011 ASL-M&S Q2)

8. An archaeologist models the presence of carbon-14 remaining in animal skulls fossil by $\frac{dA}{dt} = -kA$ where A (in grams) is the amount of carbon-14 present in the skull at time t (in years) and k is a constant. Let A_0 (in grams) be the original amount of carbon-14 in the skull. It is known that half of the carbon-14 will disappear after 5730 years.
(a) By expressing $\frac{dt}{dA}$ in terms of A, or otherwise, find the value of k correct to 3 significant figures.
(b) In an animal skull fossil, the archaeologist found that 30% of the original amount of carbon-14 is still present. Find the approximate age of the skull correct to the nearest ten years.
(6 marks) (2010 ASL-M&S Q3)

9. A scientist models the proportion, P, of the initial population of an endangered species of animal still surviving by

$$\frac{dP}{dt} = \frac{-0.09}{\sqrt{3t + 1}} \qquad (0 \le t \le T)$$

where t is time in months since the beginning of his study, and T is the number of months elapsed for the population size to decrease to 0. It is given that when $t = 0$, $P = 1$.
(a) Find the proportion of the endangered species surviving after t months from the beginning of the study.
(b) What is the proportion of the endangered species dying off within the first 5 months of the study?
(c) Determine the value of T.
(6 marks) (2009 ASL-M&S Q2)

10. The rate of change of concentration of a drug in the blood of a patient can be modelled by

$$\frac{dx}{dt}=5.3\left(\frac{1}{t+2}-\frac{1}{t+5}\right)+1.2e^{-0.1t} ,$$

where x is the concentration measured in mg/L and t is the time measured in hours after the patient has taken the drug. It is given that $x=0$ when $t=0$.

(a) Find x in terms of t .

(b) Find the concentration of the drug after a long time.

(6 marks) (2008 ASL-M&S Q3)

11. A researcher models the rate of change of the number of certain bacteria under controlled conditions by

$$\frac{dN}{dt}=\frac{800t}{(2t^2+50)^2} ,$$

where N is the number in millions of bacteria and $t(\geq 0)$ is the number of days elapsed since the start of the research. It is given that $N=4$ when $t=0$.

(a) Using the substitution $u=2t^2+50$, or otherwise, express N in terms of t .

(b) When will the number of bacteria be 6 million after the start of the research?

(7 marks) (2007 ASL-M&S Q2)

12. A researcher models the rate of change of the number of fish in a lake by

$$\frac{dN}{dt}=\frac{6}{(e^{\frac{t}{4}}+e^{-\frac{t}{4}})^2} ,$$

where N is the number in thousands of fish in the lake recorded yearly and $t(\geq 0)$ is the time measured in years from the start of the research. It is known that $N=8$ when $t=0$.

(a) Prove that $\frac{dN}{dt}=\frac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2}$. Using the substitution $u=e^{\frac{t}{2}}+1$, or otherwise, express N in terms of t .

(b) Estimate the number of fish in the lake after a very long time.

(6 marks) (2004 ASL-M&S Q2)

13. After a fixed amount of hot liquid is poured into a vessel, the rate of change of the temperature θ of the surface of the vessel can be modelled by

$$\frac{d\theta}{dt}=\frac{12(100-t)e^{\frac{-t}{100}}}{25(1+3te^{\frac{-t}{100}})} ,$$

where θ is measured in $^{\circ}C$ and $t(\geq 0)$ is the time measured in seconds. Initially $(t=0)$, the temperature of the surface of the vessel is $16^{\circ}C$.

(a) (i) Let $u=1+3te^{\frac{-t}{100}}$, find $\frac{du}{dt}$.

(ii) Using the result of (i), or otherwise, express θ in terms of t .

(b) Will the temperature of the surface of the vessel get higher than $95^{\circ}C$? Explain your answer briefly.

(7 marks) (2003 ASL-M&S Q2)

14. An engineer conducts a test for a certain brand of air-purifier in a smoke-filled room. The percentage of smoke in the room being removed by the air-purifier is given by S%. The engineer models the rate of change of S by

$$\frac{dS}{dt}=\frac{8100t}{(3t+10)^3} ,$$

where $t\ (\geq 0)$ is measured in hours from the start of the test.

(a) Using the substitution $u=3t+10$, or otherwise, find the percentage of smoke removed from the room in the first 10 hours.

(b) If the air-purifier operates indefinitely, what will the percentage of smoke removed from the room be?

(5 marks) (2002 ASL-M&S Q4)

15. The total number of visits N to a web site increases at a rate of

$$\frac{dN}{dt}=t^{\frac{1}{3}}(8+11t^{\frac{1}{2}}) \quad (0\leq t\leq 100) ,$$

where t is the time in weeks since January 1, 1999. It is known that $N=100$ when $t=1$.

(a) Express N in terms of t .

(b) Find the total number of visits to the web site when $t=64$.

(6 marks) (1999 ASL-M&S Q4)

16. A mobile phone company plans to invite a famous singer to help to promote its products. The Executive Director of the company estimates that the rate of increase of the number of customers can modeled by

$$\frac{dx}{dt} = 650e^{-0.004t} \qquad (0 \le t \le 365),$$

where x is the number of customers of the company and t is the number of days which has elapsed since the start of the promotion campaign.

(a) Suppose that at the start of the campaign, the company already has 57 000 customers. Express x in terms of t.

(b) How many days after the start of the campaign will the number of customers be doubled?

(6 marks) (1998 ASL-M&S Q4)

17. A machine depreciates with time t in years. Its value \$ $V(t)$ is initially \$20 000 and will drop to \$ 0 when $t = k$ $(k \ge 0)$. The depreciation rate at time t is

$$V'(t) = 200(t-15) \qquad \text{for } 0 \le t \le k .$$

(a) $V(t)$ for $0 \le t \le k$,

(b) the value of k, and

(c) the total depreciation in the first 5 years.

(7 marks) (1997 ASL-M&S Q4)

18. Let $y = \frac{\ln x}{x}$ $(x > 0)$, find $\frac{dy}{dx}$.

Hence or otherwise, find $\int \frac{\ln x}{x^2} dx$.

(5 marks) (1996 ASL-M&S Q2)

19. The value M (in million dollars) of a house is modeled by the equation

$$\frac{dM}{dt} = \frac{1}{3t+4} + \frac{1}{\sqrt{t+25}}$$

where t is the number of years elapsed since the end of 1994. The value of the house is 3.1 million dollars at the end of 1994.

(a) Find, in terms of t, the value of the house t years after the end of 1994.

(b) Find the rise in the value of house between the end of 1994 and the end of 2000.

(7 marks) (1995 ASL-M&S Q4)

20. The rate of spread of an epidemic can be modelled by the equation

$$\frac{dx}{dt} = 3t\sqrt{t^2+1},$$

where x is the number of people infected by the epidemic and t is the number of days which have elapsed since the outbreak of the epidemic. If $x = 10$ when $t = 0$, express x in terms of t.

(6 marks) (1994 ASL-M&S Q5)

Section B

21. In a research of the radiation intensity of a city, an expert modelled the rate of change of the radiation intensity R (in suitable units) by

$$\frac{dR}{dt} = \frac{a(30-t)+10}{(t-35)^2+b}$$

where t $(0 \le t \le T)$ is the number of days elapsed since the start of the research, a, b and T are positive constants.

It is known that the intensity increased to the greatest value of 6 units at $t = 35$, and then decreased to the level as at the start of the research at $t = T$. Moreover, the decrease of the intensity from $t = 40$ to $t = 41$ is $\ln \frac{61}{50}$ units

(a) Find the value of a.

(2 marks)

(b) Find the value of T.

(4 marks)

(c) Express R in terms of t.

(4 marks)

(d) For $0 \le t \le 35$, when would the rate of change of the radiation intensity attain its greatest value?

(4 marks)

(2012 DSE-MATH-M1 Q11)

22. The manager, Mary, of a theme park starts a promotion plan to increase **the daily number of visits** to the park. The rate of change of **the daily number of visits** to the park can be modelled by

$$\frac{dN}{dt}=\frac{k(25-t)}{e^{0.04t}+4t} \qquad (t\ge 0),$$

where N is **the daily number of visits** (in hundreds) recorded at the end of a day, t is the number of days elapsed since the start of the plan and k is a positive constant.

Mary finds that at the start of the plan, $N=10$ and $\frac{dN}{dt}=50$.

(a) (i) Let $v=1+4te^{-0.04t}$, find $\frac{dv}{dt}$.

(ii) Find the value of k, and hence express N in terms of t.

(7 marks)

(b) (i) When will **the daily number of visits** attain the greatest value?

(ii) Mary claims that there will be more than 50 hundred visits on a certain day after the start of the plan. Do you agree? Explain your answer.

(3 marks)

(c) Mary's supervisor believes that **the daily number of visits** to the park will return to the original one at the start of the plan after a long period of time. Do you agree? Explain your answer.

(Hint: $\lim_{t\to\infty} te^{-0.04t}=0$.)

(2 marks)

(SAMPLE DSE-MATH-M1 Q11)

23. A biologist studied the population of fruit fly A under limited food supply. Let t be the number of days since the beginning of the experiment and N'(t) be the number of fruit fly A at time t. The biologist modelled the rate of change of the number of fruit fly A by

$$N'(t)=\frac{20}{1+he^{-kt}} \qquad (t\ge 0)$$

where h and k are positive constants.

(a) (i) Express $\ln\left(\frac{20}{N'(t)}-1\right)$ as a linear function of t.

(ii) It is given that the intercepts on the vertical axis and the horizontal axis of the graph of the linear function in (i) are 1.5 and 7.6 respectively. Find the values of h and k.

(4 marks)

(b) Take $h=4.5$ and $k=0.2$, and assume that $N(0)=50$.

(i) Let $v=h+e^{kt}$, find $\frac{dv}{dt}$.

Hence, or otherwise, find N(t).

(ii) The population of fruit fly B can be modelled by

$$M(t)=21\left(t+\frac{h}{k}e^{-kt}\right)+b,$$

where b is a constant. It is known that $M(20)=N(20)$.

(1) Find the value of b.

(2) The biologist claims that the number of fruit fly A will be smaller than that of fruit fly B for $t>20$. Do you agree? Explain your answer.

[Hint: Consider the difference between the rates of change of the two populations.]

(11 marks)

(2008 ASL-M&S Q8)

24. An airline manager, Christine, notice that the *weekly number of passengers* of the airline is declining, so she starts a promotion plan to boost the *weekly number of passengers*. She models the rate of change of the *weekly number of passengers* by

$$\frac{dx}{dt}=\frac{30t-90}{t^2-6t+11} \qquad (t\geq 0)\ ,$$

where x is the *weekly number of passengers* recorded at the end of a week in thousands of passengers and t is the number of weeks elapsed since the start of the plan.
Christine finds that at the start of the plan (i.e. $t=0$), the *weekly number of passenger* is 40 thousand.

(a) Let $v=t^2-6t+11$, find $\frac{dv}{dt}$.

Hence, or otherwise, express x in terms of t .

(4 marks)

(b) How many weeks after the start of the plan will the *weekly number of passengers* be the same as at the start of the plan?

(2 marks)

(c) Find the least *weekly number of passengers* after the start of the plan. Give your answer correct to the nearest thousand.

(3 marks)

(d) The week when the *weekly number of passengers* drops to the least is called the *Recovery Week*.

(i) Find the change in the *weekly number of passengers* from the *Recovery Week* to its following week. Give your answer correct to the nearest thousand.

(ii) Prove that $(t+1)^2-6(t+1)+11<3(t^2-6t+11)$ for all t .

(iii) Christine's assistant claims that after the *Recovery Week*, the change in the *weekly number of passengers* from a certain week to its following week will be greater than 25 thousand. Do you agree? Explain your answer.

(6 marks)

(2006 ASL-M&S Q8)

25. A web administrator, David, launches a promotion plan to increase the *daily number of visits* to his web site. The rate of change of the *daily number of visits* to the web site can be modelled by

$$\frac{dN}{dt}=\frac{k(50-t)}{2e^{0.02t}+3t}\ ,$$

where N is the *daily number of visits* recorded at the end of a day in thousands of visits, $t(\geq 0)$ is the number of days elapsed since the start of the plan and k is a positive constant.

David finds that at the start of the plan (i.e. $t=0$), $\frac{dN}{dt}=100$ and $N=10$.

(a) (i) Let $v=2+3te^{-0.02t}$, find $\frac{dv}{dt}$.

(ii) Prove that $k=4$ and hence express N in terms of t .

(7 marks)

(b) David claims that the *daily number of visits* to his web site will be greater than 500 thousand on a certain day after the start of the plan. Do you agree? Explain your answer.

(4 marks)

(c) (i) Find $\frac{d^2N}{dt^2}$.

(ii) Describe the behaviour of N and $\frac{dN}{dt}$ during the 3rd month after the start of the plan.

(4 marks)

(2005 ASL-M&S Q9)

26. A food store manager notices that the **weekly sale** is declining, so he starts a promotion plan to boost the **weekly sale**. He models the rate of change of weekly sale G by

$$\frac{dG}{dt}=\frac{2t-8}{t^2-8t+20} \qquad (t\geq 0) ,$$

where G is the **weekly sale** recorded at the end of the week in thousands of dollars and t is the number of weeks elapsed since the start of the plan. Suppose that at the start of the plan (i.e. $t=0$), the **weekly sale** is 50 thousand dollars.

(a) (i) Express G in terms of t .

(ii) At the end of which week after the start of the plan will the **weekly sale** be the same as at the start of the plan?

(5 marks)

(b) (i) At the end of which week after the start of the plan will the **weekly sale** drop to the least?

(ii) Find the increase between the **weekly sale** of the 5th and the 6th weeks after the start of the plan.

(iii) The store manager decides that once such increase of weekly sale between two consecutive weeks is less than 0.2 thousand dollars, he will terminate the promotion plan. At the end of which week after the start of the plan will the plan be terminated?

(6 marks)

(c) Let t_1 and t_2 be the roots of $\frac{d^2G}{dt^2}=0$, where $t_1<t_2$. Find t_2 .

Briefly describe the behaviour of G and $\frac{dG}{dt}$ immediately before and after t_2 .

(4 marks)

(2002 ASL-M&S Q11)

NEW

Out of syllabus

5. Indefinite Integrals

Section A

1. (2018 DSE-MATH-M1 Q5)

5. Note that $3x^2-24x+49=3(x-4)^2+1\neq 0$.

(a) $f'(x)=0$

$\frac{12x-48}{(3x^2-24x+49)^2}=0$ — 1M

$x=4$

x	$(-\infty,4)$	4	$(4,\infty)$
$f'(x)$	−	0	+

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha=4$. — 1A

$f'(x)=0$

$\frac{12x-48}{(3x^2-24x+49)^2}=0$ — 1M

$x=4$

$f''(x)$

$=\frac{-108x^2+864x-1716}{(3x^2-24x+49)^3}$

$f''(4)$

$=12$

>0

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha=4$. — 1A

(b) (i) Let $v=3x^2-24x+49$. Then, we have $\frac{dv}{dx}=6x-24$.

$f(x)$

$=\int\frac{12x-48}{(3x^2-24x+49)^2}dx$

$=\int\frac{2}{v^2}dv$ — 1M

$=\frac{-2}{v}+C$

$=\frac{-2}{3x^2-24x+49}+C$

Since $f(x)$ has only one extreme value, we have $f(4)=5$. — 1M

$\frac{-2}{3(4)^2-24(4)+49}+C=5$

$C=7$

Thus, we have $f(x)=\frac{-2}{3x^2-24x+49}+7$. — 1A

(ii) $\lim_{x\to\infty}f(x)$

$=7$ — 1A

----------(6)

2. (2015 DSE-MATH-M1 Q8)

	Marks	Remarks
(a) $\frac{d}{dx}\left((x^6+1)\ln(x^2+1)\right)$		
$=(x^6+1)\frac{2x}{x^2+1}+6x^5\ln(x^2+1)$	1M+1A	1M for product rule
$=(x^2+1)(x^4-x^2+1)\frac{2x}{x^2+1}+6x^5\ln(x^2+1)$	1M	
$=2x^5-2x^3+2x+6x^5\ln(x^2+1)$	1A	
(b) $(x^6+1)\ln(x^2+1)=2\int(x^5-x^3+x)dx+6\int x^5\ln(x^2+1)dx$	1M	
Note that $\int(x^5-x^3+x)dx=\frac{x^6}{6}-\frac{x^4}{4}+\frac{x^2}{2}+\text{constant}$.	1A	
Thus, we have		
$\int x^5\ln(x^2+1)dx=\frac{1}{6}(x^6+1)\ln(x^2+1)-\frac{x^6}{18}+\frac{x^4}{12}-\frac{x^2}{6}+\text{constant}$.	1A	
	----------(7)	

(a)	Good. Many candidates were able to apply product rule to find $\frac{d}{dx}\left((x^6+1)\ln(x^2+1)\right)$ while some candidates did not understand the definition of polynomial and simply left $(x^6+1)\frac{2x}{x^2+1}+6x^5\ln(x^2+1)$ as the final answer instead of $(2x^5-2x^3+2x)+6x^5\ln(x^2+1)$.
(b)	Fair. Many candidates employed a wrong substitution in finding $\int(x^6+1)\frac{2x}{x^2+1}dx$, and many candidates made careless mistakes in calculating the integration.

3. (2014 DSE-MATH-M1 Q3)

(a) $\left.\frac{dy}{dx}\right|_{(1,5)} = \left(2\cdot 1 - \frac{1}{1}\right)^3$

$= 1$ — 1A

Hence the equation of tangent is $y - 5 = 1(x - 1)$.

i.e. $x - y + 4 = 0$ — 1A

(b) (i) $\left(2x - \frac{1}{x}\right)^3 = (2x)^3 - 3(2x)^2\left(\frac{1}{x}\right) + 3(2x)\left(\frac{1}{x}\right)^2 - \left(\frac{1}{x}\right)^3$ — 1M

$= 8x^3 - 12x + \frac{6}{x} - \frac{1}{x^3}$ — 1A

(ii) $y = \int\left(2x - \frac{1}{x}\right)^3 dx$

$= \int\left(8x^3 - 12x + \frac{6}{x} - \frac{1}{x^3}\right)dx$ by (i)

$= 2x^4 - 6x^2 + 6\ln|x| + \frac{1}{2x^2} + C$ — 1M

Since $P(1, 5)$ lies on S, $5 = 2(1)^4 - 6(1)^2 + 6\ln|1| + \frac{1}{2(1)^2} + C$. — 1M

i.e. $C = \frac{17}{2}$

Hence the equation of S is $y = 2x^4 - 6x^2 + 6\ln x + \frac{1}{2x^2} + \frac{17}{2}$ for $x > 0$. — 1A

(7)

(a)		Very good.
(b)	(i)	Excellent.
	(ii)	Satisfactory. Some candidates did not know $\int\frac{1}{x}dx = \ln\lvert x\rvert + C$, or wrote $\int\frac{1}{x^3}dx = -\frac{2}{x^2}$ or $\frac{1}{2x^2}$.

4. (2014 DSE-MATH-M1 Q5)

$\frac{dx}{dt} = \frac{t\sqrt{9-t^2}}{3}$

Let $u = 9 - t^2$.

$du = -2t\,dt$ — } 1M

$x = \int\frac{t\sqrt{9-t^2}}{3}dt$

$= \int\frac{u^{\frac{1}{2}}}{3}\frac{du}{-2}$ — 1M — OR $\int\frac{(9-t^2)^{\frac{1}{2}}}{3}\frac{d(9-t^2)}{-2}$

$= \frac{-1}{6}\cdot\frac{2u^{\frac{3}{2}}}{3} + C$ — 1M

$= \frac{-1}{9}(9-t^2)^{\frac{3}{2}} + C$

When $t = 0$, $x = 8$.

$\therefore\ 8 = \frac{-1}{9}(9-0)^{\frac{3}{2}} + C$ — 1M

$C = 11$

i.e. $x = \frac{-1}{9}(9-t^2)^{\frac{3}{2}} + 11$ — 1A

(5)

Satisfactory.
Some candidates substituted $t = 3$ and $x = 8$ to determine the value of the constant of integration.

5. (2012 DSE-MATH-M1 Q2)

Let $u = 4t + 1$. — 1M

$du = 4dt$

When $t = 0$, $u = 1$; when $t = 2$, $u = 9$.

The change in the value of the flat

$= \int_0^2 \frac{t}{\sqrt{4t+1}}dt$ — 1M

$= \int_1^9 \frac{1}{\sqrt{u}}\cdot\frac{u-1}{4}\frac{du}{4}$

$= \frac{1}{16}\int_1^9\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du$

$= \frac{1}{16}\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^9$ — 1A — For $\frac{1}{16}\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]$

$= \frac{5}{6}$ — 1A

Hence the percentage change $= \frac{\frac{5}{6}}{3}\times 100\%$

$= 27\frac{7}{9}\%$ — 1A — OR 27.7778%

(5)

Fair. Many candidates failed to find a suitable substitution or did wrong calculation in substitution. Some found the value of the flat at the beginning of 2014 instead of the percentage change.

6. (2012 ASL-M&S Q3)

(a) Let $u=1+e^{-0.2t}$.

$du=-0.2e^{-0.2t}dt$ — 1A

$N=\int\frac{0.3e^{-0.2t}}{(1+e^{-0.2t})^2}dt$

$N=\int\frac{0.3}{u^2}\cdot\frac{du}{-0.2}$

$=\frac{3}{2u}+C$ — 1A

$=\frac{3}{2(1+e^{-0.2t})}+C$

When $t=0$, $N=0.5$.

$\therefore\ C=\frac{-1}{4}$

i.e. $N=\frac{3}{2(1+e^{-0.2t})}-\frac{1}{4}$ — 1A

(b) $N(4)-N(0)$

$=\frac{3}{2(1+e^{-0.2\times4})}-\frac{1}{4}-0.5$ — 1M

≈ 0.284961721

Hence the increase in the number of people is 285 . — 1A

(c) $\frac{dN}{dt}=\frac{0.3e^{-0.2t}}{(1+e^{-0.2t})^2}>0$ for all $t\geq 0$

Hence N is always increasing.

(Withhold the last mark if this argument is missing)

$\lim_{t\to\infty}N=\lim_{t\to\infty}\left[\frac{3}{2(1+e^{-0.2t})}-\frac{1}{4}\right]$

$=1.25$ — 1A

Hence the number of members will never reach 1300 . — 1

OR by arguing that $e^{-0.2t}>0\Rightarrow N<\frac{3}{2}-\frac{1}{4}$

OR by arguing that $\frac{3}{2(1+e^{-0.2t})}-\frac{1}{4}=1.3$ has no real solution

(7)

Satisfactory.
Many candidates overlooked the units and did not use 0.5 to represent 500 since N was given in thousand. A number of candidates could not well explain their answer in (c) because they did not state clearly that N was an increasing function.

7. (2011 ASL-M&S Q2)

(a) $\frac{dX}{dt}=6\left(\frac{t}{0.2t^3+1}\right)^2$

$X=6\int\frac{t^2}{(0.2t^3+1)^2}dt$

Let $u=0.2t^3+1$, and therefore $du=0.6t^2\ dt$. — 1M — For $u=0.2t^3+1$ or $u=(0.2t^3+1)^2$

$\therefore\ X=6\int\frac{1}{0.6u^2}du$ — 1M

$=\frac{-10}{u}+C$ — 1A

$=\frac{-10}{0.2t^3+1}+C$

When $t=0$, $X=4$ and hence $C=14$.

i.e. $X=\frac{-10}{0.2t^3+1}+14$ — 1A

(b) $13=\frac{-10}{0.2t^3+1}+14$

$t=\sqrt[3]{45}$ months — 1A — OR 3.5569 months

(c) $X=14-\frac{10}{0.2t^3+1}<14$ for any value of t . — 1M

Hence the plan can be run for a long time. — 1A

(7)

Good.
In part (c), although most candidates found the limit of X when $t\to\infty$, the proof was incomplete without showing that the function was increasing.

8. (2010 ASL-M&S Q3)

(a) $\frac{\mathrm{d}A}{\mathrm{d}t}=-kA$

$\therefore \frac{\mathrm{d}t}{\mathrm{d}A}=\frac{-1}{kA}$ 1A

$t=\frac{-1}{k}\int\frac{\mathrm{d}A}{A}$

$=\frac{-1}{k}\ln|A|+C$ 1A

When $t=0$, $A=A_0$ and when $t=5730$, $A=\frac{A_0}{2}$.

$0=\frac{-1}{k}\ln A_0+C$ and $5730=\frac{-1}{k}\ln\frac{A_0}{2}+C$ 1M

$\therefore 5730=\frac{-1}{k}\ln\frac{A_0}{2}+\frac{1}{k}\ln A_0$

$=\frac{1}{k}\ln 2$

i.e. $k=\frac{\ln 2}{5730}$

$\approx 1.21\times 10^{-4}$ (correct to 3 significant figures) 1A

(b) $A=0.3A_0$

$\therefore t=\frac{-5730}{\ln 2}\ln(0.3A_0)+\frac{5730}{\ln 2}\ln A_0$ 1M

$=\frac{5730}{\ln 2}\ln\frac{10}{3}$

≈ 9950 years (correct to the nearest ten years) 1A

(6)

Fair. This question required candidates to deal with $\frac{\mathrm{d}t}{\mathrm{d}A}$ and integrating with respect to A which was different from the more familiar format of $\frac{\mathrm{d}A}{\mathrm{d}t}$ and integrating with respect to t. Part (b) was straightforward for candidates who could solve (a).

9. (2009 ASL-M&S Q2)

(a) $\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{-0.09}{\sqrt{3t+1}}$

$P=\int\frac{-0.09}{\sqrt{3t+1}}\mathrm{d}t$

$=-0.06\sqrt{3t+1}+C$ 1M+1A Withhold 1A if C was omitted

When $t=0$, $P=1$.

$\therefore 1=-0.06\sqrt{1}+C$

$C=1.06$

i.e. $P=-0.06\sqrt{3t+1}+1.06$ 1A

(b) When $t=5$, $P=-0.06\sqrt{3(5)+1}+1.06=0.82$ 1M

Thus, 18% of the population has died off. 1A

(c) When $P=0$, $0=-0.06\sqrt{3T+1}+1.06$

$T=103\frac{19}{27}$ 1A OR 103.7037

(6)

Good. Some candidates were not clear about the concepts of definite and indefinite integrations. Some candidates were not sure about the fact that the total population is composed of ied out_ population and the urviving_ population.

10. (2008 ASL-M&S Q3)

(a) $\frac{\mathrm{d}x}{\mathrm{d}t}=5.3\left(\frac{1}{t+2}-\frac{1}{t+5}\right)+1.2e^{-0.1t}$

$x=\int\left[5.3\left(\frac{1}{t+2}-\frac{1}{t+5}\right)+1.2e^{-0.1t}\right]\mathrm{d}t$ 1M

$=5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+C$ (since $t\geq 0$) 1A OR $5.3[\ln|t+2|-\ln|t+5|]-12e^{-0.1t}+C$

When $t=0$, $x=0$.

$\therefore 0=5.3(\ln 2-\ln 5)-12+C$ 1M

$C=5.3(\ln 5-\ln 2)+12$

≈ 16.8563

i.e. $x=5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+16.8563$ 1A OR $x=\cdots+5.3\ln 2.5+12$

(b) $\lim_{t\to\infty}\{5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+16.8563\}$ 1M

$=5.3\lim_{t\to\infty}\ln\frac{t+2}{t+5}-12\lim_{t\to\infty}e^{-0.1t}+16.8563$

$=16.8563$ 1A

i.e. the concentration of the drug after a long time = 16.8563 mg/L

(6)

Good. Some candidates could not present the mathematical notation of the limit of x properly.

11. (2007 ASL-M&S Q2)

(a) Let $u = 2t^2 + 50$.

Then, we have $\frac{du}{dt} = 4t$. **1A**

N

$= \int \frac{800t}{(2t^2+50)^2} dt$

$= \int \frac{200}{u^2} du$ **1M**

So, we have $N = \frac{-200}{u} + C$, where C is a constant. **1A**

Therefore, we have $N = \frac{-200}{2t^2+50} + C$.

Using the condition that $N = 4$ when $t = 0$, we have $4 = -4 + C$. **1M for finding C**

Hence, we have $C = 8$.

Thus, we have $N = 8 - \frac{200}{2t^2+50}$. **1A**

(b) When $N = 6$, we have $8 - \frac{200}{2t^2+50} = 6$. **1M**

So, we have $t = 5$. **1A**

The number of bacteria will be 6 million 5 days after the start of the research.

----------(7)

Good. Many candidates could handle indefinite integration, but some forgot the constant of integration.

12. (2004 ASL-M&S Q2)

(a) $\frac{dN}{dt}$

$= \frac{6}{(e^{\frac{t}{4}} + e^{\frac{-t}{4}})^2}$

$= \frac{6}{(e^{\frac{-t}{4}}(e^{\frac{t}{2}}+1))^2}$

$= \frac{6}{e^{\frac{-t}{2}}(e^{\frac{t}{2}}+1)^2}$

$= \frac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2}$ **1 must show steps**

Let $u = e^{\frac{t}{2}} + 1$.

Then, we have $\frac{du}{dt} = \frac{1}{2}e^{\frac{t}{2}}$.

Also, $dt = \frac{2du}{u-1}$. Now,

N

$= \int \frac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2} dt$

$= \int \frac{12(u-1)}{u^2(u-1)} du$

$= \int \frac{12}{u^2} du$ **1A accept $\frac{dN}{du} = \frac{12}{u^2}$.**

So, we have $N = \frac{-12}{u} + C$, where C is a constant. **1A**

Now, $N = \frac{-12}{e^{\frac{t}{2}}+1} + C$.

Using the condition that $N = 8$ when $t = 0$, we have $8 = -6 + C$. **1M for finding C**

Hence, $C = 14$.

Thus, $N = 14 - \frac{12}{e^{\frac{t}{2}}+1}$. **1A**

(b) The required number of fish

$= \lim_{t \to \infty}(14 - \frac{12}{e^{\frac{t}{2}}+1})$

$= 14 - \lim_{t \to \infty} \frac{12}{e^{\frac{t}{2}}+1}$

$= 14$ thousands **1A**

----------(6)

Fair. Many candidates were not able to relate mathematical presentations to integrations and taking limits.

13. (2003 ASL-M&S Q2)

(a) (i) Let $u = 1 + 3te^{\frac{-t}{100}}$. Then,

$\frac{du}{dt} = 3e^{\frac{-t}{100}} - \frac{3t}{100}e^{\frac{-t}{100}}$ — 1M for Product Rule + 1A

$= \frac{3}{100}(100-t)e^{\frac{-t}{100}}$

(ii) $\theta = \int \frac{\frac{12}{25}(100-t)e^{\frac{-t}{100}}}{(1+3te^{\frac{-t}{100}})}dt$

$= \int \frac{16}{u}du$ — 1A

$= 16\ln(1+3te^{\frac{-t}{100}}) + C$

When $t=0$, $\theta = 16$, we have $C = 16$ — 1M for finding C

$\therefore \quad \theta = 16\ln(1+3te^{\frac{-t}{100}}) + 16$ — 1A

(b) $\frac{d\theta}{dt} = \frac{12(100-t)e^{\frac{-t}{100}}}{25(1+3te^{\frac{-t}{100}})}$

$\begin{cases} >0 & \text{if } 0 \le t < 100 \\ =0 & \text{if } t = 100 \\ <0 & \text{if } t > 100 \end{cases}$ — 1M for testing

$\therefore$ θ attains its greatest value when $t = 100$

Note that $\theta(100) = 16\ln(1+300e^{-1}) + 16$

≈ 91.40484176

≤ 95

Thus, the temperature of the surface of the vessel will not get higher than 95°C. — 1A

----------(7)

Fair. Many candidates knew that the integration constant should not be neglected.

14. (2002 ASL-M&S Q4)

(a) $S = \int_0^{10} \frac{8100t}{(3t+10)^3}dt$. — 1A

Let $u = 3t + 10$.

$du = 3dt$.

When $t = 0$, $u = 10$.

When $t = 10$, $u = 40$.

$S = \int_{10}^{40} \frac{8100\left(\frac{u-10}{3}\right)\cdot\frac{1}{3}du}{u^3}$

$= 900\int_{10}^{40}(\frac{1}{u^2} - \frac{10}{u^3})du$ — 1M change of variable

$= 900\left[-\frac{1}{u} + \frac{5}{u^2}\right]_{10}^{40}$

$= \frac{405}{16} = 25.3125$ — 1A

The percentage of smoke removed is 25.3125%.

$S = \int_0^{10} \frac{8100t}{(3t+10)^3}dt$	1A
$= 900\int_0^{10}\left[\frac{1}{(3t+10)^2} - \frac{10}{(3t+10)^3}\right]d(3t+10)$	1M change of variable
$= 900\left[-\frac{1}{3t+10} + \frac{5}{(3t+10)^2}\right]_0^{10}$	
$= 25.3125$	1A

$S = \int \frac{8100t}{(3t+10)^3}dt = 900\left[-\frac{1}{3t+10} + \frac{5}{(3t+10)^2}\right] + C$	1M+1M for change of variable
When $t = 0$, $S = 0$. Hence, we have $C = 45$.	
So, $S = 900\left[-\frac{1}{3T+10} + \frac{5}{(3T+10)^2}\right] + 45$.	
When $t = 10$, $S = 25.3125$.	1A

(b) $S = \int_0^T \frac{8100t}{(3t+10)^3}dt$

$= 900\left[-\frac{1}{u} + \frac{5}{u^2}\right]_{10}^{3T+10}$

$= 900\left[-\frac{1}{3T+10} + \frac{5}{(3T+10)^2} + 0.05\right]$

$\lim_{T\to\infty} S = \lim_{T\to\infty} 900\left[-\frac{1}{3T+10} + \frac{5}{(3T+10)^2} + 0.05\right]$ — 1M taking limit and in terms of T

$= 45$ — 1A

$\therefore$ 45% of smoke will be removed.

----------(5)

15. (1999 ASL-M&S Q4)

(a) $N = \int (8t^{\frac{1}{3}} + 11t^{\frac{5}{6}})dt$	1M+1A	1M for integration pp−1 for missing N or dt
$= 6t^{\frac{4}{3}} + 6t^{\frac{11}{6}} + c$ for some constant c.	1A	pp−1 for missing c
$\because$ $N = 100$ when $t = 1$ $\therefore$ $100 = 6 + 6 + c \Rightarrow c = 88$	1A	
i.e. $N = 6t^{\frac{4}{3}} + 6t^{\frac{11}{6}} + 88$		
(b) When $t = 64$, $N = 6(64)^{\frac{4}{3}} + 6(64)^{\frac{11}{6}} + 88$ $= 13912$	1A (5)	

16. (1998 ASL-M&S Q4)

(a) $x = \int 650e^{-0.004t}\,dt$	1A	pp−1 for missing dt
$= -162500e^{-0.004t} + c$	1A	pp−1 for missing c pp−1 for missing x
since $x = 57000$ when $t = 0$ $c = 219500$ $\therefore$ $x = 219500 - 162500e^{-0.004t}$	1M+1A	
(b) If $57000 \times 2 = 219500 - 162500e^{-0.004t}$	1M	
then $t = \frac{1}{-0.004}\ln\left(\frac{219500-114000}{162500}\right)$ ≈ 108 (or 107.9918) $\therefore$ the number of customers will be doubled in 108 days after the start of the campaign.	1A (6)	r.t. 108

17. (1997 ASL-M&S Q4)

(a) $V(t) = \int 200(t-15)dt$	1A
$= 100t^2 - 3000t + c$	1A
$\because V(0) = 20\,000$, $\therefore c = 20\,000$	1A
Hence $V(t) = 100t^2 - 3000t + 20000$ for $0 \le t \le k$.	
(b) $\because$ $V(k) = 0$ $\therefore$ $100k^2 - 3000k + 20000 = 0$	1M
$k^2 - 30k + 200 = 0$ $(k-20)(k-10) = 0$ $k = 10$ or 20 (rejected) $k = 10$	1A
(c) $V(5) - V(0)$ $= 100(5)^2 - 3000(5) + 20000 - 20000$ $= -12500$	1M
Alternatively, $\int_0^5 200(t-15)dt$ $= \left[100t^2 - 3000t\right]_0^5$ $= -12500$	1M
$\therefore$ The total depreciation in the first 5 years is \$12 500.	1A (7)

18. (1996 ASL-M&S Q2)

$\frac{dy}{dx} = \frac{1-\ln x}{x^2}$	1M+1A	1M for quotient rule
$\int\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right)dx = \frac{\ln x}{x}$ $(+c_1)$	1M	For applying anti-differentiation
$\int \frac{\ln x}{x^2}dx = \int \frac{1}{x^2}dx - \frac{\ln x}{x}$ $(+c_1)$	1A	pp-1 for missing dx more than once
$= -\frac{1+\ln x}{x} + c$ (or $-\frac{1}{x} - \frac{\ln x}{x} + c$)	1A	No marks for missing c
	(5)	

19. (1995 ASL-M&S Q4)

(a) $M(t)=\int\left[\frac{1}{3t+4}+(t+25)^{-\frac{1}{2}}\right]dt$ 1A pp-1 for missing differential

$=\frac{1}{3}\ln(3t+4)+2(t+25)^{\frac{1}{2}}+c$ 1A + 1A 1A for integrating 1 term

$\because M(0)=3.1$

$\therefore c=3.1-\frac{1}{3}\ln4-10$ 1M

$=-\frac{2}{3}\ln2-6.9$ (or -7.3621)

$M(t)=\frac{1}{3}\ln(3t+4)+2\sqrt{t+25}-\frac{2}{3}\ln2-6.9$ 1A

(or $\frac{1}{3}\ln(3t+4)+2\sqrt{t+25}-7.3621$)

The value of the house, in million dollars, t years after the end of 1994 is $\frac{1}{3}\ln(3t+4)+2\sqrt{t+25}-\frac{2}{3}\ln2-6.9$.

(b) The rise in the value of the house, in million dollars, between the end of 1994 and the end of 2000 is

$M(6)-M(0)$

$=\frac{1}{3}\ln22+2\sqrt{31}-\frac{2}{3}\ln2-6.9-3.1$ 1M

$=\frac{1}{3}\ln\frac{11}{2}+2\sqrt{31}-10$ (or 1.7038) 1A

Alternatively,

$M(6)-M(0)=\left[\frac{1}{3}\ln(3t+4)+2\sqrt{t+25}\right]_0^6$

$=\frac{1}{3}\ln22+2\sqrt{31}-\frac{2}{3}\ln2-10$ 1M

$=\frac{1}{3}\ln\frac{11}{2}+2\sqrt{31}-10$ (or 1.7038) 1A

(7)

20. (1994 ASL-M&S Q5)

Let $u=t^2+1$, then $du=2t\,dt$. 1M

$x=\int3t(t^2+1)^{\frac{1}{2}}dt$ 1M

$=\frac{3}{2}\int u^{\frac{1}{2}}du$ 1A

$=u^{\frac{3}{2}}+c$

$=(t^2+1)^{\frac{3}{2}}+c$ 1A

Since $x=10$ when $t=0$, $10=(0^2+1)^{\frac{3}{2}}+c$ 1M

$c=9$

$\therefore x=(t^2+1)^{\frac{3}{2}}+9$. 1A

6

Section B

21. (2012 DSE-MATH-M1 Q11)

(a) When $t=35$, the intensity increased to a maximum and therefore $\frac{dR}{dt}=0$.

$\frac{a(30-35)+10}{(35-35)^2+b}=0$ 1A

$a=2$ 1A

(2)

(b) $\frac{dR}{dt}=\frac{2(30-t)+10}{(t-35)^2+b}$

Let $u=(t-35)^2+b$. 1M

$du=2(t-35)dt$

$R=\int\frac{-2t+70}{(t-35)^2+b}dt$

$=\int\frac{-2t+70}{u}\frac{du}{2(t-35)}$

$=-\ln|u|+C$

$=-\ln[(t-35)^2+b]+C$ 1A

$R|_{t=T}=R|_{t=0}$

$-\ln[(T-35)^2+b]+C=-\ln[(0-35)^2+b]+C$ 1M

$(T-35)^2=35^2$

$T=70$ or 0 (rejected) 1A

(4)

(c) $R|_{t=40} - R|_{t=41} = \ln\frac{61}{50}$

$-\ln[(40-35)^2+b]+C-\{-\ln[(41-35)^2+b]+C\} = \ln\frac{61}{50}$ — 1M

$-\ln(25+b)+\ln(36+b) = \ln\frac{61}{50}$

$\ln\frac{36+b}{25+b} = \ln\frac{61}{50}$

$b = 25$ — 1A

$\therefore \quad R = -\ln[(t-35)^2+25]+C$

$R|_{t=35} = 6$

$-\ln[(35-35)^2+25]+C = 6$ — 1M

$C = 6+\ln 25$

i.e. $R = -\ln[(t-35)^2+25]+6+\ln 25$ — 1A

(4)

(d) $\frac{dR}{dt} = \frac{2(30-t)+10}{(t-35)^2+25}$

$= \frac{70-2t}{t^2-70t+1250}$

$\frac{d^2R}{dt^2} = \frac{(t^2-70t+1250)(-2)-(70-2t)(2t-70)}{(t^2-70t+1250)^2}$ — 1M+1A

$= \frac{2t^2-140t+2400}{(t^2-70t+1250)^2}$

When the rate of change of the radiation intensity attains its greatest value, $\frac{d^2R}{dt^2} = 0$.

$2t^2-140t+2400 = 0$

$t = 30$ or 40 (rejected)

t	$0 \le t < 30$	$t = 30$	$30 < t \le 35$
$\frac{d^2R}{dt^2}$	+ve	0	-ve

1M

Hence, the rate of change of the radiation intensity would attain its greatest value when $t = 30$. — 1A

(4)

(a)	A common mistake was to mix up R with $\frac{dR}{dt}$. Fair. However, many candidates knew that maximum intensity implied $\frac{dR}{dt} = 0$
(b)	Poor. Some candidates were not able to choose a suitable substitution to solve for R, while others did not go on after substitution or made careless mistakes in further calculations.
(c)	Very poor. A common mistake was $R|_{t=41} - R|_{t=40} = \ln\frac{61}{50}$.
(d)	Very poor. Only a few candidates attempted this part. Among them, some forgot to square the denominator when applying quotient rule to calculate $\frac{d^2R}{dt^2}$.

22. (SAMPLE DSE-MATH-M1 Q11)

(a) (i) Let $v = 1+4te^{-0.04t}$. Then we have

$\frac{dv}{dt} = 4e^{-0.04t} - 0.16te^{-0.04t}$ — 1M+1A — 1M for product rule

$= 0.16e^{-0.04t}(25-t)$

(ii) When $t = 0, \frac{dN}{dt} = 50$. So we have $25k = 50$.

Thus, we have $k = 2$. — 1A

$N = \int \frac{2(25-t)}{e^{0.04t}+4t}dt$

$= 2\int \frac{e^{-0.04t}(25-t)}{1+4te^{-0.04t}}dt$ — 1M

$= \frac{2}{0.16}\int \frac{dv}{v}$ — 1M — For using (a)(i)

$= 12.5\ln|v| + C$

$= 12.5\ln(1+4te^{-0.04t})+C$

When $t = 0$, $N = 10$. So, we have $C = 10$. — 1M — For finding C

i.e. $N = 12.5\ln(1+4te^{-0.04t})+10$ — 1A

(7)

(b) (i) $\frac{dN}{dt}=\frac{2(25-t)}{e^{0.04t}+4t}$

$\begin{cases} >0 & \text{when } 0\le t<25 \\ =0 & \text{when } t=25 \\ <0 & \text{when } t>25 \end{cases}$ — 1M

So, N attains its greatest value when $t=25$. — 1A

Alternative Solution

$\frac{dN}{dt}=\frac{2(25-t)}{e^{0.04t}+4t}$

$\frac{dN}{dt}=0$ when $t=25$

$$\frac{d^2N}{dt^2}=2\left[\frac{(e^{0.04t}+4t)(-1)-(0.04e^{0.04t}+4)(25-t)}{(e^{0.04t}+4t)^2}\right]$$

$$=-4\left[\frac{(1-0.02t)e^{0.04t}+50}{(e^{0.04t}+4t)^2}\right]$$ — 1M

$$\left.\frac{d^2N}{dt^2}\right|_{t=25}=-4\left[\frac{0.5e+50}{(e+100)^2}\right]<0$$

So, N attains its greatest value when $t=25$. — 1A

(ii) $N(25)=12.5\ln(1+4te^{-0.04t})+10\approx 55.4>50$

Thus, the claim is agreed. — 1

(3)

(c) $\lim_{t\to\infty}N=\lim_{t\to\infty}[12.5\ln(1+4te^{-0.04t})+10]$

$=12.5\ln(1+0)+10$ — 1M — For using $\lim_{t\to\infty}te^{-0.04t}=0$

$=10$

Thus, the belief of Mary's supervisor is agreed. — 1

(2)

23. (2008 ASL-M&S Q8)

(a) (i) $N'(t)=\frac{20}{1+he^{-kt}}\quad (t\ge 0)$

$\ln\left[\frac{20}{N'(t)}-1\right]=-kt+\ln h$ — 1A

(ii) $\ln h=1.5$ — 1M (Either One)

$h=e^{1.5}$

≈ 4.4817 (correct to 4 d.p.) — 1A

$-k=\frac{1.5-0}{0-7.6}$ — (Either One)

$k=\frac{15}{76}$

≈ 0.1974 (correct to 4 d.p.) — 1A

(4)

(b) (i) $v=4.5+e^{0.2t}$

$\frac{dv}{dt}=0.2e^{0.2t}$ — 1A

$N(t)=\int\frac{20}{1+4.5e^{-0.2t}}dt$

$=\int\frac{100}{e^{0.2t}+4.5}(0.2e^{0.2t})dt$

$=\int\frac{100}{v}dv$ — 1M

$=100\ln|v|+C$

$=100\ln(4.5+e^{0.2t})+C\quad (\because 4.5+e^{0.2t}>0)$ — 1A

Since $N(0)=50$, so $C=50-100\ln 5.5$ — 1M

i.e. $N(t)=100\ln\frac{4.5+e^{0.2t}}{5.5}+50$ — 1A

(ii) (1) $M(20)=N(20)$

$21\left[(20)+\frac{4.5}{0.2}e^{-0.2(20)}\right]+b=100\ln\frac{4.5+e^{0.2(20)}}{5.5}+50$

$b\approx -141.2090$ — 1A

(2) Consider $M'(t)-N'(t)$

$=21(1-4.5e^{-0.2t})-\frac{20}{1+4.5e^{-0.2t}}$ — 1A — For the 1st term

$=\frac{1-425.25e^{-0.4t}}{1+4.5e^{-0.2t}}$ — 1A

$\therefore M'(t)-N'(t)>0$ when $e^{-0.4t}<\frac{1}{425.25}$ — 1M

i.e. $t>\frac{\ln 425.25}{0.4}\approx 15.1317$ — 1A

Since $M(20)=N(20)$ and $M(t)-N(t)$ is increasing when $t>20$,

so $M(t)>N(t)$ for $t>20$.

Hence the biologist's claim is correct. — 1A — Follow through

(11)

(a) (i)		Very good.
(a) (ii)		Very good, though some careless mistakes were found.
(b) (i)		Fair. A number of candidates could not apply substitution to do integration.
(ii) (1)		Fair. Some candidates were hindered by failing to complete (b) (i).
(2)		Poor. Not too many candidates attempted and those who attempted could not make use of the given hint.

24. (2006 ASL-M&S Q8)

(a) $\frac{dv}{dt}=2t-6$	1A
$x = \int \frac{30t-90}{t^2-6t+11}\,dt$	
$=15\int \frac{dv}{v}$	1A
$=15\ln\lvert v\rvert + C$	
$=15\ln(t^2-6t+11)+C \quad (\because t^2-6t+11=(t-3)^2+2>0)$	
Using the condition that $x=40$ when $t=0$, we have $C=40-15\ln 11$.	1M for finding C
Thus, we have $x=15\ln(t^2-6t+11)+40-15\ln 11$.	1A
	--------(4)
(b) $15\ln(t^2-6t+11)+40-15\ln 11=40$	
$15\ln(t^2-6t+11)=15\ln 11$	
$t^2-6t+11=11$	1M
$t(t-6)=0$	
$t=6$ or $t=0$ (rejected)	
Therefore, we have $t=6$.	1A
Thus, 6 weeks after the start of the plan, the *weekly number of passengers* will be the same as at the start of the plan.	
	--------(2)
(c) $\frac{dx}{dt}=\frac{30(t-3)}{(t-3)^2+2}$	
$\begin{cases} <0 & \text{if } 0\le t<3 \\ =0 & \text{if } t=3 \\ >0 & \text{if } t>3 \end{cases}$	1M for testing + 1A
So, x attains its least value when $t=3$.	
The least *weekly number of passengers* $=15\ln 2+40-15\ln 11$ $=40-15\ln\frac{11}{2}$ ≈ 14.42877862 ≈ 14 thousand	1A

$\frac{d^2x}{dt^2}=\frac{-30(t^2-6t+7)}{(t^2-6t+11)^2}$	
Note that $\frac{dx}{dt}=0$ when $t=3$.	
$\left.\frac{d^2x}{dt^2}\right\rvert_{t=3}=15>0$	1M for testing + 1A
Note that there is only one local minimum.	
So, x attains its least value when $t=3$.	
The least *weekly number of passengers* $=15\ln 2+40-15\ln 11$ $=40-15\ln\frac{11}{2}$ ≈ 14.42877862 ≈ 14 thousand	1A
	--------(3)
(d) By (c), note that the end of the *Recovery Week* corresponds to $t=3$.	
(i) The required change $=x(4)-x(3)$	
$=15\ln(4^2-24+11)-15\ln(3^2-18+11)$	1M
$=15(\ln 3-\ln 2)$	
$=15\ln\frac{3}{2}$ ≈ 6.081976622 ≈ 6 thousand	1A
The required change $=\int_3^4 \frac{30t-90}{t^2-6t+11}\,dt$	
$=15\left[\ln(t^2-6t+11)\right]_3^4$	
$=15\ln(4^2-24+11)-15\ln(3^2-18+11)$	1M
$=15(\ln 3-\ln 2)$	
$=15\ln\frac{3}{2}$ ≈ 6.081976622 ≈ 6 thousand	1A
(ii) $(t+1)^2-6(t+1)+11-3(t^2-6t+11)$	
$=-2t^2+14t-27$	
$=-2\left(t-\frac{7}{2}\right)^2-\frac{5}{2}$	1M accept using discriminant <0
<0	1
Thus, we have $(t+1)^2-6(t+1)+11<3(t^2-6t+11)$ for all t.	

Note that $t^2-6t+11=(t-3)^2+2>0$.

$\frac{(t+1)^2-6(t+1)+11}{t^2-6t+11}-3$

$=\frac{-2t^2+14t-27}{t^2-6t+11}$

$=\frac{-2\left(t-\frac{7}{2}\right)^2-\frac{5}{2}}{(t-3)^2+2}$ **1M accept using discriminant < 0**

<0 **1**

Thus, we have $(t+1)^2-6(t+1)+11<3(t^2-6t+11)$ for all t.

Let $f(t)=(t+1)^2-6(t+1)+11-3(t^2-6t+11)$ for all $t\ge 0$.

$\frac{df(t)}{dt}=-4t+14$

$$\frac{df(t)}{dt}\begin{cases}>0 & \text{if } 0\le t<\frac{7}{2}\\ =0 & \text{if } t=\frac{7}{2}\\ <0 & \text{if } t>\frac{7}{2}\end{cases}$$ **1M for testing**

So, $f(t)$ attains its greatest value when $t=\frac{7}{2}$.

The greatest value of $f(t)$

$=\frac{-5}{2}$

<0 **1**

Thus, we have $(t+1)^2-6(t+1)+11<3(t^2-6t+11)$ for all t.

(iii) $x(t+1)-x(t)$

$=15\ln\left((t+1)^2-6(t+1)+11\right)-15\ln(t^2-6t+11)$

$=15\ln\left(\frac{(t+1)^2-6(t+1)+11}{t^2-6t+11}\right)$

$<15\ln 3$ (by (d)(ii) and $t^2-6t+11>0$) **1M for using (d)(ii) and taking ln**

<25

Thus, the claim is incorrect. **1A f.t.**

By (d)(ii), we have $(t+1)^2-6(t+1)+11<3(t^2-6t+11)$

Note that $(t+1)^2-6(t+1)+11>0$ and $3(t^2-6t+11)>0$.

$\ln\left((t+1)^2-6(t+1)+11\right)<\ln 3+\ln(t^2-6t+11)$ **1M for using (d)(ii) and taking ln**

$15\ln\left((t+1)^2-6(t+1)+11\right)-15\ln(t^2-6t+11)<15\ln 3$

$x(t+1)-x(t)<25$

Thus, the claim is incorrect. **1A f.t.**

--------(6)

(a)		Very good
(b)		Very good.
(c)		Good. Some candidates were not able to prove that the minimum value is the least value.
(d)(i)		Very good.
(ii)		Not satisfactory. Many candidates did not know how to prove this part.
(iii)		Not satisfactory. Many candidates could not proceed due to failure in proving in (d)(ii).

25. (2005 ASL-M&S Q9)

(a) (i) Let $v=2+3te^{-0.02t}$. Then, we have

$\frac{dv}{dt}=3e^{-0.02t}-\frac{3t}{50}e^{-0.02t}$ **1M for product rule or chain rule + 1A**

$=\frac{3}{50}(50-t)e^{-0.02t}$

(ii) When $t=0$, $\frac{dN}{dt}=100$. So, we have $100=\frac{50k}{2}$.

Thus, we have $k=4$. **1**

$N=\int\frac{4(50-t)}{2e^{0.02t}+3t}dt$

$=\frac{200}{3}\int\frac{dv}{v}$ **1M for using (a)(i)**

$=\frac{200}{3}\ln v+C$

$=\frac{200}{3}\ln(2+3te^{-0.02t})+C$ **1A**

Note that when $t=0$, $N=10$. So, we have $C=10-\frac{200}{3}\ln 2$. **1M for finding C**

Thus, we have

$N=\frac{200}{3}\ln(2+3te^{-0.02t})+10-\frac{200}{3}\ln 2$ **1A**

$=\frac{200}{3}\ln\left(1+\frac{3te^{-0.02t}}{2}\right)+10$

----------(7)

(b) $\frac{dN}{dt}=\frac{4(50-t)}{2e^{0.02t}+3t}$

$$\begin{cases}>0 & \text{if } 0\le t<50\\ =0 & \text{if } t=50\\ <0 & \text{if } t>50\end{cases}$$ **1M for testing + 1A**

So, N attains its greatest value when $t=50$.

Note that $N(50)=\frac{200}{3}\ln\left(1+\frac{150}{2}e^{-1}\right)+10\approx 233.5393678<500$. **1M for comparing $N(50)$ and 500**

Thus, the claim is not correct. **1A f.t.**

$\frac{dN}{dt} = \frac{4(50-t)}{2e^{0.02t}+3t}$

$\frac{d^2N}{dt^2} = 4\left(\frac{(2e^{0.02t}+3t)(-1)-(50-t)(\frac{2}{50}e^{0.02t}+3)}{(2e^{0.02t}+3t)^2}\right)$

$= \frac{4}{25}\left(\frac{te^{0.02t}-100e^{0.02t}-3750}{(2e^{0.02t}+3t)^2}\right)$

$\frac{dN}{dt} = 0$ when $t = 50$. Also, when $t = 50$,

$\frac{d^2N}{dt^2} = \frac{4}{25}\left(\frac{50e-100e-3750}{(2e+150)^2}\right)$

$= \frac{-2}{e+75}$

< 0 — 1M for testing + 1A

So, N attains its greatest value when $t = 50$.

Note that $N(50) = \frac{200}{3}\ln(1+\frac{150}{2}e^{-1})+10 \approx 233.5393678 < 500$. — 1M for comparing $N(50)$ and 500

Thus, the claim is not correct. — 1A f.t.

---------(4)

(c) (i) $\frac{d^2N}{dt^2} = \frac{d}{dt}\left(\frac{4(50-t)}{2e^{0.02t}+3t}\right)$

$= 4\left(\frac{(2e^{0.02t}+3t)(-1)-(50-t)(\frac{2}{50}e^{0.02t}+3)}{(2e^{0.02t}+3t)^2}\right)$

$= \frac{4}{25}\left(\frac{te^{0.02t}-100e^{0.02t}-3750}{(2e^{0.02t}+3t)^2}\right)$ — 1A (accept $\frac{k}{25}\left(\frac{te^{0.02t}-100e^{0.02t}-3750}{(2e^{0.02t}+3t)^2}\right)$)

$= \frac{4}{25}\left(\frac{(t-100)e^{0.02t}-3750}{(2e^{0.02t}+3t)^2}\right)$

(ii) Note that $\frac{d^2N}{dt^2} = \frac{4}{25}\left(\frac{(t-100)e^{0.02t}-3750}{(2e^{0.02t}+3t)^2}\right)$ (by (c)(i)).

Hence, we have $\frac{d^2N}{dt^2} < 0$ for $59 \le t \le 92$. — 1M for considering the sign of numerator

So, $\frac{dN}{dt}$ decreases during the 3rd month after the start of the plan. — 1A f.t. (either)

Also note that $\frac{dN}{dt} = \frac{4(50-t)}{2e^{0.02t}+3t} < 0$ for $59 \le t \le 92$.

Therefore, N decreases during the 3rd month after the start of the plan. — 1A f.t.

---------(4)

(a) (i)		Very good.
(ii)		Fair. Many candidates could not make use of (a)(i) to express N in terms of t.
(b)		Not satisfactory. Some candidates did not know how to tackle the problem while some did not check that the stationary point is indeed a maximum point.
(c) (i)		Poor. Many candidates were not familiar with the techniques of differentiation.
(ii)		Poor. Many candidates did not know how to describe the behaviour of N and $\frac{dN}{dt}$. It is important to note that behaviour must be described with respect to t as N and $\frac{dN}{dt}$ are both functions of t.

26. (2002 ASL-M&S Q11)

(a) (i) $G = \int \frac{2t-8}{t^2-8t+20}\,dt$ — 1A

$= \ln(t^2-8t+20)+C$ — 1A

When $t = 0$, $G = 50$.

$C = 50 - \ln 20$

$G = \ln(t^2-8t+20)+50-\ln 20$ — 1A

(ii) For $G = 50$,

$\ln(t^2-8t+20)+50-\ln 20 = 50$ — 1M

$t^2-8t+20 = 20$

$t^2-8t = 0$

$t = 0$ or $t = 8$.

At the end of the 8th week, the weekly sale is the same as at the start of the promotion plan. — 1A

---------(5)

(b) (i) $\because \frac{dG}{dt} = \frac{2t-8}{t^2-8t+20} = \frac{2(t-4)}{[(t-4)^2+4]}$

$\therefore \frac{dG}{dt} = 0$ when $t = 4$

Since $\frac{dG}{dt} < 0$ when $t < 4$

and $\frac{dG}{dt} > 0$ when $t > 4$, — 1M (or: $G = \ln(t^2-8t+20)+C = \ln[(t-4)^2+4]+C$)

$\therefore$ G is least at $t = 4$.

At the end of the 4th week, the weekly sale is least. — 1A

(ii) $G(6)-G(5) = (\ln 8+50-\ln 20)-(\ln 5+50-\ln 20)$

$= \ln\frac{8}{5} \approx 0.4700$ (thousand dollars) — 1A $a-1$ for more than 4 d.p. or r.t. 0.470

(iii) $G(t+1)-G(t) < 0.2$

$\{\ln[(t+1)^2-8(t+1)+20]+50-\ln 20\} - \{\ln(t^2-8t+20)+50-\ln 20\} < 0.2$ — 1M

$\ln\frac{t^2-6t+13}{t^2-8t+20} < 0.2$ — 1A

$(e^{0.2}-1)t^2-(8e^{0.2}-6)t+(20e^{0.2}-13) > 0$ — $0.22140t^2-3.77122t+11.42806 > 0$

$t < 3.94316$ or $t > 13.09015$ — $t < 3.94315$ or $t > 13.09037$

$\because \frac{dG}{dt} < 0$ when $0 < t < 4$, G is decreasing

$\therefore$ $t < 3.94316$ is rejected.

$\therefore$ $t = 14$.

Thus the promotion plan will be terminated at the end of the 15th week. — 1A must show reasons

---------(6)

$G(t) - G(t-1) < 0.2$

$\{\ln(t^2 - 8t + 20) + 50 - \ln 20\} - \{\ln[(t-1)^2 - 8(t-1) + 20] + 50 - \ln 20\} < 0.2$ 1M

$\ln \dfrac{t^2 - 8t + 20}{t^2 - 10t + 29} < 0.2$ 1A

$(e^{0.2} - 1)t^2 - (10e^{0.2} - 8)t + (29e^{0.2} - 20) > 0$

$0.22140t^2 - 4.21403t + 15.42068 > 0$

$t < 4.94316$ or $t > 14.09015$

(c) $\dfrac{dG}{dt} = \dfrac{2t-8}{t^2 - 8t + 20}$

$\dfrac{d^2G}{dt^2} = \dfrac{2(t^2 - 8t + 20) - (2t-8)(2t-8)}{(t^2 - 8t + 20)^2}$ 1A

$= -\dfrac{2(t-2)(t-6)}{(t^2 - 8t + 20)^2}$

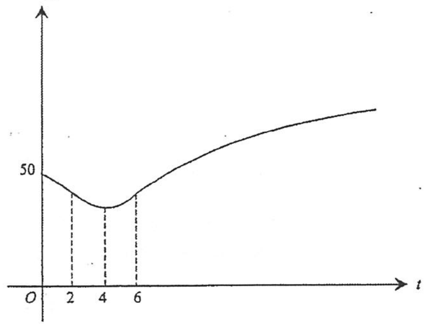
$\dfrac{d^2G}{dt^2} = 0$ when $t = 2$ or $t = 6$. $\therefore$ $t_2 = 6$ 1A

Although G keeps increasing, 1A

$\dfrac{dG}{dt}$ increases immediately before $t = 6$,
$\dfrac{dG}{dt}$ decreases immediately after $t = 6$. } 1A

----------(4)

For reference only



	$G(t)$	$\Delta G(t)$
0	50.0	
1.0	49.5692	− 0.4308
2.0	49.0837	− 0.4855
3.0	48.6137	− 0.47
4.0	48.3906	− 0.2231
5.0	48.6137	+ 0.2231
6.0	49.0837	+ 0.47
7.0	49.5692	+ 0.4855
8.0	50.0	+ 0.4308
9.0	50.3716	+ 0.3716
10.0	50.6931	+ 0.3215
11.0	50.9746	+ 0.2815
12.0	51.2238	+ 0.2492
13.0	51.4469	+ 0.2231
14.0	51.6487	+ 0.2018
15.0	51.8326	+ 0.1839
16.0	52.0015	+ 0.1689
17.0	52.1576	+ 0.1561
18.0	52.3026	+ 0.145
19.0	52.438	+ 0.1354
20.0	52.5649	+ 0.1269