4. Applications of Differentiation

Learning Unit	Learning Objective
Calculus Area	
Differentiation with Its Applications	
6. Applications of differentiation	6.1 use differentiation to solve problems involving tangents, rates of change, maxima and minima

Section A

1. Define $f(x)=\dfrac{6-x}{x+3}$ for all $x>-3$.

   (a) Prove that $f(x)$ is decreasing.

   (b) Find $\lim\limits_{x\to\infty} f(x)$.

(6 marks) (2019 DSE-MATH-M1 Q5a,b)

2. Let h be a constant. Consider the curve C: $y=x^2\sqrt{h-x}$, where $0<x<h$. It is given that $\dfrac{dy}{dx}=30$ when $x=4$.

   (a) Prove that $h=20$.

   (b) Find the maximum point(s) of C.

   (c) Write down the equation(s) of the horizontal tangent(s) to C.

(7 marks) (2018 DSE-MATH-M1 Q7)

3. Let $f(x)$ be a continuous function such that $f'(x)=\dfrac{12x-48}{(3x^2-24x+49)^2}$ for all real numbers x.

   If $f(x)$ attains its minimum value at $x=\alpha$, find α.

(3 marks) (2018 DSE-MATH-M1 Q5a)

4. Let $\mathrm{f}(x)=4x^3+mx^2+nx+615$, where m and n are constants. It is given that $(-6,33)$ is a turning point of the graph of $y=\mathrm{f}(x)$. Find

   (a) m and n ,

   (b) the minimum value(s) and the maximum value(s) of $\mathrm{f}(x)$.

(6 marks) (2017 DSE-MATH-M1 Q6)

5. Air is leaking from a spherical balloon at a constant rate of 100 cm^3 per second. Find the rate of change of the radius of the balloon at the instant when the radius is 10 cm .

(3 marks) (2014 DSE-MATH-M1 Q1)

6. Let $f(x)=\dfrac{x^x}{(2x+13)^6}$, where $x>1$.

(a) By considering $\ln f(x)$, find $f'(x)$.

(b) Show that $f(x)$ is increasing for $x>1$.

(Part a is out of Syllabus) (6 marks) (2014 DSE-MATH-M1 Q2)

7. The population p (in million) of a city at time t (in years) can be modelled by

$$p=8-\frac{2.1}{\sqrt{t+4}} \quad \text{for } t\geq 0 \ .$$

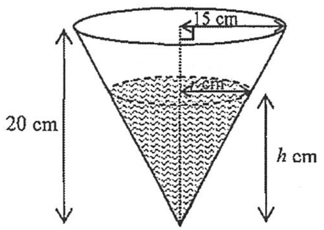
An environment study indicates that, when the population is p million, the concentration of carbon dioxide in the air is given by

$$C=2^p \text{ units} \ .$$

Find the rate of change of the concentration of carbon dioxide in the air at $t=5$.

(4 marks) (2013 DSE-MATH-M1 Q2)

8.



A glass container is in the shape of a vertically inverted right circular cone of base radius 15 cm and height 20 cm . Initially, the container is full of water. Suppose the water is running out from it at a constant rate of $2\pi\text{ cm}^3/\text{s}$. Let h cm be the depth of water remaining in the container, r cm

be the radius of the water surface (see the Figure), $V\text{ cm}^3$ be the volume of the water, and $A\text{ cm}^2$ be the area of the wet surface of the container. It is given that $V=\dfrac{1}{3}\pi r^2 h$ and $A=\pi r\sqrt{r^2+h^2}$.

(a) Express V and A in terms of r only.

(b) When $r=3$,

(i) find the rate of change of the radius of the water surface;

(ii) find the rate of change of the area of the wet surface of the container.

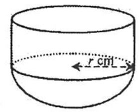
(6 marks) (PP DSE-MATH-M1 Q3)

9. When a hot air balloon is being blown up, its radius $r(t)$ (in m) will increase with time t (in hr). They are related by $r(t)=3-\dfrac{2}{2+t}$, where $t\geq 0$. It is known that the volume $V(r)$ (in m^3) of the balloon is given by $V(r)=\dfrac{4}{3}\pi r^3$.

Find the rate of change, in terms of π , of the volume of the balloon when the radius is 2.5 m .

(4 marks) (SAMPLE DSE-MATH-M1 Q2)

10. The figure shows a container (without a lid) consisting of a thin hollow hemisphere of radius r cm joined to the bottom of a right circular cylindrical thin pipe of base radius r cm. It is known that the area of the outer surface of the container is $162\pi\text{ cm}^2$.



(a) Prove that the capacity of the container is $\left(81\pi r-\dfrac{\pi r^3}{3}\right)\text{cm}^3$.

(b) As r varies, can the capacity of the container be greater than 1600 cm^3 ? Explain your answer.

(7 marks) (2004 ASL-M&S Q3)

11. At any time t (in hours), the relationship between the number N of tourists at a ski-resort and the air temperature $\theta °C$ can be modeled by

$$N=2930-(\theta+440)[\ln(\theta+49)]^2$$

where $-45\leq\theta\leq-40$.

(a) Express $\dfrac{dN}{dt}$ in terms of θ and $\dfrac{d\theta}{dt}$.

(b) At a certain moment, the air temperature is $-40°C$ and it is falling at a rate of $0.5°C$ per hour. Find, to the nearest integer, the rate of increase of the number of tourists at that moment.

(6 marks) (1996 ASL-M&S Q5)

12. An adventure estimates the volume of his hot air balloon by $V(r)=\frac{4}{3}\pi r^3+5\pi$, where r is measured in metres and V is measured in cubic metres. When the balloon is being inflated, r will increase with time $t(\geq 0)$ in such a way that,

$$r(t)=\frac{18}{3+2e^{-t}}$$

where t is measured in hours.

(a) Find the rate of change of volume of the balloon at t = 2. Give your answer correct to 2 decimal places.

(b) If the balloon is being inflated over a long period of time, what will the volume of the balloon be? Give your answer correct to 2 decimal places.

(5 marks) (2002 ASL-M&S Q2)

13. Let $P(t)$ and $C(t)$ (in suitable units) be the electric energy produced and consumed respectively in a city during the time period $[0,t]$, where t is in years and $t\geq 0$. It is known that

$P'(t)=4\left(4-e^{\frac{-t}{5}}\right)$ and $C'(t)=9\left(2-e^{\frac{-t}{10}}\right)$. The redundant electric energy being generated during the time period $[0,t]$ is $R(t)$, where $R(t)=P(t)-C(t)$ and $t\geq 0$.

(a) Find t such that $R'(t)=0$.

(3 marks)

(b) Show that $R'(t)$ decreases with t .

(3 marks)

(2013 DSE-MATH-M1 Q11(a,b))

14. The current rate of selling of a certain kind of handbags is 30 thousand per day. The sales manager decides to raise the price of the handbags. After the price of the handbags has been raised for t days, the rate of selling of handbags $r(t)$ (in thousand per day) can be modelled by

$$r(t)=20-40e^{-at}+be^{-2at} \qquad (t\geq 0),$$

where a and b are positive constants. From past experience, it is known that after the increase in the price of the handbags, the rate of selling of handbags will decrease for 9 days.

(a) Find the value of b .

(1 mark)

(b) Find the value of a correct to 1 decimal place.

(3 marks)

(c) The sales manager will start to advertise when the rate of change of the rate of selling of handbags reaches a maximum. Use the results obtained in (a) and (b) to find the rate of selling of handbags when the sales manager starts to advertise.

(4 marks)

(2012 ASL-M&S Q9(a,b,c))

15. Mr. Lee has a fish farm in Sai Kung. Last week, the fish in his farm were affected by a certain disease. An expert told Mr. Lee that the number N of fish in his farm could be modelled by the function

$$N=\frac{5000e^{\lambda t}}{t} \qquad (0<t<120) ,$$

where λ is a constant and t is the number of days elapsed since the disease began to spread. Suppose that the numbers of fish will be the same when $t=15$ and $t=95$.

(a) Find the value of λ .

(b) How many days after the start of the spread of the disease will the number of fish decrease to the minimum?

(8 marks)

(1998 ASL-M&S Q8(a))

Section B

16. A researcher, Peter, models the number of crocodiles in a lake by

$$x = 4 + \frac{3k}{2^{\lambda t} - k} ,$$

where λ and k are positive constants, x is the number in thousands of crocodiles in the lake and t (≥ 0) is the number of years elapsed since the start of the research.

(a) (i) Express $(x-4)(x-1)$ in terms of λ, k and t.

(ii) Peter claims that the number of crocodiles in the lake does not lie between 1 thousand and 4 thousand. Is the claim correct? Explain your answer.

(3 marks)

(b) Peter finds that $\frac{dx}{dt} = \frac{-\ln 2}{24}(x-4)(x-1)$.

(i) Prove that $\lambda = \frac{1}{8}$.

(ii) For each of the following conditions (1) and (2), find k. Also determine whether the crocodiles in the lake will eventually become extinct or not. If your answer is 'yes', find the time it will take for the crocodiles to become extinct; if your answer is 'no', estimate the number of crocodiles in the lake after a very long time.

(1) When $t = 0$, $x = 0.8$.

(2) When $t = 0$, $x = 7$.

(9 marks)

(2017 DSE-MATH-M1 Q12)

17. The chickens in a farm are infected by a certain bird flu. The number of chickens (in thousand) in the farm is modelled by

$$N = \frac{27}{2 + \alpha t e^{\beta t}} ,$$

where t (≥ 0) is the number of days elapsed since the start of the spread of the bird flu and α and β are constants.

(a) Express $\ln\left(\frac{27-2N}{Nt}\right)$ as a linear function of t.

(2 marks)

(b) It is given that the slope and the intercept on the horizontal axis of the graph of the linear function obtained in (a) are -0.1 and $10\ln 0.03$ respectively.

(i) Find α and β.

(ii) Will the number of chickens in the farm be less than 12 thousand on a certain day after the start of the spread of the bird flu? Explain your answer.

(iii) Describe how the rate of change of the number of chickens in the farm varies during the first 20 days after the start of the spread of the bird flu. Explain your answer.

(10 marks)

(2016 DSE-MATH-M1 Q12)

DSE Mathematics Module 1 4. Applications of Differentiation

18. The population of a kind of bacterium $p(t)$ at time t (in days elapsed since 9 am on 16/4/2010, and can be positive or negative) is modelled by

$$p(t)=\frac{a}{b+e^{-t}}+c \quad , \quad -\infty<t<\infty$$

where a , b and c are positive constants. Define the *primordial population* be the population of the bacterium long time ago and the *ultimate population* be the population of the bacterium after a long time.

(a) Find, in terms of a , b and c ,

(i) the time when the growth rate attains the maximum value;

(ii) the *primordial population*;

(iii) the *ultimate population*.

(5 marks)

(b) A scientist studies the population of the bacterium by plotting a linear graph of $\ln[p(t)-c]$ against $\ln(b+e^{-t})$ and the graph shows the intercept on the vertical axis to be $\ln 8000$. If at 9 am on 16/4/2010 the population and the growth rate of the bacterium are 6000 and 2000 per day respectively, find the values of a , b and c .

(3 marks)

(c) Another scientist claims that the population of the bacterium at the time of maximum growth rate is the mean of the primordial population and ultimate population. Do you agree? Explain your answer.

(2 marks)

(d) By expressing e^{-t} in terms of a , b , c and $p(t)$, express $p'(t)$ in the form of $\frac{-b}{a}[p(t)-\alpha][p(t)-\beta]$, where $\alpha<\beta$.

Hence express α and β in terms of a , b and c .

Sketch $p'(t)$ against $p(t)$ for $\alpha<p(t)<\beta$ and hence verify your answer in (c).

(5 marks)

(2010 ASL-M&S Q9)

19. In a certain year, the amount of water (in million cubic metres) stored in a reservoir can be modeled by

$$A(t)=(-t^2+5t+a)e^{kt}+7 \qquad (0\le t\le 12) \ ,$$

where a and k are constants and t is the time measured in months from the start of the year. The amount of water stored in the reservoir is the greatest when $t=2$. It is found that $A(0)=3$.

(a) Find the value of a .

Hence find the amount of water stored in the reservoir when $t=1$.

(2 marks)

(b) Find the value of k .

(3 marks)

(c) In that year, the period during which the amount of water stored in the reservoir is 7 million cubic metres or more is terms *adequate*.

(i) How long does the *adequate* period last?

(ii) Find the least amount of water stored in the reservoir, within that year, after the *adequate* period has ended.

(iii) Find $\frac{d^2 A(t)}{dt^2}$.

(iv) Describe the behavior of $A(t)$ and $\frac{dA(t)}{dt}$, within that year, after the *adequate* period has ended for 6 months.

(10 marks)

(2007 ASL-M&S Q9)

20. A researcher monitors the process of using micro-organisms to decompose food waste to fertilizer. He records daily the pH value of the waste and models its pH value by

$$P(t)=a+\frac{1}{5}(t^2-8t-8)e^{-kt},$$

where $t(\ge 0)$ is the time measured in days, a and k are positive constants.

When the decomposition process starts (i.e. $t=0$) , the pH value of the waste is 5.9. Also, the researcher finds that $P(8)-P(4)=1.83$.

(a) Find the values of a and k correct to 1 decimal place.

(5 marks)

(b) Using the value of k obtained in (a) ,

(i) determine on which days the maximum pH value and the minimum pH value occurred respectively;

(ii) prove that $\frac{d^2P}{dt^2}>0$ for all $t\ge 23$.

(8 marks)

(c) Estimate the pH value of the waste after a very long time.

[Note: Candidates may use $\lim_{t\to\infty}(t^2e^{-kt})=0$ without proof.]

(2 marks) (2003 ASL-M&S Q9)

21. A chemical factory continually discharges a constant amount of biochemical waste into a river. The microorganisms in the waste material flow down the river and remove dissolved oxygen from the water during biodegradation. The concentration of dissolved oxygen (CDO) of the river is given by

$$G(x)=2a-12e^{-kx}+(a+12)e^{-2kx} \ ,$$

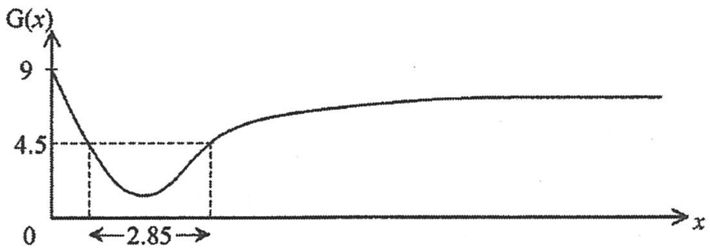
where $G(x)$ mg/L is the CDO of the river at position x km downstream from the location of discharge of the waste, and a , k are positive constants.

At the location of the discharge of waste (i.e. $x=0$) , the CDO of the river is 9 mg/L .

(a) (i) Show that $a=3$.

(ii) Find the minimum CDO of the river.

(b) The figure shows a sketch of the graph of $G(x)$ against x . It is found that downstream from the location of the discharge of waste, a stretch of 2.85 km of the river has a CDO of 4.5 mg/L or below.



(i) Find the value of k correct to 1 decimal place.

(ii) Find $G''(x)$.

Hence determine the position of the river, to the nearest 0.1 km, where the rate of change of the CDO is greatest.

(iii) A river is said to be *healthy* if the CDO of the river is 5.5 mg/L or above. Will the river in this case become *healthy*? If yes, find the position of the river, to the nearest 0.1 km, where it becomes *healthy* again.

(2001 ASL-M&S Q8)

22. A researcher studied the commercial fishing situation in a certain fishing zone. Denoting the total catch of coral fish in that zone in t years time from January 1, 1992 by $N(t)$ (in thousand tonnes), he obtained the following data:

t	2	4
$N(t)$	55	98

The researcher modelled $N(t)$ by $\ln N(t)=a-e^{1-kt}$ where a and k are constants.

(a) Show that $e^{-4k}-e^{-2k}+\frac{1}{e}\ln\frac{98}{55}=0$.

Hence find, to 2 decimal places, two sets of values of a and k .

(4 marks)

(b) The researcher later found out that $N(7)=170$. Determine which set of values of a and k obtained in (a) will make the model fit for the known data.

Hence estimate, to the nearest thousand tonnes, the total possible catch of coral fish in that zone since January 1, 1992.

(4 marks)

(c) The rate of change of the total catch of coral fish in that zone since January 1, 1992 by at time t is given by $\frac{dN(t)}{dt}$.

(i) Show that $\frac{dN(t)}{dt}=kN(t)e^{1-kt}$.

(ii) Using the values of a and k chosen in (b), determine in which year the maximum rate of change occurred.

Hence find, to the nearest integer, the volume of fish caught in that year.

(7 marks)

(Part c is out of Syllabus) (2000 ASL-M&S Q11)

23. A vehicle tunnel company wants to raise the tunnel fees. An expert predicts that after the increase in the tunnel fees, the number of vehicles passing through the tunnel each day will drop drastically in the first week and on the t-th day after the first week, the number $N(t)$ (in thousands) of vehicles passing through the tunnel can be modelled by

$$N(t)=\frac{40}{1+be^{-rt}} \qquad (t\geq 0)$$

where b and r are positive constants.

(a) Suppose that by the end of the first week after the increase in the tunnel fees, the number of vehicles passing through the tunnel each day drops to 16 thousand and by the end of the second week, the number increases to 17.4 thousand, find b and r correct to 2 decimal places.

(5 marks)

(b) Show that $N(t)$ is increasing.

(3 marks)

(c) As time passes, $N(t)$ will approach the average number N_a of vehicles passing through the tunnel each day before the increase in the tunnel fees. Find N_a .

(2 marks)

(d) The expert suggests that the company should start to advertise on the day when the rate of increase of the number of cars passing through the tunnel per day is the greatest. Using the values of b and r obtained in (a),

(i) find $N''(t)$, and

(ii) hence determine when the company should start to advertise.

(5 marks)

(1997 ASL-M&S Q8)

24. A merchant sells compact discs (CDs). A market researcher suggests that if each CD is sold for $\$x$, the number N($x$) of CDs sold per week can be modeled by

$$N(x)=ae^{-bx}$$

where a and b are constants.

The merchant wants to determine the values of a and b based on the following results obtained from a survey:

x	20	30	40	50
N(x)	450	301	202	136

(a) (i) Express ln N(x) as a linear function of x .

(ii) Use a graph paper to estimate graphically the values of a and b correct to 2 decimal places.

(7 marks)

(b) Suppose the merchant wishes to sell 400 CDs in the next week. Use the values of a and b estimated in (a) to determine the price of each CD. Give your answer correct to 1 decimal place.

(2 marks)

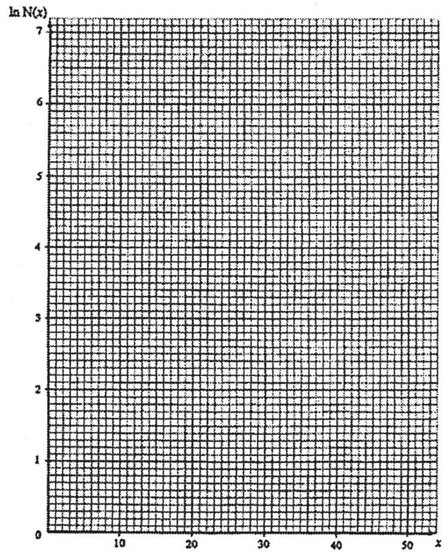
(c) It is known that the merchant obtains CDs at a cost of \$10 each. Let G(x) dollars denote the weekly profit. Using the values of a and b estimated in (a),

(i) express G(x) in terms of x.

(ii) find $G'(x)$ and hence determine the selling price for each CD in order to maximize the profit.

(6 marks)

(1995 ASL-M&S Q8)



2021 DSE Q8

Let $f(x)$ be a function such that $f'(x)=\frac{2^{kx}}{1+2^{kx}}$, where k is a constant. The straight line $8x-9y+10=0$ touches the curve $y=f(x)$ at the point A. It is given that the x-coordinate of A is 1. Find

(a) k,

(b) $f(x)$.

(7 marks)

2021 DSE Q12

A tank is used for collecting rain water. During a certain shower, rain water flows into the tank for 7 minutes. Let $V\text{ m}^3$ be the volume of rain water in the tank. It is given that

$$\frac{dV}{dt}=\sqrt{t+1}\sqrt{3-\sqrt{t+1}} \quad (0\le t\le 7),$$

where t is the number of minutes elapsed since rain water starts flowing into the tank. The tank is empty at $t=0$ and the rate of change of the volume of rain water in the tank attains its maximum value when $t=T$.

(a) Find T. (4 marks)

(b) Find the exact value of V when $t=T$. (5 marks)

(c) The tank is in the shape of an inverted right circular cone of height 1 m and base radius 6 m. The tank is held vertically. Let h m be the depth of rain water in the tank. Find

(i) the constant Q such that $\frac{dV}{dt}=Qh^2\frac{dh}{dt}$,

(ii) $\left.\frac{dh}{dt}\right|_{t=T}$.

(5 marks)

4. Application of Differentiation

Section A

1. (2019 DSE-MATH-M1 Q5a,b)

(a) For all $x>-3$, $f'(x)=\frac{(x+3)(-1)-(6-x)(1)}{(x+3)^2}=\frac{-9}{(x+3)^2}<0$ Thus, $f(x)$ is decreasing.	1	
Note that $f(x)=\frac{9}{x+3}-1$ for all $x>-3$. Thus, $f(x)$ is decreasing.	1	
(b) $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\frac{6}{x}-1}{1+\frac{3}{x}}=-1$	1A	
$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\left(\frac{9}{x+3}-1\right)=-1$	1A	

(a)	Good. Some candidates were unable to show that $f'(x)<0$ to complete the proof.
(b)	Good. Some candidates were unable to consider $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\frac{6}{x}-1}{1+\frac{3}{x}}$ to obtain the required limit.

2. (2018 DSE-MATH-M1 Q7)

3. (2018 DSE-MATH-M1 Q5a)

Note that $3x^2-24x+49=3(x-4)^2+1\neq 0$.

(a) $f'(x)=0$

$\frac{12x-48}{(3x^2-24x+49)^2}=0$ 1M

$x=4$

x	$(-\infty,4)$	4	$(4,\infty)$
$f'(x)$	$-$	0	$+$

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha=4$. 1A

$f'(x)=0$

$\frac{12x-48}{(3x^2-24x+49)^2}=0$ 1M

$x=4$

$f''(x)$

$=\frac{-108x^2+864x-1716}{(3x^2-24x+49)^3}$

$f''(4)$

$=12$

>0

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha=4$. 1A

Very good. Over 85% of the candidates were able to find the value of α .

4. (2017 DSE-MATH-M1 Q6)

(a) $f(6)=-33$

$4(6^3)+m(6^2)+n(6)+615=-33$ 1M

$6m+n=-252$

$f'(x)=12x^2+2mx+n$

$f'(6)=0$

$12(6^2)+2m(6)+n=0$ 1M

$12m+n=-432$

Solving, we have $m=-30$ and $n=-72$. 1A for both correct

(b) $f'(x)=12x^2-60x-72$

$f'(x)=0$ when $x=-1$ or $x=6$. 1M

x	$(-\infty,-1)$	-1	$(-1,6)$	6	$(6,\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	653	↘	-33	↗

1M for testing

Thus, the minimum value is -33 and the maximum value is 653 . 1A for both correct

----------(6)

(a)	Very good. Most candidates were able to find the values of m and n .
(b)	Very good. Many candidates were able to find the maximum value and the minimum value.

5. (2014 DSE-MATH-M1 Q1)

Let V and r be the volume and radius of the spherical balloon respectively.

$V=\frac{4}{3}\pi r^3$

$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ 1M

$\therefore\ -100=4\pi\cdot 10^2\frac{dr}{dt}$ 1M

$\frac{dr}{dt}=\frac{-1}{4\pi}$ 1A OR -0.0796

Hence the rate of change of the radius is $\frac{-1}{4\pi}$ cm s^{-1} .

(3)

Satisfactory.

Many candidates set $\frac{dV}{dt}$ equal to 100 rather than -100 .

6. (2014 DSE-MATH-M1 Q2)

(a) $f(x)=\frac{x^x}{(2x+13)^6}$

$\ln f(x)=x\ln x-6\ln(2x+13)$ 1A

$\frac{f'(x)}{f(x)}=\ln x+x\cdot\frac{1}{x}-6\cdot\frac{2}{2x+13}$ 1M+1A

$f'(x)=\left(\ln x+1-\frac{12}{2x+13}\right)f(x)$

$=\left(\ln x+\frac{2x+1}{2x+13}\right)\frac{x^x}{(2x+13)^6}$ 1A Accept $\left(\ln x+\frac{2x+1}{2x+13}\right)f(x)$

(b) For $x>1$, we have $\ln x>0$, $\frac{2x+1}{2x+13}>0$ and $\frac{x^x}{(2x+13)^6}>0$. 1M

$\therefore\ f'(x)>0$ OR $f'(x)\geq 0$

Hence $f(x)$ is an increasing function. 1

(6)

(a)	Good.
(b)	Very poor. Most candidates failed to show clearly why $f'(x)>0$.

7. (2013 DSE-MATH-M1 Q2)

$p = 8 - \dfrac{2.1}{\sqrt{t+4}}$

$\dfrac{dp}{dt} = \dfrac{2.1}{2(t+4)^{\frac{3}{2}}}$ 1A

$C = 2^p$

$\dfrac{dC}{dp} = 2^p \ln 2$ 1A

$\dfrac{dC}{dt} = \dfrac{dC}{dp} \cdot \dfrac{dp}{dt}$

$= 2^p \ln 2 \cdot \dfrac{2.1}{2(t+4)^{\frac{3}{2}}}$ 1M

When $t = 5$, $p = 7.3$ and hence

$\dfrac{dC}{dt} = 2^{7.3} \ln 2 \cdot \dfrac{2.1}{2(5+4)^{\frac{3}{2}}}$

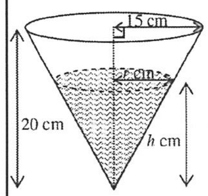
≈ 4.2479 1A

i.e. the rate of change of the concentration of carbon dioxide ≈ 4.2479 units/year.

(4)

Satisfactory. Some candidates found $\frac{dp}{dt}$ or $\frac{dC}{dp}$ wrongly (for example, writing $\frac{dC}{dp} = 2^p \ln p$ or $p2^{p-1}$), while some others obtained $\frac{dC}{dt}$ correctly but did not substitute 5 for t.

8. (PP DSE-MATH-M1 Q3)



(a) By similar triangles, we have $\dfrac{h}{r} = \dfrac{20}{15}$. 1M

$h = \dfrac{4r}{3}$

$\therefore \ V = \dfrac{1}{3}\pi r^2 \left(\dfrac{4r}{3}\right)$

$= \dfrac{4}{9}\pi r^3$ 1A

$A = \pi r\sqrt{r^2 + \left(\dfrac{4r}{3}\right)^2}$

$= \dfrac{5}{3}\pi r^2$ 1A

(b) (i) $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$ 1M (Either one)

$= \dfrac{4}{3}\pi r^2 \dfrac{dr}{dt}$

$-2\pi = \dfrac{4}{3}\pi(3)^2 \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{-1}{6}$ 1A

Hence the rate of change of the radius of the water surface is $\dfrac{-1}{6}$ cm/s.

(ii) $\dfrac{dA}{dt} = \dfrac{dA}{dr} \cdot \dfrac{dr}{dt}$

$= \dfrac{10}{3}\pi r \dfrac{dr}{dt}$

$= \dfrac{10}{3}\pi(3)\left(\dfrac{-1}{6}\right)$

$= \dfrac{-5}{3}\pi$ 1A

Hence the rate of change of the area of the wet surface is $\dfrac{-5}{3}\pi \text{ cm}^2/\text{s}$.

(6)

(a)	甚佳。部分學生未能利用相似三角形的特性。
(b)	平平。很多學生誤以爲 $\frac{dV}{dt} = +2\pi$ 。

9. (SAMPLE DSE-MATH-M1 Q2)

$\because \ r = 3 - \dfrac{2}{2+t}$

$\therefore \ \dfrac{dr}{dt} = \dfrac{2}{(2+t)^2}$ — 1A

$V = \dfrac{4}{3}\pi r^3$

$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$ — 1A

When $r = 2.5$, $2.5 = 3 - \dfrac{2}{2+t}$ — 1M

i.e. $t = 2$

$\therefore \ \left.\dfrac{dV}{dt}\right|_{r=2.5} = 4\pi(2.5)^2 \cdot \dfrac{2}{(2+2)^2}$

$= \dfrac{25}{8}\pi$ — 1A

$\therefore$ the rate of change of volume of the balloon is $\dfrac{25}{8}\pi \text{ m}^3/\text{hr}$.

(4)

10. (2004 ASL-M&S Q3)

(a) Since $2\pi rh + 2\pi r^2 = 162\pi$, we have

$rh + r^2 = 81$. — 1A (either one)

Therefore,

The required capacity

$= \pi r^2 h + \dfrac{2}{3}\pi r^3$ — (either one)

$= \pi r(81 - r^2) + \dfrac{2}{3}\pi r^3$

$= (81\pi r - \dfrac{1}{3}\pi r^3)\ \text{cm}^3$ — 1

(b) Let $f(r) = 81\pi r - \dfrac{1}{3}\pi r^3$ for all $r \ge 0$. Then, we have

$\dfrac{df(r)}{dr} = 81\pi - \pi r^2$. — 1A

$\dfrac{df(r)}{dr} = 0$ when $r = 9$ — 1M

$\dfrac{df(r)}{dr} \begin{cases} > 0 & \text{if } 0 \le r < 9 \\ = 0 & \text{if } r = 9 \\ < 0 & \text{if } r > 9 \end{cases}$ — 1M for testing

So, $f(r)$ attains its greatest value when $r = 9$. — 1A

Note that $f(9) = 486\pi$

[redacted]

≤ 1600

Thus, by (a), the capacity of the container cannot be greater than $1\,600\ \text{cm}^3$. — 1A

Let $f(r) = 81\pi r - \dfrac{1}{3}\pi r^3$ for all $r \ge 0$. Then, we have	
$\dfrac{df(r)}{dr} = 81\pi - \pi r^2$.	1A
$\dfrac{df(r)}{dr} = 0$ when $r = 9$	1M
$\dfrac{d^2 f(r)}{dr^2} = -2\pi r < 0$ for any $r > 0$	1M for testing
[redacted]	
So, $f(r)$ attains its greatest value when $r = 9$.	1A
Note that $f(9) = 486\pi$ [redacted] ≤ 1600	
Thus, by (a), the capacity of the container cannot be greater than $1\,600\ \text{cm}^3$.	1A

(7)

Good. Some candidates failed to prove that the extreme value is the greatest value.

11. (1996 ASL-M&S Q5)

(a) $\because$ $\dfrac{dN}{d\theta} = -[\ln(\theta+49)]^2 - \dfrac{2(\theta+440)\ln(\theta+49)}{\theta+49}$ — 1M+1A+1A (1M for product rule; 1A for diff. of log.)

$= -\ln(\theta+49)\left[\ln(\theta+49) + \dfrac{2(\theta+440)}{\theta+49}\right]$

$\therefore$ $\dfrac{dN}{dt} = \dfrac{dN}{d\theta}\cdot\dfrac{d\theta}{dt}$

$= -\ln(\theta+49)\left[\ln(\theta+49) + \dfrac{2(\theta+440)}{\theta+49}\right]\dfrac{d\theta}{dt}$ — 1A

(b) $\theta = -40$, $\dfrac{d\theta}{dt} = -0.5$

$\dfrac{dN}{dt} = -\ln(-40+49)\left[\ln(-40+49) + \dfrac{2(-40+440)}{-40+49}\right](-0.5)$ — 1M

≈ 100 — 1A

$\therefore$ The rate of increase of the number of tourists is 100 per hour.

(6)

12. (2002 ASL-M&S Q2)

(a) At $t=2$, $r(2) = 5.5035$(m)

$\dfrac{dV}{dr} = 4\pi r^2$

$\dfrac{dr}{dt} = \dfrac{18\times 2e^{-t}}{(3+2e^{-t})^2} = \dfrac{36e^{-t}}{(3+2e^{-t})^2}$ — 1A

At $t=2$, $\dfrac{dV}{dr} = 380.6109$

$\dfrac{dr}{dt} = 0.45545$

$\therefore$ $\dfrac{dV}{dt} = \dfrac{dV}{dr}\dfrac{dr}{dt}$

At $t=2$, $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$ — 1M

$= 380.6109\times 0.45545$

$= 173.35\ (m^3/h)$ — 1A (Accept : 173.31–173.39) $a-1$ for more than 2 d.p.

$\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$ — 1M

$\dfrac{dr}{dt} = \dfrac{36e^{-t}}{(3+2e^{-t})^2}$ — 1A

$\dfrac{dV}{dt} = 4\pi r^2\cdot\dfrac{36e^{-t}}{(3+2e^{-t})^2}$

$= \dfrac{144\pi r^2 e^{-t}}{(3+2e^{-t})^2}$ — (Accept : $\dfrac{dV}{dt} = \dfrac{46656\pi e^{-t}}{(3+2e^{-t})^4}$)

At $t=2$,

$r \approx 5.50346$

$\therefore$ $\dfrac{dV}{dt} \approx 173.35\ (m^3/h)$ — 1A (Accept : 173.31–173.39) $a-1$ for more than 2 d.p.

(b) $\lim_{t\to\infty} r(t) = \lim_{t\to\infty}\dfrac{18}{3+2e^{-t}} = 6$ (m) — 1M

$\therefore$ the volume of the balloon will be

$V = \dfrac{4}{3}\pi(6)^3 + 5\pi$

$= 293\ \pi$

$= 920.49\ (m^3)$ — 1A $a-1$ for more than 2 d.p.

(5)

13. (2013 DSE-MATH-M1 Q11(a,b))

(a) $R'(t)=0$

$P'(t)-C'(t)=0$

$4(4-e^{\frac{-t}{5}})-9(2-e^{\frac{-t}{10}})=0$ — 1A

$-4\left(e^{\frac{-t}{10}}\right)^2+9e^{\frac{-t}{10}}-2=0$ — 1M — For $e^{\frac{-t}{5}}=\left(e^{\frac{-t}{10}}\right)^2$

$e^{\frac{-t}{10}}=0.25$ or 2

$t=20\ln 2$ or $-10\ln 2$ (rejected as $t\ge 0$) — 1A — OR $t\approx 13.8629$

(3)

(b) $R'(t)=-4e^{\frac{-t}{5}}+9e^{\frac{-t}{10}}-2$ — 1A

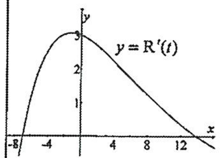
$R''(t)=\frac{4}{5}e^{\frac{-t}{5}}-\frac{9}{10}e^{\frac{-t}{10}}$ — 1M

$=\frac{1}{10}e^{\frac{-t}{10}}\left(8e^{\frac{-t}{10}}-9\right)$

<0 for $t\ge 0$ (since $e^{\frac{-t}{10}}\le 1$ for $t\ge 0$) — 1

Therefore $R'(t)$ decreases with t.

(3)



(a)	Fair. Some candidates confused $R(t)$ with $R'(t)$, or found $R(t)=P(t)-C(t)$ by integration first and then obtained the expression for $R'(t)=P'(t)-C'(t)$ by differentiation. Many candidates failed to make use of knowledge about quadratic equations to solve for t. Some got wrong answers such as '$e^{\frac{-t}{5}}=0.25$ or 2' or did not reject $t=-10\ln 2$.
(b)	Very poor. Many candidates failed to find $R''(t)$ correctly. Among those who were able to find $R''(t)$, only few provided sufficient reasons to conclude that '$R'(t)$ decreases with t'.

14. (2012 ASL-M&S Q9)

(a) $r(t)=20-40e^{-at}+be^{-2at}$

$\because$ $r(0)=20-40e^{0}+be^{0}=30$

$\therefore$ $b=50$ — 1A

(1)

(b) $r'(t)<0$ for 9 days

$40ae^{-at}-100ae^{-2at}<0$ for $t<9$ — 1M

$20ae^{-2at}(2e^{at}-5)<0$

$e^{at}<2.5$

$t<\frac{\ln 2.5}{a}$ — 1A

$\therefore$ $\frac{\ln 2.5}{a}=9$

i.e. $a\approx 0.1$ (correct to 1 decimal place) — 1A

(3)

(c) The rate of change of the rate of selling of handbags is $r'(t)=4e^{-0.1t}-10e^{-0.2t}$.

$\frac{d}{dt}r'(t)=-0.4e^{-0.1t}+2e^{-0.2t}$

$\frac{d}{dt}r'(t)=0$ when $0.4e^{-0.1t}=2e^{-0.2t}$ — 1M

$e^{0.1t}=5$

$t=10\ln 5$ — 1A — OR 16.0944

$\frac{d^2}{dt^2}r'(t)=0.04e^{-0.1t}-0.4e^{-0.2t}$

When $t=10\ln 5$, $\frac{d^2}{dt^2}r'(t)=-0.008<0$ — 1M — OR by using sign test

Hence $r'(t)$ is maximum when $t=10\ln 5$

$r(10\ln 5)=20-40e^{-0.1(10\ln 5)}+50e^{-0.2(10\ln 5)}=14$

The rate of selling $=14$ thousand per day — 1A — OR 14000 per day

(4)

(a)		Very good.
(b)		Satisfactory. Many candidates used an equation rather than an inequality to solve for the value of a.
(c)		Fair. Some candidates overlooked that the given condition was for the rate of change of the rate of selling. When consider the maximum rate of change, candidates should set the second derivative $\frac{d^2r}{dt^2}$ zero.

15. (1998 ASL-M&S Q8)

Solution	Marks	Remarks
(i) If $\dfrac{5000e^{15\lambda}}{15}=\dfrac{5000e^{95\lambda}}{95}$	1A	
then $e^{80\lambda}=\dfrac{19}{3}$ $\lambda=\dfrac{1}{80}\ln(\dfrac{19}{3})$ ≈ 0.0231	1M+1A	
(ii) $N=\dfrac{5000e^{\lambda t}}{t}\approx\dfrac{5000e^{0.0231t}}{t}$ $\dfrac{dN}{dt}=5000\left(\dfrac{\lambda te^{\lambda t}-e^{\lambda t}}{t^2}\right)$ $=\dfrac{5000e^{\lambda t}(\lambda t-1)}{t^2}$	1M+1A	
$\begin{cases} <0 & \text{when } 0<t<\dfrac{1}{\lambda} \\ =0 & \text{when } t=\dfrac{1}{\lambda} \quad (\approx 43.3410) \\ >0 & \text{when } \dfrac{1}{\lambda}<t<120 \end{cases}$	1M+1A	
$\therefore$ N attains its minimum when $t\approx 43.3410$ (The number of fish decreased to the minimum in about 43 days after the spread of the disease.)	1A	r.t. 43

Section B

16. (2017 DSE-MATH-M1 Q12)

Solution	Marks	Remarks
(a) (i) $x-4=\dfrac{3k}{2^{\lambda t}-k}$ $x-1=\dfrac{3(2^{\lambda t})}{2^{\lambda t}-k}$ $(x-4)(x-1)=\dfrac{9k2^{\lambda t}}{(2^{\lambda t}-k)^2}$	1A	
(ii) $\dfrac{9k2^{\lambda t}}{(2^{\lambda t}-k)^2}>0$ (as $k>0$) $(x-4)(x-1)>0$ (by (a)(i))	1M	
$x>4$ or $x<1$ Thus, the claim is correct.	1A ---------(3)	f.t.
(b) (i) $\dfrac{dx}{dt}=\dfrac{-3(\ln 2)k\lambda 2^{\lambda t}}{(2^{\lambda t}-k)^2}$	1A	
$\dfrac{-\ln 2}{24}(x-4)(x-1)=\dfrac{-3(\ln 2)k2^{\lambda t}}{8(2^{\lambda t}-k)^2}$ $\lambda=\dfrac{1}{8}$	1	
(ii) (1) When $t=0$, $x=0.8$. $-3.2=\dfrac{3k}{1-k}$ $k=16$	1A	
When $x=0$, we have $4+\dfrac{48}{2^{\frac{t}{8}}-16}=0$.	1M	either one
So, we have $2^{\frac{t}{8}}=4$.	1M	either one
Solving, we have $t=16$. Thus, the crocodiles in the lake will eventually become extinct in 16 years.	1A	
(2) When $t=0$, $x=7$. $3=\dfrac{3k}{1-k}$ $k=0.5$	1A	
When $x=0$, we have $4+\dfrac{1.5}{2^{\frac{t}{8}}-0.5}=0$.		
So, we have $2^{\frac{t}{8}}=0.125$.		
It is impossible as $2^{\frac{t}{8}}>1$ for $t>0$. Thus, the crocodiles in the lake will never become extinct.	1A	f.t.
Note that $\lim\limits_{t\to\infty}x=\lim\limits_{t\to\infty}\left(4+\dfrac{1.5}{2^{\frac{t}{8}}-0.5}\right)=4$. After a very long time, the estimated number of crocodiles in the lake is 4 000 .	1A ---------(9)	

(a) (i)	Poor. Only a few candidates were able to express $(x-4)(x-1)$ in terms of λ, k and t.
(ii)	Poor. Only a few candidates were able to use the result in (a)(i) to finish the argument.
(b) (i)	Fair. Many candidates were unable to find $\frac{dx}{dt}$.
(ii) (1)	Fair. Only some candidates were able to find the value of k.
(2)	Fair. Many candidates estimated the number of crocodiles in the lake after a very long time without first determining that the crocodiles in the lake will not become extinct eventually.

17. (2016 DSE-MATH-M1 Q12)

(a) $N=\frac{27}{2+\alpha te^{\beta t}}$

$\frac{27-2N}{Nt}=\alpha e^{\beta t}$ — 1M

$\ln\left(\frac{27-2N}{Nt}\right)=\ln\alpha+\beta t$ — 1A

----------(2)

(b) (i) $\beta=-0.1$ — 1A

$0=-0.1(10\ln 0.03)+\ln\alpha$

$\ln\alpha=\ln 0.03$

$\alpha=0.03$ — 1A

(ii) $\frac{dN}{dt}$

$=-27(2+0.03te^{-0.1t})^{-2}(0.03)(e^{-0.1t}-0.1te^{-0.1t})$ — 1M — for $\frac{d}{dt}e^{\beta t}=\beta e^{\beta t}$

$=\frac{0.081(t-10)e^{-0.1t}}{(2+0.03te^{-0.1t})^2}$ — 1A

For $\frac{dN}{dt}=0$, we have $t=10$.

t	$0\le t<10$	$t=10$	$t>10$
$\frac{dN}{dt}$	$-$	0	$+$

— 1M

So, N attains its least value when $t=10$. — 1A

The least value of $N=\frac{27}{2+0.3e^{-1}}\approx 12.79400243>12$.

Thus, the number of chickens will not be less than 12 thousand on a certain day after the start of the spread of the bird flu. — 1A — f.t.

(iii) $\frac{d^2N}{dt^2}$

$=\frac{d}{dt}\left(\frac{dN}{dt}\right)$

$=\frac{0.081(2+0.03te^{-0.1t})^2(e^{-0.1t}-0.1(t-10)e^{-0.1t})}{(2+0.03te^{-0.1t})^4}-\frac{0.081(t-10)e^{-0.1t}(2)(2+0.03te^{-0.1t})(0.03)(e^{-0.1t}-0.1te^{-0.1t})}{(2+0.03te^{-0.1t})^4}$ — 1M — for quotient rule

$=0.0081\left(\frac{(2+0.03te^{-0.1t})(20-t)e^{-0.1t}+0.06(t-10)^2e^{-0.2t}}{(2+0.03te^{-0.1t})^3}\right)$ — 1A

Hence, we have $\frac{d^2N}{dt^2}>0$ for $0\le t\le 20$.

So, $\frac{dN}{dt}$ increases for $0\le t\le 20$. — 1A — f.t.

Thus, the rate of change of the number of chickens increases. ----------(10)

(a)	Very good. More than 70% of the candidates were able to express $\ln\left(\frac{27-2N}{Nt}\right)$ as a linear function of t.
(b) (i)	Good. Many candidates were able to use the slope of the linear function to find β, while a few candidates wrongly took the given horizontal intercept as the vertical intercept to find α.
(ii)	Fair. Many candidates wrongly gave the limiting value of N instead of the least value of N as the answer. Some candidates were unable to evaluate $\frac{d}{dt}te^{-0.1t}$ when finding $\frac{dN}{dt}$.
(iii)	Poor. Most candidates were unable to find the derivative of $\frac{dN}{dt}$ to describe how the rate of change of the number of chickens varies. Only a very small number of candidates were able to determine the sign of $\frac{d^2N}{dt^2}$ for $0\le t\le 20$.

18. (2010 ASL-M&S Q9)

(a) (i) $p(t)=\dfrac{a}{b+e^{-t}}+c$

$p'(t)=\dfrac{ae^{-t}}{(b+e^{-t})^2}$ 1A

$p''(t)=\dfrac{(b+e^{-t})^2(-ae^{-t})-(ae^{-t})2(b+e^{-t})(-e^{-t})}{(b+e^{-t})^4}$

$=\dfrac{ae^{-t}(e^{-t}-b)}{(b+e^{-t})^3}$ 1A

Hence $p''(t)=0$ when $e^{-t}-b=0$.

i.e. $t=-\ln b$

t	$t<-\ln b$	$t=-\ln b$	$t>-\ln b$
$p''(t)$	+	0	−

Hence the growth rate attains the maximum value when $t=-\ln b$ 1A Follow through

(ii) *primordial population* $=\lim\limits_{t\to-\infty}\left(\dfrac{a}{b+e^{-t}}+c\right)=c$ 1A

(iii) *ultimate population* $=\lim\limits_{t\to\infty}\left(\dfrac{a}{b+e^{-t}}+c\right)=\dfrac{a}{b}+c$ 1A

(5)

(b) $\ln[p(t)-c]=-\ln(b+e^{-t})+\ln a$

$\therefore$ $\ln a=\ln 8000$

$a=8000$ 1A

$\therefore$ $p'(0)=\dfrac{8000}{(b+1)^2}=2000$

$b=1$ or -3 (rejected) 1A

$\therefore$ $p(0)=\dfrac{8000}{1+1}+c=6000$

$c=2000$ 1A

(3)

(c) The population at the time of maximum growth rate is

$p(-\ln b)=\dfrac{a}{2b}+c$ 1A

The mean of the *primordial population* and *ultimate population* is

$\dfrac{1}{2}\left[c+\left(\dfrac{a}{b}+c\right)\right]=\dfrac{a}{2b}+c$

Hence the scientist's claim is agreed. 1

(2)

(d) $p(t)=\dfrac{a}{b+e^{-t}}+c$

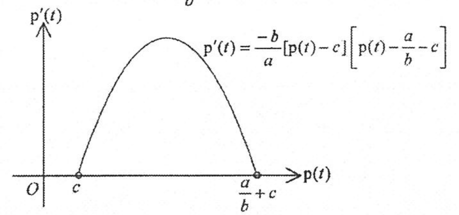
$e^{-t}=\dfrac{a}{p(t)-c}-b$ 1A

$\therefore$ $p'(t)=\dfrac{a\left[\dfrac{a}{p(t)-c}-b\right]}{\left[b+\left(\dfrac{a}{p(t)-c}-b\right)\right]^2}$ 1M

$=\dfrac{a[p(t)-c]\{a-b[p(t)-c]\}}{a^2}$

$=\dfrac{-b}{a}[p(t)-c]\left[p(t)-\dfrac{a}{b}-c\right]$

Hence $\alpha=c$ and $\beta=\dfrac{a}{b}+c$. 1A



1A

From the graph, we can see that $p'(t)$ is maximum when $p(t)$ is the mean of c and $\dfrac{a}{b}+c$, i.e. the mean of the *primordial population* and *ultimate population* 1 Follow through

(5)

(a) (i)		Fair. Many candidates confused the maximum growth rate and the maximum population and hence could not determine the time required.
(ii) (iii)		Fair. Some candidates mistook $p(0)$ to be the primordial population.
(b)		Fair.
(c)		Poor. Most candidates did not understand the question.
(d)		Very poor. Most candidates could not go beyond expressing e^{-t} in terms of a, b, c and $p(t)$.

19. (2007 ASL-M&S Q9)

Solution	Marks
(a) $A(t)=(-t^2+5t+a)e^{kt}+7$	
Since $A(0)=3$, we have $a+7=3$.	
Thus, we have $a=-4$.	1A
The required amount of water stored	
$=(-1^2+5-4)e^k+7$	
$=7$ million cubic metres	1A
	----------(2)
(b) $A(t)=(-t^2+5t-4)e^{kt}+7$	
$\frac{dA(t)}{dt}$	
$=(-2t+5)e^{kt}+(-t^2+5t-4)(ke^{kt})$	1M for product rule
$=\left(-kt^2+(5k-2)t+5-4k\right)e^{kt}$	
Note that when $t=2$, $\frac{dA(t)}{dt}=0$.	
So, we have $2k+1=0$.	1M
Thus, we have $k=\frac{-1}{2}$.	1A
	----------(3)
(c) (i) When $A(t)\geq 7$, we have	
$-t^2+5t-4\geq 0$	
$t^2-5t+4\leq 0$	1M accept setting quadratic equation
$1\leq t\leq 4$	1A (accept $t=1\to t=4$)
Thus, the *adequate* period lasts for 3 months.	
(ii) Note that $A(t)=(-t^2+5t-4)e^{\frac{-t}{2}}+7$.	
So, $\frac{dA(t)}{dt}=\left(\frac{t^2}{2}-\frac{9t}{2}+7\right)e^{\frac{-t}{2}}$	
and $\frac{dA(t)}{dt}=0$ when $t=2$ (rejected since $t>4$) or $t=7$.	1A
$\frac{dA(t)}{dt}\begin{cases}<0 & \text{if } 4<t<7\\ =0 & \text{if } t=7\\ >0 & \text{if } 7<t\leq 12\end{cases}$	1M for testing + 1A
So, $A(t)$ attains its least value when $t=7$.	
The least amount of water stored	
$=A(7)$	
≈ 6.456447098	
≈ 6.4564 million cubic metres	1A $a-1$ for r.t. 6.456 million cubic metres

Solution	Marks
Note that $A(t)=(-t^2+5t-4)e^{\frac{-t}{2}}+7$.	
So, $\frac{dA(t)}{dt}=\left(\frac{t^2}{2}-\frac{9t}{2}+7\right)e^{\frac{-t}{2}}$	
and $\frac{dA(t)}{dt}=0$ when $t=2$ (rejected since $t>4$) or $t=7$.	1A
$\frac{d^2A(t)}{dt^2}$	
$=\left(\frac{-t^2}{4}+\frac{9t}{4}-\frac{7}{2}\right)e^{\frac{-t}{2}}+\left(t-\frac{9}{2}\right)e^{\frac{-t}{2}}$	
$=\left(\frac{-t^2}{4}+\frac{13t}{4}-8\right)e^{\frac{-t}{2}}$	1A
Therefore, we have $\left.\frac{d^2A(t)}{dt^2}\right\rvert_{t=7}=\frac{5}{2}e^{\frac{-7}{2}}>0$.	1M for testing + 1A
Note that there is only one local minimum after the *adequate* period.	
So, $A(t)$ attains its least value when $t=7$.	
The least amount of water stored	
$=A(7)$	
≈ 6.456447098	
≈ 6.4564 million cubic metres	1A $a-1$ for r.t. 6.456 million cubic metres
	either one
(iii) $\frac{d^2A(t)}{dt^2}$	
$=\left(\frac{-t^2}{4}+\frac{9t}{4}-\frac{7}{2}\right)e^{\frac{-t}{2}}+\left(t-\frac{9}{2}\right)e^{\frac{-t}{2}}$	
$=\left(\frac{-t^2}{4}+\frac{13t}{4}-8\right)e^{\frac{-t}{2}}$	
(iv) Since $\frac{dA(t)}{dt}=\frac{1}{2}\left((t-\frac{9}{2})^2-\frac{25}{4}\right)e^{\frac{-t}{2}}>0$ for $10<t<12$,	1M for considering the sign
$A(t)$ increases, within that year, after the *adequate* period has ended for 6 months.	1A f.t. (either one)
Since $\frac{d^2A(t)}{dt^2}=\frac{-1}{4}\left((t-\frac{13}{2})^2-\frac{41}{4}\right)e^{\frac{-t}{2}}<0$ for $10<t<12$,	
$\frac{dA(t)}{dt}$ decreases, within that year, after the *adequate* period has ended for 6 months.	1A f.t.
	----------(10)

(a)		Very good.
(b)		Good.
(c) (i)		Good.
(ii)		Fair. Many candidates neglected the range of values of the variable and hence could not complete the solution.
(iii)		Good.
(iv)		Poor. Many candidates could not make use of the value of $A''(t)$ to explain the behaviour of $A'(t)$ and many did not specify the time period.

20. (2003 ASL-M&S Q9)

(a) $\because$ $P(0) = 5.9$

$\therefore$ $a + \frac{1}{5}(0 - 0 - 8) = 5.9$

So, $a = 7.5$ — 1A

$P(t) = 7.5 + \frac{1}{5}(t^2 - 8t - 8)e^{-kt}$

$\because$ $P(8) - P(4) = 1.83$

$\therefore$ $-1.6e^{-8k} + 4.8e^{-4k} = 1.83$ — 1M+1A

$160(e^{-4k})^2 - 480e^{-4k} + 183 = 0$ — 1M can be absorbed

$e^{-4k} \approx 2.551784198$ or $e^{-4k} \approx 0.448215801$

$k \approx -0.2341982$ or $k \approx 0.200620116$

$\because$ $k > 0$

$\therefore$ $k \approx 0.2$ (correct to 1 decimal place) — 1A

----------(5)

(b) $P(t) = \frac{15}{2} + \frac{1}{5}(t^2 - 8t - 8)e^{-0.2t}$

(i) $\frac{dP(t)}{dt} = \frac{-1}{25}\left[(t^2 - 8t - 8) - 5(2t - 8)\right]e^{-0.2t}$ — 1M for Product Rule or Chain Rule

$= \frac{-1}{25}(t^2 - 18t + 32)e^{-0.2t}$ — 1A independent of the obtained value of a

$= \frac{-1}{25}(t - 2)(t - 16)e^{-0.2t}$

For $\frac{dP(t)}{dt} = 0$, we have $t = 2$ or $t = 16$.

$$\frac{dP(t)}{dt}\begin{cases} < 0 & \text{if } 0 \le t < 2 \\ = 0 & \text{if } t = 2 \\ > 0 & \text{if } 2 < t < 16 \end{cases}$$

So, the minimum pH value occurred at $t = 2$. — 1M+1A

$$\frac{dP(t)}{dt}\begin{cases} > 0 & \text{if } 2 < t < 16 \\ = 0 & \text{if } t = 16 \\ < 0 & \text{if } t > 16 \end{cases}$$

So, the maximum pH value occurred at $t = 16$. — 1M+1A accept max at $t = 0$ and at $t = 16$

$\frac{dP(t)}{dt} = \frac{-1}{25}\left[(t^2 - 8t - 8) - 5(2t - 8)\right]e^{-0.2t}$ — 1M for Product Rule or Chain Rule

$= \frac{-1}{25}(t^2 - 18t + 32)e^{-0.2t}$ — 1A independent of the obtained value of a

$= \frac{-1}{25}(t - 2)(t - 16)e^{-0.2t}$

For $\frac{dP(t)}{dt} = 0$, we have $t = 2$ or $t = 16$.

$\frac{d^2P(t)}{dt^2} = \frac{1}{125}[t^2 - 18t + 32 - 5(2t - 18)]e^{-0.2t}$

$= \frac{1}{125}(t^2 - 28t + 122)e^{-0.2t}$

$\left.\frac{d^2P(t)}{dt^2}\right|_{t=2} \approx 0.375379225 > 0$

So, the minimum pH value occurred at $t = 2$. — 1M+1A

$\left.\frac{d^2P(t)}{dt^2}\right|_{t=16} \approx -0.022826834 < 0$

So, the maximum pH value occurred at $t = 16$. — 1M+1A accept max at $t = 0$ and at $t = 16$

(ii) $\frac{d^2P}{dt^2} = \frac{1}{125}[t^2 - 18t + 32 - 5(2t - 18)]e^{-0.2t}$

$= \frac{1}{125}(t^2 - 28t + 122)e^{-0.2t}$ — 1A

$\because$ $\frac{d^2P}{dt^2} = \frac{1}{125}\left(t - (14 - \sqrt{74})\right)\left(t - (14 + \sqrt{74})\right)e^{-0.2t}$

$5 < 14 - \sqrt{74} < 6$ and $22 < 14 + \sqrt{74} < 23$

$\therefore$ $\frac{d^2P}{dt^2} > 0$ for all $t \ge 23$. — 1

----------(8)

(c) The required pH value

$= \lim_{t \to \infty}\left(7.5 + \frac{1}{5}(t^2 - 8t - 8)e^{-0.2t}\right)$

$= 7.5 + \frac{1}{5}\lim_{t \to \infty}\left(t^2 e^{-0.2t}\right) - \frac{8}{5}\lim_{t \to \infty}\left(te^{-0.2t}\right) - \frac{8}{5}\lim_{t \to \infty} e^{-0.2t}$

$= 7.5 + \frac{1}{5}(0) - \frac{8}{5}(0) - \frac{8}{5}(0)$ $\left(\because \lim_{t \to \infty}(te^{-0.2t}) = \left(\lim_{t \to \infty}\frac{1}{t}\right)\left(\lim_{t \to \infty} t^2 e^{-0.2t}\right) = (0)(0) = 0\right)$ — 1A for $\lim_{t \to \infty}(te^{-0.2t}) = 0$ (can be absorbed)

$= 7.5$ — 1M accept the required pH value $= a$

----------(2)

(a)		Good. Some candidates were unable to transform the equation '$-1.6e^{-8k} + 4.8e^{-4k} = 1.83$' into a quadratic equation.
(b)		Good. Most candidates were able to differentiate functions involving 'exp' function.
(c)		Satisfactory. Some candidates had difficulty in finding the limit.

21. (2001 ASL-M&S Q8)

(a) (i) Since $G(0)=9$,

$\therefore\quad 2a-12+(a+12)=9$

$a=3$ — 1

(ii) $G(x)=6-12e^{-kx}+15e^{-2kx}$

$G'(x)=12ke^{-kx}-30ke^{-2kx}$ — 1A

$=6ke^{-kx}(2-5e^{-kx})$

$G'(x)=0$ when $e^{-kx}=\frac{2}{5}$ or $x=\frac{1}{k}\ln\frac{5}{2}$ $\boxed{\frac{0.9163}{k}}$ — 1A

and $G'(x)\begin{cases}<0 & \text{when } 0\le x<\frac{1}{k}\ln\frac{5}{2}\\ >0 & \text{when } x>\frac{1}{k}\ln\frac{5}{2}\end{cases}$ — 1M

$\therefore$ $G(x)$ is minimum when $e^{-kx}=\frac{2}{5}$.

$G''(x)=-12k^2e^{-kx}+60k^2e^{-2kx}$	
When $e^{-kx}=\frac{2}{5}$, $G''(x)=\frac{24}{5}k^2>0$	
Since $G(x)$ has only one stationary point for $x\ge 0$,	
$G(x)$ is minimum when $e^{-kx}=\frac{2}{5}$.	$\boxed{1M}$

(ii) $G(x)=6-12e^{-kx}+15e^{-2kx}$	
$=15(e^{-2kx}-\frac{4}{5}e^{-kx})+6$	
$=15(e^{-kx}-\frac{2}{5})^2+\frac{18}{5}$	$\boxed{1M}$+$\boxed{1A}$
$G(x)$ is minimum when $e^{-kx}=\frac{2}{5}$.	$\boxed{1M}$

The minimum CDO $=\left[6-12\left(\frac{2}{5}\right)+15\left(\frac{2}{5}\right)^2\right]$ mg/L

$=3.6$ mg/L — 1A f.t.

---------(5)

(b) (i) Solving $G(x)=4.5$, we have

$6-12e^{-kx}+15e^{-2kx}=4.5$ — 1M

$10(e^{-kx})^2-8e^{-kx}+1=0$

$e^{-kx}=\frac{4\pm\sqrt{6}}{10}$ — 1A

$x=-\frac{1}{k}\ln\frac{4\pm\sqrt{6}}{10}$

Hence $-\frac{1}{k}\ln\frac{4-\sqrt{6}}{10}+\frac{1}{k}\ln\frac{4+\sqrt{6}}{10}=2.85$ — 1M+1A

$\frac{1}{k}\ln\frac{4+\sqrt{6}}{4-\sqrt{6}}=2.85$

$k\approx 0.5$ (1 d.p.) — 1A

(ii) $G'(x)=6e^{-0.5x}-15e^{-x}$ $\boxed{12ke^{-kx}-30ke^{-2kx}}$

$G''(x)=-3e^{-0.5x}+15e^{-x}$ $\boxed{-12k^2e^{-kx}+60k^2e^{-2kx}}$ — 1M

$=3e^{-0.5x}(5e^{-0.5x}-1)$ $\boxed{12k^2e^{-kx}(5e^{-kx}-1)}$

$G''(x)=0$ when $x=-\frac{1}{0.5}\ln\frac{1}{5}$ (≈ 3.2) $\boxed{x=-\frac{1}{k}\ln\frac{1}{5}}$

and $G''(x)\begin{cases}<0 & \text{when } x>-\frac{1}{0.5}\ln\frac{1}{5} & \left[x>-\frac{1}{k}\ln\frac{1}{5}\right]\\ >0 & \text{when } 0\le x<-\frac{1}{0.5}\ln\frac{1}{5} & \left[0\le x<-\frac{1}{k}\ln\frac{1}{5}\right]\end{cases}$ — 1M

$G'''(x)=1.5e^{-0.5x}(1-10e^{-0.5x})$ $\boxed{12k^3e^{-kx}(1-10e^{-kx})}$	
When $e^{-kx}=\frac{1}{5}$, $G'''(x)=-0.3<0$ $\boxed{-2.4k^3<0}$	
Since $G'(x)$ has only one stationary point for $x\ge 0$,	
$G'(x)$ is greatest when $e^{-kx}=\frac{1}{5}$.	$\boxed{1M}$

$\therefore$ 3.2 km downstream from the location of discharge of the waste, the rate of change of the CDO is greatest. — 1A

(iii) Solving $G(x)=5.5$, we have

$30e^{-x}-24e^{-.5x}+1=0$

$e^{-0.5x}=\frac{12\pm\sqrt{114}}{30}$

$x=-\frac{1}{0.5}\ln\frac{12\pm\sqrt{114}}{30}$

$x\approx 0.6$ or 6.2 — 1A

$\therefore$ The river will return to be healthy 6.2 km downstream form the location of discharge of waste. — 1

Since $\lim_{x\to\infty}G(x)=\lim_{x\to\infty}\left(6-12e^{-0.5x}+15e^{-x}\right)=6>5.5$	
$\therefore$ The river will return to be healthy.	$\boxed{1}$
Solving $G(x)=5.5$, we have $x\approx 0.6$ or 6.2	
$\therefore$ The river will return to be healthy 6.2 km downstream form the location of discharge of waste.	$\boxed{1A}$

---------(10)

22. (2000 ASL-M&S Q11)

Solution	Marks
(a) $\begin{cases} \ln 55 = a - e^{1-2k} \\ \ln 98 = a - e^{1-4k} \end{cases}$	
Eliminating a, we have $e^{1-4k} - e^{1-2k} + \ln 98 - \ln 55 = 0$	
$e^{-4k} - e^{-2k} + \frac{1}{e}\ln\frac{98}{55} = 0$	1
$(e^{-2k})^2 - e^{-2k} + \frac{1}{e}\ln\frac{98}{55} = 0$	1M quadratic equation
$e^{-2k} = \dfrac{1 \pm \sqrt{1 - \frac{4}{e}\ln\frac{98}{55}}}{2}$ ≈ 0.30635 or 0.69365 ≈ 0.306 or 0.694	1A r.t. 0.306 , 0.694
$\begin{cases} k \approx 0.5915 \\ a \approx 4.8401 \end{cases}$ or $\begin{cases} k \approx 0.1829 \\ a \approx 5.8929 \end{cases}$	
$\begin{cases} k \approx 0.59 \\ a \approx 4.84 \end{cases}$ or $\begin{cases} k \approx 0.18 \\ a \approx 5.89 \text{ (or 5.90)} \end{cases}$ (2 d.p.)	1A a–1 for more than 2 d.p.
(b) Using $\begin{cases} k \approx 0.59 \\ a \approx 4.84 \end{cases}$, $\ln N(7) \approx 4.80$,	1M r.t. 4.80
$N(7) \approx 121$.	r.t. 121
Using $\begin{cases} k \approx 0.18 \\ a \approx 5.89 \end{cases}$, $\ln N(7) \approx 5.12$,	r.t. 5.12 – 5.14
$N(7) \approx 167$. (or comparing $\ln 170 \approx 5.1358$)	r.t. 167 – 170
$\therefore$ $\begin{cases} k \approx 0.18 \\ a \approx 5.89 \end{cases}$ will make the model fit for the known data.	1A follow through
$\because$ $N(t) = e^{\ln N(t)} \approx e^{5.89 - e^{1-0.18t}}$	1M
$\therefore$ $N(t) \to e^{5.89} \approx 361$ as $t \to \infty$ The total possible catch of coral fish in that area since January 1, 1992 is 361 thousand tonnes.	1A r.t. 361 – 365 *pp*–1 for wrong/missing unit

Solution	Marks
(c) (i) $\because$ $\ln N(t) = a - e^{1-kt}$	
$\therefore$ $\frac{N'(t)}{N(t)} = ke^{1-kt}$	
$N'(t) = k\,N(t)e^{1-kt}$	1
Alternatively, $N(t) = e^{a - e^{1-kt}}$ $N'(t) = -e^{1-kt}(-k)e^{a-e^{1-kt}} = ke^{1-kt}\,N(t)$	1
(ii) $N''(t) = k[N'(t)e^{1-kt} - k\,N(t)e^{1-kt}]$ $= k^2\,N(t)e^{1-kt}(e^{1-kt} - 1)$	1A
$\begin{cases} > 0 & \text{when } t < \frac{1}{k} \\ = 0 & \text{when } t = \frac{1}{k} \\ < 0 & \text{when } t > \frac{1}{k} \end{cases}$	1M
$\therefore$ $N'(t)$ is maximum at $t = \frac{1}{k}$	1A
≈ 5.56	$t \in [5.47, 5.56]$
The maximum rate of change of the total catch of coral fish in that area since January 1, 1992 occurred in 1997.	1A
$\ln N(6) \approx 4.97$, $N(6) \approx 143.6$ $\ln N(5) \approx 4.78$, $N(5) \approx 119.7$	$\ln N(6) \in [4.97, 4.99]$ $N(6) \in [143.6, 146.3]$ $\ln N(5) \in [4.78, 4.80]$ $N(5) \in [119.7, 122.0]$
$\therefore$ The volume of fish caught in 1997 $= [N(6) - N(5)]$ thousand tonnes	1M
≈ 24 thousand tonnes	1A *pp*–1 for wrong/missing unit

23. (1997 ASL-M&S Q8)

(a) $\because$ $N(0) = 16$

$\therefore$ $\dfrac{40}{1+b} = 16$ — 1M

$b = 1.5$ — 1A

$\because$ $N(7) = 17.4$

$\therefore$ $\dfrac{40}{1+1.5e^{-7r}} = 17.4$ — 1M

$e^{-7r} = \dfrac{1}{1.5}\left(\dfrac{40}{17.4} - 1\right)$

$r = \dfrac{1}{-7}\ln\left[\dfrac{1}{1.5}\left(\dfrac{40}{17.4} - 1\right)\right]$ — 1M

≈ 0.02 — 1A

(b) $N(t) = \dfrac{40}{1+be^{-rt}}$ (or $\dfrac{40}{1+1.5e^{-0.02t}}$)

$N'(t) = \dfrac{-40(-bre^{-rt})}{(1+be^{-rt})^2}$ (or $\dfrac{-40(-1.5)(0.02)e^{-0.02t}}{(1+1.5e^{-0.02t})^2}$) — 1M+1A

$= \dfrac{40bre^{-0.02t}}{(1+be^{-rt})^2}$ (or $\dfrac{1.2e^{-0.02t}}{(1+1.5e^{-0.02t})^2}$)

> 0

$\therefore$ $N(t)$ is increasing. — 1

(c) $\because$ $\lim\limits_{t\to\infty} e^{-rt} = 0$

$\therefore$ $N_a = \lim\limits_{t\to\infty}\dfrac{40}{1+be^{-rt}}$ (or $\lim\limits_{t\to\infty}\dfrac{40}{1+1.5e^{-0.02t}}$) — 1M

$= 40$ — 1A

(d) (i) $N''(t)$

$= \dfrac{[(1+1.5e^{-0.02t})(1.2) - 1.2e^{-0.02t}(2)(1.5)](1+1.5e^{-0.02t})(-0.02)e^{-0.02t}}{(1+1.5e^{-0.02t})^4}$ — 1M

$= \dfrac{0.012e^{-0.02t}(3e^{-0.02t} - 2)}{(1+1.5e^{-0.02t})^3}$ — 1A

(ii) From (i), $N''(t)\begin{cases} >0 & \text{when } t < t_0 \\ =0 & \text{when } t = t_0 \\ <0 & \text{when } t > t_0 \end{cases}$ — 1M — For Solving $N''(t) = 0$

where $t_0 = -\dfrac{1}{0.02}\ln\dfrac{2}{3} \approx 20.2733$

$\therefore$ The rate of increase is the greatest when $t = t_0 \approx 20.2733$ — 1M — For checking maximum

$\because$ $N'(20) \approx 0.199999$

$N'(21) \approx 0.199989$

$\therefore$ The company should start to advertise on the 20th day after the first week. — 1A

24. (1995 ASL-M&S Q8)

(a) (i) $N(x) = ae^{-bx}$

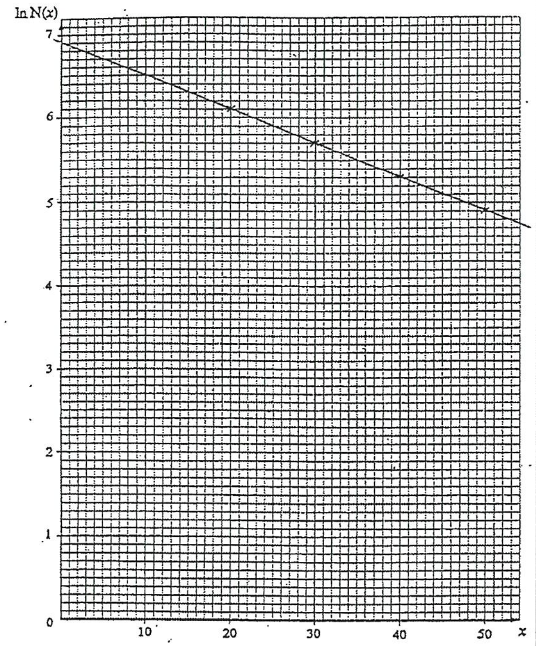
$\ln N(x) = \ln a + \ln e^{-bx}$ — 1A

$= \ln a - bx$ — 1A

(ii)

x	20	30	40	50
$\ln N(x)$	6.11	5.71	5.31	4.91

1A — At least 1 d.p.



1A + 1A — 1A for the points; 1A for the line

From the graph,

$\ln a = 6.9$ — Accept 6.85 - 6.95

$\therefore$ $a = 992.27$ — 1A — Accept 943.88 - 1043.15

$-b = \dfrac{4.91-6.11}{50-20}$

$b = 0.04$ — 1A — no mark for b=slope or b=-0.04 in any calculation.

(b) $992.27e^{-0.04x}=400$	1M	
$-0.04x=\ln\frac{400}{992.27}$		
$x=22.7$		
In general, accept $ae^{-0.04x}=400$ where $a\in(943.88,1043.15)$ $-0.04x=\ln\frac{400}{a}$ $x\in(21.5,24.0)$	1M	
∴ The price of each CD should be \$22.7.	1A	Accept \$21.5 - \$ 24.0
(c) (i) $G(x)=992.27(x-10)e^{-0.04x}$	1A	
In general, accept $G(x)=a(x-10)e^{-0.04x}$ where $a\in(943.88,1043.15)$	1A	
(ii) $G'(x)=992.27[e^{-0.04x}+(-0.04)(x-10)e^{-0.04x}]$	1M	
$=992.27e^{-0.04x}[1.4-0.04x]$	1A	
In general, accept $G'(x)=a[e^{-0.04x}+(-0.04)(x-10)e^{-0.04x}]$ $=ae^{-0.04x}[1.4-0.04x]$ where $a\in(943.88,1043.15)$	1M 1A	
$G'(x)=0$ when $x=35$ and	1M	for $G'(x)=0$ and solving
$G'(x)\begin{cases}>0 & \text{if } x<35\\ <0 & \text{if } x>35\end{cases}$	1M	
Alternatively $G''(x)=1.59xe^{-0.04x}-95.18e^{-0.04x}$ $G''(35)=-9.750$	1M	
Therefore $G(x)$ is maximum when $x=35$. For maximum profit, the selling price for each CD should be \$35.	1A	