3. Derivatives and Differentiation of Functions

Learning Unit	Learning Objective
Calculus Area	
Differentiation with Its Applications	
3. Derivative of a function	3.1 recognise the intuitive concept of the limit of a function 3.2 find the limits of algebraic functions, exponential functions and logarithmic functions 3.3 recognise the concept of the derivative of a function from first principles 3.4 recognise the slope of the tangent of the curve $y=f(x)$ at a point $x=x_0$
4. Differentiation of a function	4.1 understand the addition rule, product rule, quotient rule and chain rule of differentiation 4.2 find the derivatives of algebraic functions, exponential functions and logarithmic functions
5. Second derivative	5.1 recognise the concept of the second derivative of a function 5.2 find the second derivative of an explicit function

Implicit differentiation is **not** required.

~~Logarithmic differentiation is required.~~

Logarithmic differentiation is **not** required.

- $\cancel{\dfrac{dx}{dy}=\dfrac{1}{\frac{dy}{dx}}}$

Section A

1. Consider the curve C: $y=\dfrac{x}{\sqrt{x-2}}$, where $x>2$.

(a) Find $\dfrac{dy}{dx}$.

(b) A tangent to C passes through the point $(9,0)$. Find the slope of this tangent.

(7 marks) (2017 DSE-MATH-M1 Q7)

2. Consider the curve C: $y=(2x+8)^{\frac{3}{2}}+3x^2$, where $x>-4$.

(a) Find $\dfrac{dy}{dx}$.

(b) Someone claims that two of the tangents to C are parallel to the straight line $6x+y+4=0$. Do you agree? Explain your answer.

(7 marks) (2016 DSE-MATH-M1 Q7)

3. Consider the curve C: $y=x\sqrt{2x^2+1}$.

(a) Find $\dfrac{dy}{dx}$.

(b) Two of the tangents to C are perpendicular to the straight line $3x+17y=0$. Find the equations of the two tangents.

(7 marks) (2015 DSE-MATH-M1 Q7)

4. Consider the curve C: $y=x(x-2)^{\frac{1}{3}}$ and the straight line L that passes through the origin and is parallel to the tangent to C at $x=3$.

(a) Find the equation of L.

(b) Find the x-coordinates of the two intersecting points of C and L.

(4 marks) (2013 DSE-MATH-M1 Q3a, b)

5. It is given that $t=y^3+2y^{\frac{-1}{2}}+1$ and $e^t=x^{x^2+1}$.

(a) Find $\dfrac{dt}{dy}$.

(b) By expressing t in terms of x, find $\dfrac{dt}{dx}$.

(c) Find $\dfrac{dy}{dx}$ in terms of x and y.

(5 marks) (PP DSE-MATH-M1 Q2)

6. Consider the curve C: $y=x(2x-1)^{\frac{1}{2}}$, where $x>\dfrac{1}{2}$.

(a) Find $\dfrac{dy}{dx}$.

(b) Using (a), find the equations of the two tangents to the curve C which are parallel to the straight line $2x-y=0$.

(6 marks) (PP DSE-MATH-M1 Q4)

7. Let $x=\ln\dfrac{1+t}{1-t}$, where $-1<t<1$.

(a) Find $\dfrac{dx}{dt}$.

(b) Let $y=1+e^{-x}-e^{-2x}$.

(i) Find $\dfrac{dy}{dx}$.

(ii) Find the value of $\dfrac{dy}{dt}$ when $t=\dfrac{1}{2}$.

(6 marks) (2013 ASL-M&S Q2)

8. Let $y=\dfrac{1-e^{4x}}{1+e^{8x}}$.

(a) Find the value of $\dfrac{dy}{dx}$ when $x=0$.

(b) Let $(z^2+1)e^{3z}=e^{\alpha+\beta x}$, where α and β are constants.

(i) Express $\ln(z^2+1)+3z$ as a liner function of x .

(ii) It is given that the graph of the linear function obtained in (b)(i) passes through the origin and the slope of the graph is 2. Find the values of α and β .

(iii) Using the values of α and β obtained in (b)(ii), find the value of $\dfrac{dy}{dz}$ when $z=0$.

(7 marks) (2007 ASL-M&S Q3)

9. A chemical X is continuously added to a solution to form a substance Y . The total amount of Y formed is given by

$$y=3+\frac{4x-9}{\sqrt{4x^2+3x+9}} ,$$

where x grams and y grams are the total amount of X added and the total amount of Y formed respectively.

(a) Find $\dfrac{dy}{dx}$ when 10 grams of X is added to the solution.

(b) Estimate the total amount of Y formed if X is indefinitely added to the solution.

(6 marks) (2003 ASL-M&S Q3)

10. Let $u=e^{2x}$, and $\dfrac{dy}{du}=\dfrac{1}{u}-2u$.

(a) Express $\dfrac{du}{dx}$ and $\dfrac{dy}{dx}$ in terms of x .

(b) It is known that $y=1$ when $x=0$. Express y in terms of x .

(5 marks) (2001 ASL-M&S Q2)

11. Let $y=xe^{\frac{1}{x}}$ where $x>0$. Show that $x^4\dfrac{d^2y}{dx^2}-y=0$.

(5 marks) (1998 ASL-M&S Q1)

12. The population size x of an endangered species of animals is modeled by the equation

$$\frac{d^2x}{dt^2}-2\frac{dx}{dt}-3x=0 ,$$

where t denotes the time.

It is known that $x=100e^{kt}$ where k is a negative constant. Determine the value of k .

(5 marks) (1994 ASL-M&S Q2)

13. Let $x=-\dfrac{5}{t^2}+2e^{-3t}$ and $y=\dfrac{10}{t^2}+e^{2t}$ $(t\neq 0)$. It is given that $\dfrac{dy}{dx}=-2$. By considering $\dfrac{dy}{dt}=\dfrac{dy}{dx}\cdot\dfrac{dx}{dt}$, find the value of t .

(4 marks) (*modified from* 2002 ASL-M&S Q1)

14. It is given that $\begin{cases}x=\ln(2t+4)\\ y=e^{t^2+4t+4}\end{cases}$, where $t>-2$.

(a) By considering $\dfrac{dy}{dt}=\dfrac{dy}{dx}\cdot\dfrac{dx}{dt}$, express $\dfrac{dy}{dx}$ in terms of y only.

(b) Find the value of $\dfrac{d^2y}{dx^2}$ when $x=0$.

(6 marks) (*modified from* 2012 ASL-M&S Q2)

Limit at infinity (Section A)

15. Define $f(x)=\dfrac{6-x}{x+3}$ for all $x>-3$.

(a) Prove that $f(x)$ is decreasing.

(b) Find $\lim\limits_{x\to\infty} f(x)$.

(c) Find the exact value of the area of the region bounded by the graph of $y=f(x)$, the x-axis and the y-axis.

(6 marks) (2019 DSE-MATH-M1 Q5)

16. Let $f(x)$ be a continuous function such that $f'(x)=\dfrac{12x-48}{(3x^2-24x+49)^2}$ for all real numbers x.

(a) If $f(x)$ attains its minimum value at $x=\alpha$, find α.

(b) It is given that the extreme value of $f(x)$ is 5. Find

(i) $f(x)$,

(ii) $\lim\limits_{x\to\infty} f(x)$.

(6 marks) (2018 DSE-MATH-M1 Q5)

17. The value $R(t)$, in thousand dollars, of a machine can be modelled by

$$R(t)=Ae^{-0.5t}+B,$$

where $t\ (\geq 0)$ is the time, in years, since the machine has been purchased. At $t=0$, its value is 500 thousand dollars and in the long run, its value is 10 thousand dollars.

(a) Find the values of A and B.

(b) The machine can generate revenue at a rate of $P'(t)=600e^{-0.3t}$ thousand dollars per year, where t is the number since the machine has been purchased. Richard purchased the machine for his factory and used it for 5 years before he sold it. How much did he gain in this process? Correct your answer to the nearest thousand dollars.

(6 marks) (2013 ASL-M&S Q3)

18. An advertising company starts a media advertisement to recruit new members for a club. Past experience shows that the rate of change of the number of members N (in thousand) is given by

$$\frac{dN}{dt}=\frac{0.3e^{-0.2t}}{\left(1+e^{-0.2t}\right)^2},$$

where $t\ (\geq 0)$ is the number of weeks elapsed after the launch of the advertisement. The club has 500 members before the launch of the advertisement.

(a) Using the substitution $u=1+e^{-0.2t}$, express N in terms of t.

(b) Find the increase in the number of members of the club 4 weeks after the launch of the advertisement. Correct your answer to the nearest integer.

(c) Will the number of members of the club ever reach 1300 after the launch of the advertisement? Explain your answer.

(7 marks) (2012 ASL-M&S Q3)

19. A company launches a promotion plan to raise revenue. The total amount of money X (in million dollars) invested in the plan can be modelled by

$$\frac{dX}{dt}=6\left(\frac{t}{0.2t^3+1}\right)^2,\quad t\geq 0,$$

where t is the number of months elapsed since the launch of the plan.
Initially, 4 million dollars are invested in the plan.

(a) Using the substitution $u=0.2t^3+1$, or otherwise, express X in terms of t.

(b) Find the number of months elapsed since the launch of the plan if a total amount of 13 million dollars are invested in the plan.

(c) If the company has a budget of 14.5 million dollars only, can the plan be run for a long time? Explain your answer.

(7 marks) (2011 ASL-M&S Q2)

20. The rate of change of concentration of a drug in the blood of a patient can be modelled by

$$\frac{dx}{dt}=5.3\left(\frac{1}{t+2}-\frac{1}{t+5}\right)+1.2e^{-0.1t},$$

where x is the concentration measured in mg/L and t is the time measured in hours after the patient has taken the drug. It is given that $x=0$ when $t=0$.

(a) Find x in terms of t.

(b) Find the concentration of the drug after a long time.

(6 marks) (2008 ASL-M&S Q3)

21. A researcher models the rate of change of the number of fish in a lake by

$$\frac{dN}{dt}=\frac{6}{(e^{\frac{t}{4}}+e^{-\frac{t}{4}})^2},$$

where N is the number in thousands of fish in the lake recorded yearly and $t(\geq 0)$ is the time measured in years from the start of the research. It is known that $N=8$ when $t=0$.

(a) Prove that $\dfrac{dN}{dt}=\dfrac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2}$. Using the substitution $u=e^{\frac{t}{2}}+1$, or otherwise, express N in terms of t.

(b) Estimate the number of fish in the lake after a very long time.

(6 marks) (2004 ASL-M&S Q2)

22. An engineer conducts a test for a certain brand of air-purifier in a smoke-filled room. The percentage of smoke in the room being removed by the air-purifier is given by S %. The engineer models the rate of change of S by

$$\frac{dS}{dt}=\frac{8100t}{(3t+10)^3},$$

where $t\ (\geq 0)$ is measured in hours from the start of the test.

(a) Using the substitution $u = 3t + 10$, or otherwise, find the percentage of smoke removed from the room in the first 10 hours.

(b) If the air-purifier operates indefinitely, what will the percentage of smoke removed from the room be?

(5 marks) (2002 ASL-M&S Q4)

23. An adventure estimates the volume of his hot air balloon by $V(r)=\frac{4}{3}\pi r^3+5\pi$, where r is measured in metres and V is measured in cubic metres. When the balloon is being inflated, r will increase with time $t(\geq 0)$ in such a way that,

$$r(t)=\frac{18}{3+2e^{-t}}$$

where t is measured in hours.

(a) Find the rate of change of volume of the balloon at $t = 2$. Give your answer correct to 2 decimal places.

(b) If the balloon is being inflated over a long period of time, what will the volume of the balloon be? Give your answer correct to 2 decimal places.

(5 marks) (2002 ASL-M&S Q2)

Limit at infinity (Section B)

24. Let y be the amount (in suitable units) of suspended particulate in a laboratory. It is given that

$$(E):\quad y=\frac{340}{2+e^{-t}-2e^{-2t}}\qquad (t\geq 0),$$

where t is the time (in hours) which has elapsed since an experiment started.

(a) Will the value of y exceed 171 in the long run? Justify your answer.

(2 marks)

(b) Find the greatest value and least value of y.

(6 marks)

(c) (i) Rewrite (E) as a quadratic equation in e^{-t}.

(ii) It is known that the amounts of suspended particulate are the same at the time $t=\alpha$ and $t=3-\alpha$. Given that $0\leq\alpha<3-\alpha$, find α.

(4 marks)

(2014 DSE-MATH-M1 Q11)

25. A researcher models the rate of change of the population size of a kind of insects in a forest by

$$P'(t)=kte^{\frac{a}{20}t},$$

where $P(t)$, in thousands, is the population size, $t\ (\geq 0)$ is the time measured in weeks since the start of the research, and a, k are integers.

The following table shows some values of t and $P'(t)$.

t	1	2	3	4
$P'(t)$	22.83	43.43	61.97	78.60

(a) Express $\ln\frac{P'(t)}{t}$ as a linear function of t.

(1 mark)

(b) By plotting a suitable straight line on the graph paper on next page, estimate the integers a and k.

(5 marks)

(c) Suppose that $P(0)=30$. Using the estimates in (b),

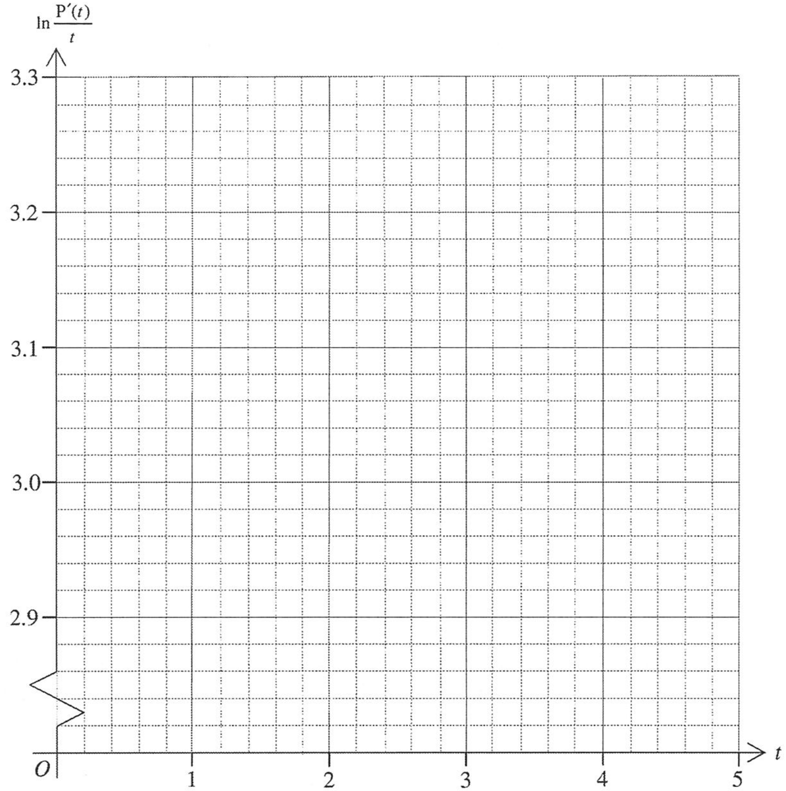
(i) find the value of t such that the rate of change of the population size of the insect is the greatest;

(ii) find $\frac{d}{dt}\left(te^{\frac{a}{20}t}\right)$ and hence, or otherwise, find $P(t)$;

(iii) estimate the population size after a very long time.

[Hint: You may use the fact that $\lim_{t\to\infty}\frac{t}{e^{mt}}=0$ for any positive constant m.]

(9 marks)



(PP DSE-MATH-M1 Q11)

26. The manager, Mary, of a theme park starts a promotion plan to increase **the daily number of visits** to the park. The rate of change of **the daily number of visits** to the park can be modelled by

$$\frac{dN}{dt}=\frac{k(25-t)}{e^{0.04t}+4t} \qquad (t\geq 0),$$

where N is **the daily number of visits** (in hundreds) recorded at the end of a day, t is the number of days elapsed since the start of the plan and k is a positive constant.

Mary finds that at the start of the plan, $N=10$ and $\frac{dN}{dt}=50$.

(a) (i) Let $v=1+4te^{-0.04t}$, find $\frac{dv}{dt}$.

(ii) Find the value of k, and hence express N in terms of t.

(7 marks)

(b) (i) When will **the daily number of visits** attain the greatest value?

(ii) Mary claims that there will be more than 50 hundred visits on a certain day after the start of the plan. Do you agree? Explain your answer.

(3 marks)

(c) Mary's supervisor believes that **the daily number of visits** to the park will return to the original one at the start of the plan after a long period of time. Do you agree? Explain your answer.

(Hint: $\lim\limits_{t\to\infty} te^{-0.04t}=0$.)

(2 marks)

(SAMPLE DSE-MATH-M1 Q11)

27. The population of a kind of bacterium $p(t)$ at time t (in days elapsed since 9 am on 16/4/2010, and can be positive or negative) is modelled by

$$p(t)=\frac{a}{b+e^{-t}}+c\ ,\quad -\infty<t<\infty$$

where a, b and c are positive constants. Define the *primordial population* be the population of the bacterium long time ago and the *ultimate population* be the population of the bacterium after a long time.

(a) Find, in terms of a, b and c,

(i) the time when the growth rate attains the maximum value;

(ii) the *primordial population*;

(iii) the *ultimate population*.

(5 marks)

(b) A scientist studies the population of the bacterium by plotting a linear graph of $\ln[p(t)-c]$ against $\ln(b+e^{-t})$ and the graph shows the intercept on the vertical axis to be $\ln 8000$. If at 9 am on 16/4/2010 the population and the growth rate of the bacterium are 6000 and 2000 per day respectively, find the values of a, b and c.

(3 marks)

(c) Another scientist claims that the population of the bacterium at the time of maximum growth rate is the mean of the primordial population and ultimate population. Do you agree? Explain your answer.

(2 marks)

(d) By expressing e^{-t} in terms of a, b, c and $p(t)$, express $p'(t)$ in the form of $\frac{-b}{a}[p(t)-\alpha][p(t)-\beta]$, where $\alpha<\beta$. Hence express α and β in terms of a, b and c. Sketch $p'(t)$ against $p(t)$ for $\alpha<p(t)<\beta$ and hence verify your answer in (c).

(5 marks)

(2010 ASL-M&S Q9)

28. A shop owner wants to launch two promotion plans A and B to raise the revenue. Let R and Q (in million dollars) be the respective cumulative weekly revenues of the shop after the launching of the promotion plans A and B. It is known that R and Q can be modelled by

$$\frac{dR}{dt}=\begin{cases}\ln(2t+1) & \text{when } 0\le t\le 6\\ 0 & \text{when } t>6\end{cases},$$

and

$$\frac{dQ}{dt}=\begin{cases}45t(1-t)+\dfrac{1.58}{t+1} & \text{when } 0\le t\le 1\\ \dfrac{30e^{-t}}{(3+2e^{-t})^2} & \text{when } t>1\end{cases}$$

respectively, where t is the number of weeks elapsed since the launching of a promotion plan.

(a) Suppose plan A is adopted.

(i) Using the trapezoidal rule with 6 sub-intervals, estimate the total amount of revenue in the first 6 weeks since the start of the plan.

(ii) Is the estimate in (a)(i) an over-estimate or under-estimate? Explain your answer briefly.

(4 marks)

(b) Suppose plan B is adopted.

(i) Find the total amount of revenue in the first week since the start of the plan.

(ii) Using the substitution $u=3+2e^{-t}$, or otherwise, find the total amount of revenue in the first n weeks, where $n>1$, since the start of the plan. Express your answer in terms of n.

(6 marks)

(c) Which of the plans will produce more revenue in the long run? Explain your answer briefly.

(5 marks)

(2009 ASL-M&S Q9)

29. A researcher studied the soot reduction effect of a petrol additive on soot emission of a car. Let t be the number of hours elapsed after the petrol additive has been used and $r(t)$, measured in ppm per hour, be the rate of change of the amount of soot reduced. The researcher suggested that $r(t)$ can be modeled by $r(t)=\alpha te^{-\beta t}$, where α and β are positive constants.

(a) Express $\ln\dfrac{r(t)}{t}$ as a linear function of t.

(1 mark)

(b) It is given that the slope and the intercept on the vertical axis of the graph of the linear function obtained in (a) are –0.50 and 2.3 respectively. Find the values of α and β correct to 1 significant figure.

Hence find the greatest rate of change of the amount of soot reduced after the petrol additive has been used. Give your answer correct to 1 significant figure.

(6 marks)

(c) Using the values of α and β obtained in (b) correct to 1 significant figure,

(i) find $\dfrac{d}{dt}\left((t+\dfrac{1}{\beta})e^{-\beta t}\right)$ and hence find, in terms of T, the total amount of soot reduced when the petrol additive has been used for T hours;

(ii) estimate the total amount of soot reduced when the petrol additive has been used for a very long time.

[Note: Candidates may use $\lim\limits_{T\to\infty}(Te^{-\beta T})=0$ without proof.]

(8 marks)

(2005 ASL-M&S Q8)

30. A researcher monitors the process of using micro-organisms to decompose food waste to fertilizer. He records daily the pH value of the waste and models its pH value by

$$P(t)=a+\frac{1}{5}\left(t^2-8t-8\right)e^{-kt},$$

where $t(\geq 0)$ is the time measured in days, a and k are positive constants.

When the decomposition process starts (i.e. $t=0$), the pH value of the waste is 5.9. Also, the researcher finds that $P(8)-P(4)=1.83$.

(a) Find the values of a and k correct to 1 decimal place.

(5 marks)

(b) Using the value of k obtained in (a),

(i) determine on which days the maximum pH value and the minimum pH value occurred respectively;

(ii) prove that $\dfrac{d^2P}{dt^2}>0$ for all $t\geq 23$.

(8 marks)

(c) Estimate the pH value of the waste after a very long time.

[Note: Candidates may use $\lim\limits_{t\to\infty}\left(t^2e^{-kt}\right)=0$ without proof.]

(2 marks)

(2003 ASL-M&S Q9)

31. The spread of an epidemic in a town can be measured by the value of PPI (the proportion of population infected). The value of PPI will increase when the epidemic breaks out and will stabilize when it dies out.

The spread of the epidemic in town A last year could be modelled by the equation

$P'(t)=\dfrac{0.04ake^{-kt}}{1-a}$, where a , $k>0$ and $P(t)$ was the PPI t days after the outbreak of the epidemic. The figure shows the graph of $\ln P'(t)$ against t, which was plotted based on some observed data obtained last year. The initial value of PPI is 0.09 (i.e. $P(0)=0.09$) .

(a) (i) Express $\ln P'(t)$ as a linear function of t and use the figure to estimate the values of a and k correct to 2 decimal places.

Hence find $P(t)$.

(ii) Let μ be the PPI 3 days after the outbreak of the epidemic. Find μ .

(iii) Find the stabilized PPI.

(8 marks)

(b) In another town B , the health department took precautions so as to reduce the PPI of the epidemic. It is predicted that the rate of spread of the epidemic will follow the equation

$Q'(t)=6(b-0.05)(3t+4)^{\frac{-3}{2}}$, where $Q(t)$ is the PPI t days after the outbreak of the epidemic in town B and b is the initial value of PPI.

(i) Suppose $b=0.09$.

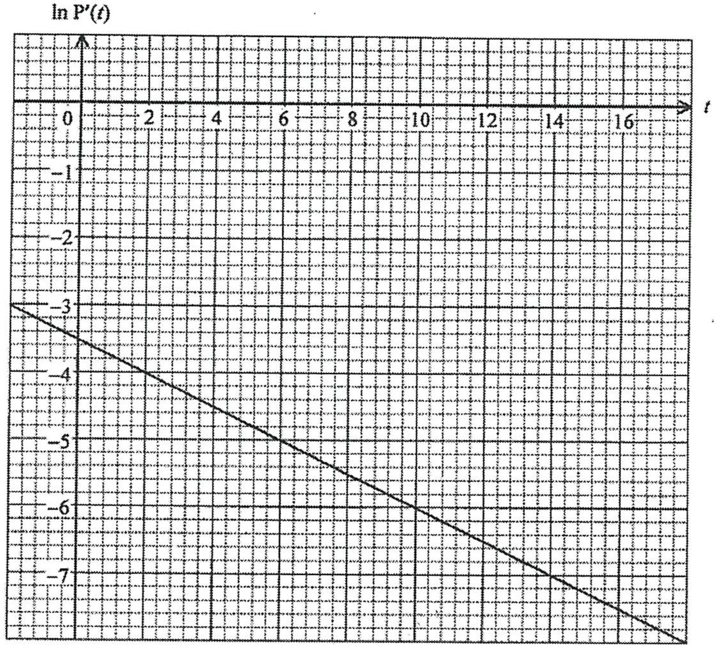
(I) Determine whether the PPI in town B will reach the value of μ in (a)(ii).

(II) How much is the stabilized PPI reduced in town B as compared with that in town A?

(ii) Find the range of possible values of b if the epidemic breaks out in town B . Explain your answer briefly.

(7 marks)

The graph of $\ln P'(t)$ against t



(2001 ASL-M&S Q9)

32. A department store has two promotion plans, F and G, designed to increase its profit, from which only one will be chosen. A marketing agent forecasts that if x hundred thousand dollars is spent on a promotion plan, the respective rates of change of its profit with respect to x can be modelled by

$$f(x) = 16 + 4xe^{-0.25x} \quad \text{and} \quad g(x) = 16 + \frac{6x}{\sqrt{1+8x}} .$$

(a) Suppose that promotion plan F is adopted.

(i) Show that $f(x) \le f(4)$ for $x > 0$.

(ii) If six hundred thousand dollars is spent on the plan, use the trapezoidal rule with 6 sub-intervals to estimate the expected increase in profit to the nearest hundred thousand dollars.

(6 marks)

(b) Suppose that promotion plan G is adopted.

(i) Show that $g(x)$ is strictly increasing for $x > 0$.
As x tends to infinity, what value would $g(x)$ tend to?

(ii) If six hundred thousand dollars is spent on the plan, use the substitution $u = \sqrt{1+8x}$, or otherwise, to find the expected increase in profit to the nearest hundred thousand dollars.

(7 marks)

(c) The manager of the department store notices that if six hundred thousand dollars is spent on promotion, plan F will result in a bigger profit than G . Determine which plan will eventually result in a bigger profit if the amount spent on promotion increases indefinitely. Explain your answer briefly.

(2 marks)

(2000 ASL-M&S Q9)

33. A researcher studied the commercial fishing situation in a certain fishing zone. Denoting the total catch of coral fish in that zone in t years time from January 1, 1992 by $N(t)$ (in thousand tonnes), he obtained the following data:

t	2	4
$N(t)$	55	98

The researcher modelled $N(t)$ by $\ln N(t) = a - e^{1-kt}$ where a and k are constants.

(a) Show that $e^{-4k} - e^{-2k} + \frac{1}{e}\ln\frac{98}{55} = 0$.

Hence find, to 2 decimal places, two sets of values of a and k.

(4 marks)

(b) The researcher later found out that $N(7) = 170$. Determine which set of values of a and k obtained in (a) will make the model fit for the known data.

Hence estimate, to the nearest thousand tonnes, the total possible catch of coral fish in that zone since January 1, 1992.

(4 marks)

(c) The rate of change of the total catch of coral fish in that zone since January 1, 1992 by at time t is given by $\frac{dN(t)}{dt}$.

(i) Show that $\frac{dN(t)}{dt} = kN(t)e^{1-kt}$.

(ii) Using the values of a and k chosen in (b), determine in which year the maximum rate of change occurred.

Hence find, to the nearest integer, the volume of fish caught in that year.

(7 marks)

(Part c is out of Syllabus) (2000 ASL-M&S Q11)

34. A vehicle tunnel company wants to raise the tunnel fees. An expert predicts that after the increase in the tunnel fees, the number of vehicles passing through the tunnel each day will drop drastically in the first week and on the t-th day after the first week, the number $N(t)$ (in thousands) of vehicles passing through the tunnel can be modelled by

$$N(t) = \frac{40}{1+be^{-rt}} \qquad (t \geq 0)$$

where b and r are positive constants.

(a) Suppose that by the end of the first week after the increase in the tunnel fees, the number of vehicles passing through the tunnel each day drops to 16 thousand and by the end of the second week, the number increases to 17.4 thousand, find b and r correct to 2 decimal places.

(5 marks)

(b) Show that $N(t)$ is increasing.

(3 marks)

(c) As time passes, $N(t)$ will approach the average number N_a of vehicles passing through the tunnel each day before the increase in the tunnel fees. Find N_a .

(2 marks)

(d) The expert suggests that the company should start to advertise on the day when the rate of increase of the number of cars passing through the tunnel per day is the greatest. Using the values of b and r obtained in (a),

(i) find $N''(t)$, and

(ii) hence determine when the company should start to advertise.

(5 marks)

(1997 ASL-M&S Q8)

Summary of Limit at Infinity

Limits	Question
$\lim_{x\to\infty}\frac{6-x}{x+3}=-1$	(2019 DSE-MATH-M1 Q5)
$\lim_{t\to\infty}\left(3+\frac{4x-9}{\sqrt{4x^2+3x+9}}\right)=5$	(2003 ASL-M&S Q3)
$\lim_{t\to\infty}\frac{-2}{3x^2-24x+49}+7=7$	(2018 DSE-MATH-M1 Q5)
$\lim_{t\to\infty}Ae^{-0.5t}+B=B$	(2013 ASL-M&S Q3)
$\lim_{t\to\infty}\left[\frac{3}{2(1+2e^{-0.2t})}-\frac{1}{4}\right]=1.25$	(2012 ASL-M&S Q3)
$\lim_{t\to\infty}\left(\frac{-10}{0.2t^3+1}+14\right)=14$	(2011 ASL-M&S Q2)
$\lim_{t\to\infty}\{5.3[\ln(t+2)-\ln(t+5)]-12^{-0.1t}+16.8563\}=16.8563$	(2008 ASL-M&S Q3)
$\lim_{t\to\infty}\left(14-\frac{12}{e^{\frac{t}{2}}+1}\right)=14$	(2004 ASL-M&S Q2)
$\lim_{T\to\infty}\left(\frac{-1}{3T+10}+\frac{5}{(3T+10)^2}+0.05\right)=0.05$	(2002 ASL-M&S Q4)
$\lim_{t\to\infty}\left(\frac{18}{3+2e^{-t}}\right)=6$	(2002 ASL-M&S Q2)
$\lim_{t\to\infty}\left(\frac{340}{2+e^{-t}-2e^{-2t}}\right)=170$	(2014 DSE-MATH-M1 Q11)
$\lim_{t\to\infty}\left(9630-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}\right)=9630$	(PP DSE-MATH-M1 Q11)
$\lim_{t\to\infty}[12.5\ln(1+4te^{-0.04t})+10]=10$	(SAMPLE DSE-MATH-M1 Q11)
$\lim_{t\to-\infty}\left(\frac{a}{b+e^{-t}}+c\right)=c$	(2010 ASL-M&S Q9)
$\lim_{t\to\infty}\left(\frac{a}{b+e^{-t}}+c\right)=\frac{a}{b}+c$	
$\lim_{n\to\infty}\left(\frac{15}{2}+1.58\ln 2-\frac{15}{3+2e^{-1}}+\frac{15}{3+2e^{-n}}\right)\approx 9.5799$	(2009 ASL-M&S Q9)
$\lim_{T\to\infty}[40-20(T+2)e^{-0.5T}]=40$	(2005 ASL-M&S Q8)

$\lim_{t\to\infty}\left(7.5+\frac{1}{5}(t^2-8t-8)e^{-0.2t}\right)=7.5$	(2003 ASL-M&S Q9)

$\lim_{t\to\infty}\left(-0.12e^{-0.25t}+0.21\right)=0.21$	(2001 ASL-M&S Q9)
$\lim_{t\to\infty}\left(-0.16(3t+4)^{\frac{-1}{2}}+0.17\right)=0.17$	
$\lim_{x\to\infty}\left(16+\frac{6x}{\sqrt{1+8x}}\right)=\infty$	(2000 ASL-M&S Q9)
$\ln N(t)=5.89-e^{1-0.18t}$ $\lim_{t\to\infty}N(t)=\lim_{t\to\infty}\left(e^{5.89-e^{1-0.18t}}\right)=e^{5.89}\approx 361$	(2000 ASL-M&S Q11)
$\lim_{t\to\infty}\left(\frac{40}{1+be^{-rt}}\right)=40$	(1997 ASL-M&S Q8)

Out of syllabus

35. Let $y=\sqrt[3]{\frac{3x-1}{x-2}}$, where $x>2$.

(a) Use logarithmic differentiation to express $\frac{1}{y}\cdot\frac{dy}{dx}$ in terms of x .

(b) Using the result of (a), find $\frac{d^2y}{dx^2}$ when $x=3$.

(Part a is out of Syllabus) (6 marks) (2012 DSE-MATH-M1 Q4)

36. Let $u=\sqrt{\frac{2x+3}{(x+1)(x+2)}}$, where $x>-1$.

(a) Use logarithmic differentiation to express $\frac{du}{dx}$ in terms of u and x.

(b) Suppose $u=3^y$, express $\frac{dy}{dx}$ in terms of x.

(Part a is out of Syllabus) (5 marks) (SAMPLE DSE-MATH-M1 Q6)

37. It is given that $\begin{cases} x=\ln(2t+4) \\ y=e^{t^2+4t+4} \end{cases}$, where $t>-2$.

(a) Express $\frac{dy}{dx}$ in terms of y only.

(b) Find the value of $\frac{d^2y}{dx^2}$ when $x=0$.

(Out of Syllabus) (6 marks) (2012 ASL-M&S Q2)

38. Let C be the curve $x=y^4-y$.

(a) Find $\frac{dy}{dx}$.

(b) Find the equation of the tangent to C if the slope of the tangent is $\frac{1}{3}$.

(Out of Syllabus) (7 marks) (2009 ASL-M&S Q3)

39. Suppose $y^3-uy=1$ and $u=2^{x^2}$.

(a) Find $\frac{dy}{du}$ in terms of u and y .

(b) Find $\frac{du}{dx}$ in terms of x .

(c) Find $\frac{dy}{dx}$ in terms of x and y .

(Out of Syllabus) (7 marks) (2008 ASL-M&S Q2)

40. Let $w=\sqrt{\frac{(x-1)^3}{(x+2)(2x+1)}}$, where $x>1$.

(a) Express $\ln w$ in the form $a\ln(x-1)+b\ln(x+2)+c\ln(2x+1)$ where a , b and c are constants.

Hence find $\frac{dw}{dx}$.

(b) Suppose $w=2^y$.

Express $\frac{dy}{dw}$ in terms of w .

Hence express $\frac{dy}{dx}$ in terms of x .

(Out of Syllabus) (7 marks) (2005 ASL-M&S Q3)

41. Let $x=-\frac{5}{t^2}+2e^{-3t}$ and $y=\frac{10}{t^2}+e^{2t}$ $(t\neq 0)$. If $\frac{dy}{dx}=-2$, find the value of t .

(Out of Syllabus) (4 marks) (2002 ASL-M&S Q1)

42. Let $\ln(xy)=\frac{x}{y}$ where x , $y>0$. Show that $\frac{dy}{dx}=\frac{xy-y^2}{xy+x^2}$.

(Out of Syllabus) (4 marks) (2000 ASL-M&S Q1)

43. It is given that $e^{xy}=\frac{x(x+1)^3}{x^2+1}$, where $x>0$.

(a) Find the value of y when $x=1$.

(b) Find the value of $\frac{dy}{dx}$ when $x=1$.

(Out of Syllabus) (5 marks) (1999 ASL-M&S Q1)

44. (a) If $e^x+e^y=xy$, find $\frac{dy}{dx}$.

(b) If $y=\frac{1}{x+1}\sqrt{\frac{(x-2)(x+3)}{x+1}}$ where $x>2$, use logarithmic differentiation to find $\frac{dy}{dx}$.

(Out of Syllabus) (6 marks) (1995 ASL-M&S Q2)

2021 DSE Q5

Let $f(x)=e^{-x^{\frac{1}{3}}}$.

(a) Let $g(u)=e^{-u}(u^2+2u+2)$, where $u=x^{\frac{1}{3}}$. Find the constant β such that $\frac{dg(u)}{dx}=\beta f(x)$.

(b) Express, in terms of e , the area of the region bounded by the curve $y=f(x)$, the x-axis, the y-axis and the straight line $x=8$.

(6 marks)

2021 DSE Q7

Let $y=\frac{e^x}{x^3-x+2}$, where $0\leq x\leq 5$. Find

(a) $\frac{dy}{dx}$,

(b) the greatest value and the least value of y .

3. Derivative and Differentiation of Functions

Section A

1. (2017 DSE-MATH-M1 Q7)

(a) $y=\dfrac{x}{\sqrt{x-2}}$		
$\dfrac{dy}{dx}=\dfrac{\sqrt{x-2}-x\left(\dfrac{1}{2}\right)(x-2)^{\frac{-1}{2}}}{x-2}$	1M	for quotient rule
$\dfrac{dy}{dx}=\dfrac{x-4}{2(x-2)^{\frac{3}{2}}}$	1A	
(b) Let (h,k) be the coordinates of the point of contact. So, the slope of this tangent is $\dfrac{h-4}{2(h-2)^{\frac{3}{2}}}$.		
$\dfrac{k-0}{h-9}=\dfrac{h-4}{2(h-2)^{\frac{3}{2}}}$	1M+1A	1M for using (a)
$\dfrac{h}{\sqrt{h-2}}2(h-2)^{\frac{3}{2}}=(h-4)(h-9)$		
$h^2+9h-36=0$	1M	
$h=3$ or $h=-12$ (rejected)		
The slope of this tangent $=\dfrac{3-4}{2(3-2)^{\frac{3}{2}}}$	1M	
$=\dfrac{-1}{2}$	1A	
	----------(7)	

(a)	Good. Many candidates were able to find $\dfrac{dy}{dx}$ but some candidates did not simplify the answer.
(b)	Fair. Many candidates wrongly thought that $(9,0)$ was the point of contact.

2. (2016 DSE-MATH-M1 Q7)

(a) $\dfrac{dy}{dx}$		
$=\left(\dfrac{3}{2}\right)(2x+8)^{\frac{1}{2}}(2)+6x$	1M	for chain rule
$=3\sqrt{2x+8}+6x$	1A	
(b) Note that the slope of the straight line $6x+y+4=0$ is -6 . So, the slope of the tangent is -6 .		
$3\sqrt{2x+8}+6x=-6$	1M+1A	1M for using (a)
$\sqrt{2x+8}=-2(x+1)$		
$2x+8=4(x+1)^2$		
$2x^2+3x-2=0$	1M	for $ax^2+bx+c=0$
$x=-2$ or $x=\dfrac{1}{2}$ (rejected)	1A	for ' $x=-2$ or $x=\dfrac{1}{2}$ '
Hence, there is only one tangent to C parallel to the straight line $6x+y+4=0$. Thus, the claim is disagreed.	1A	f.t.
	----------(7)	

(a)	Very good. Nearly all of the candidates were able to apply chain rule to find $\dfrac{dy}{dx}=3\sqrt{2x+8}+6x$.
(b)	Good. Some candidates were unable to solve the equation involving radical $3\sqrt{2x+8}+6x=-6$, and many candidates were unable to reject the inappropriate root $x=\dfrac{1}{2}$.

3. (2015 DSE-MATH-M1 Q7)

	Marks	Remarks
(a) $y = x\sqrt{2x^2+1}$		
$\frac{dy}{dx} = \sqrt{2x^2+1} + x\left(\frac{1}{2}\right)(2x^2+1)^{\frac{-1}{2}}(4x)$	1M	for chain rule
$\frac{dy}{dx} = \frac{4x^2+1}{\sqrt{2x^2+1}}$	1A	
(b) Note that the slope of the straight line is $\frac{-3}{17}$.		
So, the slope of each tangent is $\frac{17}{3}$.		
$\frac{4x^2+1}{\sqrt{2x^2+1}} = \frac{17}{3}$	1M+1A	1M for using (a)
$3(4x^2+1) = 17\sqrt{2x^2+1}$		
$9(4x^2+1)^2 = 289(2x^2+1)$		
$72x^4 - 253x^2 - 140 = 0$	1M	for $ax^4+bx^2+c=0$
$x = 2$ or $x = -2$		
For $x=2$, we have $y=6$.		
The equation of the tangent to C at the point $(2, 6)$ is		
$y-6 = \frac{17}{3}(x-2)$	1M	either one
$17x - 3y - 16 = 0$		
For $x=-2$, we have $y=-6$.		
The equation of the tangent to C at the point $(-2, -6)$ is		
$y+6 = \frac{17}{3}(x+2)$		
$17x - 3y + 16 = 0$	1A	for both
	(7)	

(a)	Very good. Most candidates were able to apply chain rule to find $\frac{dy}{dx}$.
(b)	Good. Some candidates made careless mistakes in simplifying the equation involving radical, and some candidates failed to write a quadratic equation in x^2.

4. (2013 DSE-MATH-M1 Q3a,b)

	Marks	Remarks
(a) $y = x(x-2)^{\frac{1}{3}}$		
$\frac{dy}{dx} = (x-2)^{\frac{1}{3}} + \frac{1}{3}(x-2)^{\frac{-2}{3}}x$	1M	For product rule
When $x=3$, $\frac{dy}{dx} = 2$.		
Hence the equation of L is $y = 2x$.	1A	
(b) Solving C and L:		
$x(x-2)^{\frac{1}{3}} = 2x$	1M	
$x\left[(x-2)^{\frac{1}{3}} - 2\right] = 0$		
$x = 0$ or 10	1A	

(a)	Good. Some candidates found the equation of the tangent to C at $x=3$ instead of the equation of L.
(b)	Good. Some candidates did not know how to solve equations with fraction exponents or missed out the root $x=0$ by dividing both sides of an equation by x.

5. (PP DSE-MATH-M1 Q2)

	Marks	Remarks
(a) $t = y^3 + 2y^{\frac{-1}{2}} + 1$		
$\frac{dt}{dy} = 3y^2 - y^{\frac{-3}{2}}$	1A	
(b) $e^t = x^{x^2+1}$		
$t = (x^2+1)\ln x$	1A	
$\frac{dt}{dx} = \frac{x^2+1}{x} + 2x\ln x$	1A	
(c) $\frac{dy}{dx} = \frac{dt}{dx} \div \frac{dt}{dy}$	1M	
$= \frac{(x^2+1+2x^2\ln x)y^{\frac{3}{2}}}{x\left(3y^{\frac{7}{2}}-1\right)}$	1A	OR $\frac{\frac{x^2+1}{x}+2x\ln x}{3y^2-y^{\frac{-3}{2}}}$
	(5)	

(a)	甚佳。少數學生誤以爲 $\frac{dt}{dy} = \frac{dy}{dt}$。
(b)	平平。部分學生未能正確運算 $\frac{d}{dx}[(x^2+1)\ln x]$。
(c)	甚佳。少數學生未能正確應用鏈式法則。

6. (PP DSE-MATH-M1 Q4)

Solution	Marks	Remarks
(a) $y=x(2x-1)^{\frac{1}{2}}$		
$\frac{dy}{dx}=(2x-1)^{\frac{1}{2}}+x\cdot\frac{1}{2}(2x-1)^{\frac{-1}{2}}(2)$	1M	For product rule
$=\frac{3x-1}{(2x-1)^{\frac{1}{2}}}$	1A	
(b) For tangents parallel to $2x-y=0$, we need $\frac{dy}{dx}=2$.		
$\frac{3x-1}{(2x-1)^{\frac{1}{2}}}=2$	1M	
$9x^2-6x+1=4(2x-1)$		
$9x^2-14x+5=0$		
$x=1$ or $\frac{5}{9}$	1A	
For $x=1$, $y=1$ and hence the equation of the tangent is		
$y-1=2(x-1)$		
$2x-y-1=0$	1A	
For $x=\frac{5}{9}$, $y=\frac{5}{27}$ and hence the equation of the tangent is		
$y-\frac{5}{27}=2\left(x-\frac{5}{9}\right)$		
$54x-27y-25=0$	1A	
	(6)	

(a)	甚佳。 大數學生明白函數的微分法。
(b)	平平。 部分學生未能求出切線的方程。

7. (2013 ASL-M&S Q2)

Solution	Marks	Remarks
(a) $x=\ln\frac{1+t}{1-t}$		
$\frac{dx}{dt}=\frac{1-t}{1+t}\cdot\frac{(1-t)(1)-(1+t)(-1)}{(1-t)^2}$	1M	
Alternative Solution $x=\ln(1+t)-\ln(1-t)$; $\frac{dx}{dt}=\frac{1}{1+t}+\frac{1}{1-t}$	1M	
$=\frac{2}{1-t^2}$	1A	
(b) (i) $y=1+e^{-x}-e^{-2x}$		
$\frac{dy}{dx}=-e^{-x}+2e^{-2x}$	1A	
(ii) $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$		
$=(-e^{-x}+2e^{-2x})\left(\frac{2}{1-t^2}\right)$	1M	
When $t=\frac{1}{2}$, $x=\ln 3$.	1A	
$\frac{dy}{dt}=(-e^{-\ln 3}+2e^{-2\ln 3})\left[\frac{2}{1-\left(\frac{1}{2}\right)^2}\right]$		
$=\frac{-8}{27}$	1A	OR -0.2963
	(6)	

Good.
Some candidates were not able to apply the quotient rule.

8. (2007 ASL-M&S Q3)

Solution	Marks
(a) $y=\frac{1-e^{4x}}{1+e^{8x}}$	
$\frac{dy}{dx}=\frac{(1+e^{8x})(-4e^{4x})-(1-e^{4x})(8e^{8x})}{(1+e^{8x})^2}$	1M for quotient rule or product rule
When $x=0$, we have $\frac{dy}{dx}=-2$.	1A
(b) (i) Since $(z^2+1)e^{3z}=e^{\alpha+\beta x}$, we have $\ln(z^2+1)+3z=\alpha+\beta x$.	1A
(ii) Since the graph of the linear function passes through the origin and the slope of the graph is 2, we have $\alpha=0$ and $\beta=2$.	1A for both correct
(iii) $\ln(z^2+1)+3z=2x$	
$\frac{2z}{z^2+1}+3=2\frac{dx}{dz}$	
Therefore, we have $\left.\frac{dx}{dz}\right\|_{z=0}=\frac{3}{2}$.	1A
Note that $x=0$ when $z=0$.	
Also note that $\left.\frac{dy}{dx}\right\|_{x=0}=-2$.	
$\left.\frac{dy}{dz}\right\|_{z=0}$	
$=\left(\left.\frac{dy}{dx}\right\|_{x=0}\right)\left(\left.\frac{dx}{dz}\right\|_{z=0}\right)$	
$=(-2)\left(\frac{3}{2}\right)$	1M for chain rule
$=-3$	1A
$y=\frac{1-e^{6z+2\ln(z^2+1)}}{1+e^{12z+4\ln(z^2+1)}}$	1A
$y=\frac{1-(z^2+1)^2e^{6z}}{1+(z^2+1)^4e^{12z}}$	
$\frac{dy}{dz}=\frac{\left(1+(z^2+1)^4e^{12z}\right)\left(-6(z^2+1)^2e^{6z}-2(z^2+1)(2z)e^{6z}\right)-\left(1-(z^2+1)^2e^{6z}\right)\left(12(z^2+1)^4e^{12z}+4(z^2+1)^3(2z)e^{12z}\right)}{\left(1+(z^2+1)^4e^{12z}\right)^2}$	1M for quotient rule or product rule
$\left.\frac{dy}{dz}\right\|_{z=0}=-3$	1A
	---------(7)

Good. Most candidates could handle quotient rule and product rule. It is more efficient to apply $\frac{dy}{dz}=\frac{dy}{dx}\cdot\frac{dx}{dz}$ but some candidates went through the tedious way by expressing the function of y in terms of z.

9. (2003 ASL-M&S Q3)

(a) $\dfrac{dy}{dx}=\dfrac{4(4x^2+3x+9)^{\frac{1}{2}}-\frac{1}{2}(4x^2+3x+9)^{\frac{-1}{2}}(8x+3)(4x-9)}{4x^2+3x+9}$	1M for quotient rule + 1M for chain rule + 1A
$=\dfrac{8(4x^2+3x+9)-(8x+3)(4x-9)}{2(4x^2+3x+9)^{\frac{3}{2}}}$	
$=\dfrac{84x+99}{2(4x^2+3x+9)^{\frac{3}{2}}}$	
When $x=10$, $\dfrac{dy}{dx}=\dfrac{84(10)+99}{2(4(10)^2+3(10)+9)^{\frac{3}{2}}}$	
≈ 0.051043308	
≈ 0.0510	1A a–1 for r.t. 0.051
(b) The required amount	
$=\lim\limits_{x\to\infty}\left(3+\dfrac{4x-9}{\sqrt{4x^2+3x+9}}\right)$	
$=3+\lim\limits_{x\to\infty}\dfrac{4-\frac{9}{x}}{\sqrt{4+\frac{3}{x}+\frac{9}{x^2}}}$	1M can be absorbed
$=3+2$	
$=5$ grams	1A
	----------(6)

Fair. Handling quotient rule and chain rule together needs more practice. Some candidates were not familiar with the techniques of taking limits.

10. (2001 ASL-M&S Q2)

(a) Since $u=e^{2x}$, $\therefore \dfrac{du}{dx}=2e^{2x}$.	1A
$\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=\left(\dfrac{1}{u}-2u\right)\cdot 2u=2-4u^2=2-4e^{4x}$	1M+1A 1M for $\left(\dfrac{1}{u}-2u\right)\cdot 2u$
$\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=\left(\dfrac{1}{u}-2u\right)\cdot 2e^{2x}=\left(\dfrac{1}{e^{2x}}-2e^{2x}\right)\cdot 2e^{2x}=2-4e^{4x}$	$\boxed{1M}+\boxed{1A}$
$y=\ln u-u^2+c$	
$y=\ln e^{2x}-(e^{2x})^2+c$	$\boxed{1M}$
$y=2x-e^{4x}+c$	
$\dfrac{dy}{dx}=2-4e^{4x}$	$\boxed{1A}$ f.t.
(b) Using (a), $y=\int(2-4e^{4x})dx$	1M
$=2x-e^{4x}+c$ for some constant c.	
Putting $x=0$ and $y=1$, we have $c=2$.	
$\therefore y=2x-e^{4x}+2$	1A
	----------(5)

11. (1998 ASL-M&S Q1)

$y=xe^{\frac{1}{x}}$		
$\dfrac{dy}{dx}=e^{\frac{1}{x}}+xe^{\frac{1}{x}}(-\dfrac{1}{x^2})$	1M+1M	1M for product rule 1M for chain rule
$=e^{\frac{1}{x}}-\dfrac{1}{x}e^{\frac{1}{x}}$	1A	
$\dfrac{d^2y}{dx^2}=-\dfrac{1}{x^2}e^{\frac{1}{x}}+\dfrac{1}{x^2}e^{\frac{1}{x}}+\dfrac{1}{x^3}e^{\frac{1}{x}}$	1A	
$=\dfrac{1}{x^3}e^{\frac{1}{x}}$		
$\therefore x^4\dfrac{d^2y}{dx^2}-y=x^4\left(\dfrac{1}{x^3}e^{\frac{1}{x}}\right)-xe^{\frac{1}{x}}$		
$=0$	1	accept $\dfrac{d^2y}{dx^2}=\dfrac{y}{x^4}$
	(5)	

12. (1994 ASL-M&S Q2)

$\dfrac{dx}{dt}=100ke^{kt}$	1A
$\dfrac{d^2x}{dt^2}=100k^2e^{kt}$	1A
Hence $100k^2e^{kt}-200ke^{kt}-300e^{kt}=0$	1M
$k^2-2k-3=0$	1A
$(k-3)(k+1)=0$	
$k=-1$ or 3	
$\because k$ is negative	
$\therefore k=-1$	1A
	5

13. (2002 ASL-M&S Q1)

Solution	Marks
$\frac{dx}{dt}=\frac{10}{t^3}-6e^{-3t}$ $\frac{dy}{dt}=-\frac{20}{t^3}+2e^{2t}$	1M+1A (1M for $(e^{at})'=ae^{at}$)
$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\left(\frac{dy}{dt}\right)\left(\frac{1}{\frac{dx}{dt}}\right)=\frac{-\frac{20}{t^3}+2e^{2t}}{\frac{10}{t^3}-6e^{-3t}}$	1M for Chain Rule and Inverse Function Rule
For $\frac{dy}{dx}=-2$ $\frac{-\frac{20}{t^3}+2e^{2t}}{\frac{10}{t^3}-6e^{-3t}}=-2$	1M
$e^{5t}=6$ $t=\frac{1}{5}\ln 6(\approx 0.3584)$	1A a–1 for r.t. 0.358
	---------(5)

14. (2012 ASL-M&S Q2)

Solution	Marks	Remarks
(a) $y=e^{t^2+4t+4}$ and $x=\ln(2t+4)$		
$\frac{dy}{dt}=e^{t^2+4t+4}(2t+4)$ and $\frac{dx}{dt}=\frac{1}{t+2}$	1A	For both
$\frac{dy}{dx}=\frac{dy}{dt}+\frac{dx}{dt}$ $=2e^{t^2+4t+4}(t+2)\cdot(t+2)$	1M	
<u>Alternative Solution</u> $\ln y=(t+2)^2$ and $x=\ln 2+\ln(t+2)$		OR $\cdots$ and $t+2=\frac{1}{2}e^{x}$
$\therefore\ x=\ln 2+\frac{1}{2}\ln(\ln y)$	1A	OR $\ln y=\frac{1}{4}e^{2x}$
$\frac{dx}{dy}=\frac{1}{2}\cdot\frac{1}{\ln y}\cdot\frac{1}{y}$	1M	OR $\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{4}e^{2x}\cdot 2$
$\frac{dy}{dx}=2y\ln y$	1A	
(b) $\frac{d^2y}{dx^2}=\left(2y\cdot\frac{1}{y}+2\ln y\right)\frac{dy}{dx}$ $=4y\ln y(1+\ln y)$	1M	For chain rule
When $x=0$, $t=\frac{-3}{2}$ and so $y=e^{\frac{1}{4}}$.	1A	
$\therefore\ \frac{d^2y}{dx^2}=4e^{\frac{1}{4}}\left(\frac{1}{4}\right)\left(1+\frac{1}{4}\right)$ $=\frac{5}{4}e^{\frac{1}{4}}$	1A	OR 1.6050
	(6)	

Limit at infinity (Section A)

15. (2019 DSE-MATH-M1 Q5)

Solution	Marks
(a) For all $x>-3$, $f'(x)$ $=\frac{(x+3)(-1)-(6-x)(1)}{(x+3)^2}$ $=\frac{-9}{(x+3)^2}$ <0 Thus, $f(x)$ is decreasing.	1
Note that $f(x)=\frac{9}{x+3}-1$ for all $x>-3$. Thus, $f(x)$ is decreasing.	1
(b) $\lim_{x\to\infty} f(x)$ $=\lim_{x\to\infty}\frac{\frac{6}{x}-1}{1+\frac{3}{x}}$ $=-1$	1A
$\lim_{x\to\infty} f(x)$ $=\lim_{x\to\infty}\left(\frac{9}{x+3}-1\right)$ $=-1$	1A
(c) For $y=0$, we have $x=6$. The required area $=\int_0^6 f(x)\,dx$	1M
$=\int_0^6 \frac{6-x}{x+3}\,dx$ $=\int_0^6\left(\frac{9}{x+3}-1\right)dx$	1M
$=[9\ln(x+3)-x]_0^6$	1M
$=9\ln 3-6$	1A
For $y=0$, we have $x=6$. The required area $=\int_0^6 f(x)\,dx$	1M
$=\int_0^6 \frac{6-x}{x+3}\,dx$ $=\int_3^9 \frac{6-(u-3)}{u}\,du$ (by letting $u=x+3$)	1M
$=\int_3^9\left(\frac{9}{u}-1\right)du$ $=[9\ln u-u]_3^9$	1M
$=9\ln 3-6$	1A
	---------(6)

(a)	Good. Some candidates were unable to show that $f'(x) < 0$ to complete the proof.
(b)	Good. Some candidates were unable to consider $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{6}{x}-1}{1+\frac{3}{x}}$ to obtain the required limit.
(c)	Good. Many candidates were able to use integration to obtain the required area, but some candidates were unable to give the answer in exact value.

16. (2018 DSE-MATH-M1 Q5)

Note that $3x^2-24x+49 = 3(x-4)^2+1 \neq 0$.

(a) $f'(x)=0$

$\frac{12x-48}{(3x^2-24x+49)^2}=0$

$x=4$ 1M

x	$(-\infty, 4)$	4	$(4, \infty)$
$f'(x)$	$-$	0	$+$

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha = 4$. 1A

> $f'(x)=0$
>
> $\frac{12x-48}{(3x^2-24x+49)^2}=0$
>
> $x=4$ 1M
>
> $f''(x)$
>
> $=\frac{-108x^2+864x-1716}{(3x^2-24x+49)^3}$
>
> $f''(4)$
>
> $=12$
>
> >0
>
> So, $f(x)$ attains its minimum value at $x=4$.
>
> Thus, we have $\alpha = 4$. 1A

(b) (i) Let $v = 3x^2-24x+49$. Then, we have $\frac{dv}{dx}=6x-24$.

$f(x)$

$=\int \frac{12x-48}{(3x^2-24x+49)^2}dx$

$=\int \frac{2}{v^2}dv$ 1M

$=\frac{-2}{v}+C$

$=\frac{-2}{3x^2-24x+49}+C$

Since $f(x)$ has only one extreme value, we have $f(4)=5$. 1M

$\frac{-2}{3(4)^2-24(4)+49}+C=5$

$C=7$

Thus, we have $f(x)=\frac{-2}{3x^2-24x+49}+7$. 1A

(ii) $\lim_{x\to\infty} f(x)$

$=7$ 1A

---------(6)

(a)	Very good. Over 85% of the candidates were able to find the value of α .
(b) (i)	Good. Many candidates were able to find $f(x)$ by indefinite integral but some candidates were unable to use a suitable substitution.
(ii)	Fair. Only some candidates were able to find the constant of integration in (b)(i), and thus the required limit.

17. (2013 ASL-M&S Q3)

(a) $R(t)=Ae^{-0.5t}+B$

$R(t)\to 10$ when $t\to\infty$ 1M

$\therefore\ B=10$ 1A

$R(0)=500$

$500=A+B$

$\therefore\ A=490$ 1A

(b) $\int_0^5 P'(t)\,dt+R(5)-R(0)$ 1M

$=\int_0^5 600e^{-0.3t}dt+\left[490e^{-0.5(5)}+10\right]-500$

$=[-2000e^{-0.3t}]_0^5+490e^{-2.5}-490$ 1A For $[-2000e^{-0.3t}]_0^5$

$=-2000e^{-1.5}+490e^{-2.5}+1510$

≈ 1104 1A

Hence Richard gains 1104 thousand dollars in the process.

(6)

Good.
In (b), some candidates did not consider the depreciation of the value of the machine in five years.

18. (2012 ASL-M&S Q3)

(a) Let $u=1+e^{-0.2t}$.

$du=-0.2e^{-0.2t}dt$ 1A

$N=\int\frac{0.3e^{-0.2t}}{(1+e^{-0.2t})^2}dt$

$N=\int\frac{0.3}{u^2}\cdot\frac{du}{-0.2}$

$=\frac{3}{2u}+C$ 1A

$=\frac{3}{2(1+e^{-0.2t})}+C$

When $t=0$, $N=0.5$.

$\therefore\ C=\frac{-1}{4}$

i.e. $N=\frac{3}{2(1+e^{-0.2t})}-\frac{1}{4}$ 1A

(b) $N(4)-N(0)$

$=\frac{3}{2(1+e^{-0.2\times 4})}-\frac{1}{4}-0.5$ 1M

≈ 0.284961721

Hence the increase in the number of people is 285. 1A

(c) $\frac{dN}{dt}=\frac{0.3e^{-0.2t}}{(1+e^{-0.2t})^2}>0$ for all $t\ge 0$

Hence N is always increasing.

} Withhold the last mark if this argument is missing

$\lim_{t\to\infty}N=\lim_{t\to\infty}\left[\frac{3}{2(1+e^{-0.2t})}-\frac{1}{4}\right]$

OR by arguing that $e^{-0.2t}>0\Rightarrow N<\frac{3}{2}-\frac{1}{4}$

$=1.25$ 1A

Hence the number of members will never reach 1300. 1

OR by arguing that $\frac{3}{2(1+e^{-0.2t})}-\frac{1}{4}=1.3$ has no real solution

(7)

Satisfactory.
Many candidates overlooked the units and did not use 0.5 to represent 500 since N was given in thousand. A number of candidates could not well explain their answer in (c) because they did not state clearly that N was an increasing function.

19. (2011 ASL-M&S Q2)

(a) $\dfrac{dX}{dt}=6\left(\dfrac{t}{0.2t^3+1}\right)^2$

$X=6\displaystyle\int\frac{t^2}{(0.2t^3+1)^2}\,dt$

Let $u=0.2t^3+1$, and therefore $du=0.6t^2\ dt$. — 1M — For $u=0.2t^3+1$ or $u=(0.2t^3+1)^2$

$\therefore\ X=6\displaystyle\int\frac{1}{0.6u^2}\,du$ — 1M

$=\dfrac{-10}{u}+C$ — 1A

$=\dfrac{-10}{0.2t^3+1}+C$

When $t=0$, $X=4$ and hence $C=14$.

i.e. $X=\dfrac{-10}{0.2t^3+1}+14$ — 1A

(b) $13=\dfrac{-10}{0.2t^3+1}+14$

$t=\sqrt[3]{45}$ months — 1A — OR 3.5569 months

(c) $X=14-\dfrac{10}{0.2t^3+1}<14$ for any value of t. — 1M

Hence the plan can be run for a long time. — 1A

(7)

Good.
In part (c), although most candidates found the limit of X when $t\to\infty$, the proof was incomplete without showing that the function was increasing.

20. (2008 ASL-M&S Q3)

(a) $\dfrac{dx}{dt}=5.3\left(\dfrac{1}{t+2}-\dfrac{1}{t+5}\right)+1.2e^{-0.1t}$

$x=\displaystyle\int\left[5.3\left(\frac{1}{t+2}-\frac{1}{t+5}\right)+1.2e^{-0.1t}\right]dt$ — 1M

$=5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+C$ (since $t\geq 0$) — 1A — OR $5.3[\ln|t+2|-\ln|t+5|]-12e^{-0.1t}+C$

When $t=0$, $x=0$.

$\therefore 0=5.3(\ln 2-\ln 5)-12+C$ — 1M

$C=5.3(\ln 5-\ln 2)+12$

≈ 16.8563

i.e. $x=5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+16.8563$ — 1A — OR $x=\cdots+5.3\ln 2.5+12$

(b) $\displaystyle\lim_{t\to\infty}\{5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+16.8563\}$ — 1M

$=5.3\displaystyle\lim_{t\to\infty}\ln\frac{t+2}{t+5}-12\lim_{t\to\infty}e^{-0.1t}+16.8563$

$=16.8563$ — 1A

i.e. the concentration of the drug after a long time = 16.8563 mg/L

(6)

Good. Some candidates could not present the mathematical notation of the limit of x properly.

21. (2004 ASL-M&S Q2)

(a) $\dfrac{dN}{dt}$

$=\dfrac{6}{(e^{\frac{t}{4}}+e^{\frac{-t}{4}})^2}$

$=\dfrac{6}{(e^{\frac{-t}{4}}(e^{\frac{t}{2}}+1))^2}$

$=\dfrac{6}{e^{\frac{-t}{2}}(e^{\frac{t}{2}}+1)^2}$

$=\dfrac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2}$ — 1 must show steps

Let $u=e^{\frac{t}{2}}+1$.

Then, we have $\dfrac{du}{dt}=\dfrac{1}{2}e^{\frac{t}{2}}$.

Also, $dt=\dfrac{2du}{u-1}$. Now,

N

$=\displaystyle\int\frac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2}\,dt$

$=\displaystyle\int\frac{12(u-1)}{u^2(u-1)}\,du$

$=\displaystyle\int\frac{12}{u^2}\,du$ — 1A accept $\dfrac{dN}{du}=\dfrac{12}{u^2}$.

So, we have $N=\dfrac{-12}{u}+C$ [redacted] — 1A

Now, $N=\dfrac{-12}{e^{\frac{t}{2}}+1}+C$.

Using the condition that $N=8$ when $t=0$, we have $8=-6+C$. — 1M for finding C

Hence, $C=14$.

Thus, $N=14-\dfrac{12}{e^{\frac{t}{2}}+1}$. — 1A

(b) The required number of fish

$=\displaystyle\lim_{t\to\infty}\left(14-\frac{12}{e^{\frac{t}{2}}+1}\right)$

$=14-\displaystyle\lim_{t\to\infty}\frac{12}{e^{\frac{t}{2}}+1}$

$=14$ thousands — 1A

(6)

Fair. Many candidates were not able to relate mathematical presentations to integrations and taking limits.

22. (2002 ASL-M&S Q4)

(a) $S=\int_0^{10}\frac{8100t}{(3t+10)^3}\,dt$. — 1A

Let $u=3t+10$.
$du=3dt$.
When $t=0$, $u=10$.
When $t=10$, $u=40$.

$S=\int_{10}^{40}\frac{8100\left(\frac{u-10}{3}\right)\cdot\frac{1}{3}du}{u^3}$

$=900\int_{10}^{40}\left(\frac{1}{u^2}-\frac{10}{u^3}\right)du$ — 1M change of variable

$=900\left[-\frac{1}{u}+\frac{5}{u^2}\right]_{10}^{40}$

$=\frac{405}{16}=25.3125$ — 1A

The percentage of smoke removed is 25.3125%.

$S=\int_0^{10}\frac{8100t}{(3t+10)^3}dt$ — 1A

$=900\int_0^{10}\left[\frac{1}{(3t+10)^2}-\frac{10}{(3t+10)^3}\right]d(3t+10)$ — 1M change of variable

$=900\left[-\frac{1}{3t+10}+\frac{5}{(3t+10)^2}\right]_0^{10}$

$=25.3125$ — 1A

$S=\int\frac{8100t}{(3t+10)^3}dt=900\left[-\frac{1}{3t+10}+\frac{5}{(3t+10)^2}\right]+C$ — 1M+1M for change of variable

When $t=0$, $S=0$. Hence, we have $C=45$.

So, $S=900\left[-\frac{1}{3T+10}+\frac{5}{(3T+10)^2}\right]+45$.

When $t=10$, $S=25.3125$. — 1A

(b) $S=\int_0^{T}\frac{8100t}{(3t+10)^3}dt$

$=900\left[-\frac{1}{u}+\frac{5}{u^2}\right]_{10}^{3T+10}$

$=900\left[-\frac{1}{3T+10}+\frac{5}{(3T+10)^2}+0.05\right]$

$\lim_{T\to\infty}S=\lim_{T\to\infty}900\left[-\frac{1}{3T+10}+\frac{5}{(3T+10)^2}+0.05\right]$ — 1M taking limit and in terms of T

$=45$ — 1A

$\therefore$ 45% of smoke will be removed.

---------(5)

23. (2002 ASL-M&S Q2)

(a) At $t=2$, $r(2)=5.5035$(m)

$\frac{dV}{dr}=4\pi r^2$

$\frac{dr}{dt}=\frac{18\times 2e^{-t}}{(3+2e^{-t})^2}=\frac{36e^{-t}}{(3+2e^{-t})^2}$ — 1A

At $t=2$, $\frac{dV}{dr}=380.6109$

$\frac{dr}{dt}=0.45545$

$\therefore\ \frac{dV}{dt}=\frac{dV}{dr}\cdot\frac{dr}{dt}$

At $t=2$, $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ — 1M

$=380.6109\times 0.45545$

$=173.35$ (m³/h) — 1A (Accept: 173.31–173.39) a–1 for more than 2 d.p.

$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ — 1M

$\frac{dr}{dt}=\frac{36e^{-t}}{(3+2e^{-t})^2}$ — 1A

$\frac{dV}{dt}=4\pi r^2\cdot\frac{36e^{-t}}{(3+2e^{-t})^2}$

$=\frac{144\pi r^2e^{-t}}{(3+2e^{-t})^2}$ — (Accept: $\frac{dV}{dt}=\frac{46656\pi e^{-t}}{(3+2e^{-t})^4}$)

At $t=2$,
$r\approx 5.50346$

$\therefore\ \frac{dV}{dt}\approx 173.35$ (m^3/h) — 1A (Accept: 173.31–173.39) a–1 for more than 2 d.p.

(b) $\lim_{t\to\infty}r(t)=\lim_{t\to\infty}\frac{18}{3+2e^{-t}}=6$ (m) — 1M

$\therefore$ the volume of the balloon will be

$V=\frac{4}{3}\pi(6)^3+5\pi$

$=293\pi$

$=920.49$ (m³) — 1A a–1 for more than 2 d.p.

---------(5)

Limit at infinity (Section B)

24. (2014 DSE-MATH-M1 Q11)

(a) $\lim_{t\to\infty}\frac{340}{2+e^{-t}-2e^{-2t}}=\frac{340}{2+0-2\cdot 0}$ — 1M
$=170$
Hence the value of y will not exceed 171 in the long run. — 1A
(2)

(b) $\frac{dy}{dt}=340[-(2+e^{-t}-2e^{-2t})^{-2}](-e^{-t}+4e^{-2t})$ — 1A
$=\frac{340(e^{-t}-4e^{-2t})}{(2+e^{-t}-2e^{-2t})^2}$
$\therefore\ \frac{dy}{dt}=0$ when $e^{-t}-4e^{-2t}=0$ — 1M — For $\frac{dy}{dt}=0$
i.e. $t=\ln 4$ — 1A

t	$0\le t<\ln 4$	$t=\ln 4$	$t>\ln 4$
$\frac{dy}{dt}$	-ve	0	+ve

— 1M

Hence y is minimum when $t=\ln 4$.
The minimum value of $y=\frac{340}{2+e^{-\ln 4}-2e^{-2\ln 4}}$
$=160$ — 1A

When $t=0$, $y=\frac{340}{2+e^0-2e^0}$
$=340$

As the graph of y is continuous, and by (a), the greatest value of y is 340 and the least value of y is 160. — 1A
(6)

(c) (i) $y=\frac{340}{2+e^{-t}-2e^{-2t}}$
$2y+ye^{-t}-2ye^{-2t}=340$
$2y(e^{-t})^2-ye^{-t}+340-2y=0$ — 1A

(ii) Since $e^{-\alpha}$ and $e^{\alpha-3}$ are roots of the equation in (i),
$\frac{340-2y}{2y}=e^{-\alpha}e^{\alpha-3}$ — 1M
$340-2y=2ye^{-3}$
Hence the equation becomes $2y(e^{-t})^2-ye^{-t}+2ye^{-3}=0$
i.e. $2(e^{-t})^2-e^{-t}+2e^{-3}=0$ — 1A
$\therefore\ e^{-\alpha}=\frac{1+\sqrt{1-16e^{-3}}}{4}$ or $\frac{1-\sqrt{1-16e^{-3}}}{4}$ (rejected as $e^{-\alpha}$ is the greater root)
i.e. $\alpha=-\ln\frac{1+\sqrt{1-16e^{-3}}}{4}$ — 1A — OR $\ln\frac{1-\sqrt{1-16e^{-3}}}{4}+3$ OR 1.0140
(4)

(a)	Good. Some candidates thought that $\lim_{t\to+\infty}e^{-t}=1$.
(b)	Fair. Quite a lot of candidates failed to consider both the value of y at $t=0$ and the limit found in (a).
(c)	Very poor. Most candidates wrote wrongly the equation required in (i).

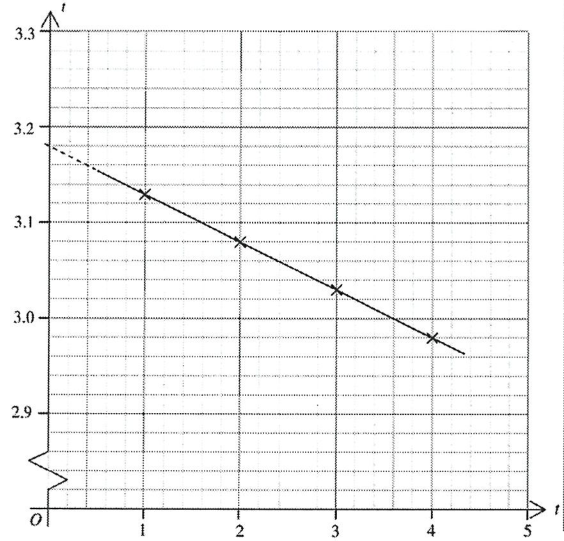
25. (PP DSE-MATH-M1 Q11)

(a) $P'(t)=kte^{\frac{a}{20}t}$
$\ln\frac{P'(t)}{t}=\frac{a}{20}t+\ln k$ — 1A
(1)

(b)

t	1	2	3	4
$P'(t)$	22.83	43.43	61.97	78.60
$\ln\frac{P'(t)}{t}$	3.13	3.08	3.03	2.98

— 1A



— 1A

From the graph, $\frac{a}{20}\approx\frac{2.98-3.13}{4-1}$ — 1M
$a\approx -1$ — 1A
From the graph, $\ln k\approx 3.18$ — 1A
$k\approx 24$
Either one
(5)

(c) (i) $\frac{d}{dt}P'(t)=\frac{d}{dt}\left(24te^{\frac{-t}{20}}\right)$

$=24e^{\frac{-t}{20}}\left(1-\frac{t}{20}\right)$ 1A

$\therefore\ \frac{d}{dt}P'(t)=0$ when $t=20$

t	<20	20	>20
$\frac{d}{dt}P'(t)$	+ve	0	-ve

1M

Alternative Solution

$\frac{d^2}{dt^2}P'(t)=24e^{\frac{-t}{20}}\left[\frac{-1}{20}\left(1-\frac{t}{20}\right)+\frac{-1}{20}\right]$

$=\frac{6}{5}e^{\frac{-t}{20}}\left(\frac{t}{20}-2\right)$

$\therefore\ \frac{d^2}{dt^2}P'(t)<0$ when $t=20$ 1M

Hence the rate of change of the population size is greatest when $t=20$. 1A

(ii) $\frac{d}{dt}\left(te^{\frac{-t}{20}}\right)=e^{\frac{-t}{20}}-\frac{1}{20}te^{\frac{-t}{20}}$ 1A

$24te^{\frac{-t}{20}}=480e^{\frac{-t}{20}}-480\frac{d}{dt}\left(te^{\frac{-t}{20}}\right)$

$\int 24te^{\frac{-t}{20}}\,dt=-9600e^{\frac{-t}{20}}-480te^{\frac{-t}{20}}+C$ 1M

$P(t)=C-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}$ 1A

Since $P(0)=30$, we have

$C-480(0)e^{0}-9600e^{0}=30$ 1M

$C=9630$

$\therefore\ P(t)=9630-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}$ 1A

(iii) $\lim_{t\to\infty}P(t)=\lim_{t\to\infty}\left(9630-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}\right)$

$=9630$ 1A

$\therefore$ the population size after a very long time is estimated to be 9630 thousands.

(9)

(a)	良好。 大部分學生能正確表達線性函數。
(b)	良好。 部分學生看漏了精確度的要求。
(c) (i)	平平。 部分學生在取得第一導數並使該導數相等於零之後，沒有作出極大測試。
(ii)	平平。 大部分學生不懂如何利用 $\frac{d}{dt}\left(te^{\frac{u}{20}t}\right)$ 的結果求 $P(t)$。
(iii)	甚差。 大部分學生因爲在前部分出錯而得出錯誤的結論。

26. (SAMPLE DSE-MATH-M1 Q11)

(a) (i) Let $v=1+4te^{-0.04t}$. Then we have

$\frac{dv}{dt}=4e^{-0.04t}-0.16te^{-0.04t}$ 1M+1A 1M for product rule

$=0.16e^{-0.04t}(25-t)$

(ii) When $t=0,\frac{dN}{dt}=50$. So we have $25k=50$.

Thus, we have $k=2$. 1A

$N=\int\frac{2(25-t)}{e^{0.04t}+4t}\,dt$

$=2\int\frac{e^{-0.04t}(25-t)}{1+4te^{-0.04t}}\,dt$ 1M

$=\frac{2}{0.16}\int\frac{dv}{v}$ 1M For using (a)(i)

$=12.5\ln|v|+C$

$=12.5\ln(1+4te^{-0.04t})+C$

When $t=0$, $N=10$. So, we have $C=10$. 1M For finding C

i.e. $N=12.5\ln(1+4te^{-0.04t})+10$ 1A

(7)

(b) (i) $\dfrac{dN}{dt}=\dfrac{2(25-t)}{e^{0.04t}+4t}$

$\begin{cases} >0 & \text{when } 0\le t<25 \\ =0 & \text{when } t=25 \\ <0 & \text{when } t>25 \end{cases}$ — 1M

So, N attains its greatest value when $t=25$. — 1A

Alternative Solution

$\dfrac{dN}{dt}=\dfrac{2(25-t)}{e^{0.04t}+4t}$

$\dfrac{dN}{dt}=0$ when $t=25$

$\dfrac{d^2N}{dt^2}=2\left[\dfrac{(e^{0.04t}+4t)(-1)-(0.04e^{0.04t}+4)(25-t)}{(e^{0.04t}+4t)^2}\right]$

$=-4\left[\dfrac{(1-0.02t)e^{0.04t}+50}{(e^{0.04t}+4t)^2}\right]$ — 1M

$\left.\dfrac{d^2N}{dt^2}\right|_{t=25}=-4\left[\dfrac{0.5e+50}{(e+100)^2}\right]<0$

So, N attains its greatest value when $t=25$. — 1A

(ii) $N(25)=12.5\ln(1+4te^{-0.04t})+10\approx 55.4>50$

Thus, the claim is agreed. — 1

(3)

(c) $\lim\limits_{t\to\infty}N=\lim\limits_{t\to\infty}[12.5\ln(1+4te^{-0.04t})+10]$

$=12.5\ln(1+0)+10$ — 1M — For using $\lim\limits_{t\to\infty}te^{-0.04t}=0$

$=10$

Thus, the belief of Mary's supervisor is agreed. — 1

(2)

27. (2010 ASL-M&S Q9)

(a) (i) $p(t)=\dfrac{a}{b+e^{-t}}+c$

$p'(t)=\dfrac{ae^{-t}}{(b+e^{-t})^2}$ — 1A

$p''(t)=\dfrac{(b+e^{-t})^2(-ae^{-t})-(ae^{-t})2(b+e^{-t})(-e^{-t})}{(b+e^{-t})^4}$

$=\dfrac{ae^{-t}(e^{-t}-b)}{(b+e^{-t})^3}$ — 1A

Hence $p''(t)=0$ when $e^{-t}-b=0$.

i.e. $t=-\ln b$

t	$t<-\ln b$	$t=-\ln b$	$t>-\ln b$
$p''(t)$	+	0	−

Hence the growth rate attains the maximum value when $t=-\ln b$ — 1A — Follow through

(ii) *primordial population* $=\lim\limits_{t\to-\infty}\left(\dfrac{a}{b+e^{-t}}+c\right)=c$ — 1A

(iii) *ultimate population* $=\lim\limits_{t\to\infty}\left(\dfrac{a}{b+e^{-t}}+c\right)=\dfrac{a}{b}+c$ — 1A

(5)

(b) $\ln[p(t)-c]=-\ln(b+e^{-t})+\ln a$

$\therefore\ \ln a=\ln 8000$

$a=8000$ — 1A

$\therefore\ p'(0)=\dfrac{8000}{(b+1)^2}=2000$

$b=1$ or -3 (rejected) — 1A

$\therefore\ p(0)=\dfrac{8000}{1+1}+c=6000$

$c=2000$ — 1A

(3)

(c) The population at the time of maximum growth rate is

$p(-\ln b)=\dfrac{a}{2b}+c$ — 1A

The mean of the *primordial population* and *ultimate population* is

$\dfrac{1}{2}\left[c+\left(\dfrac{a}{b}+c\right)\right]=\dfrac{a}{2b}+c$

Hence the scientist's claim is agreed. — 1

(2)

(d) $p(t)=\dfrac{a}{b+e^{-t}}+c$

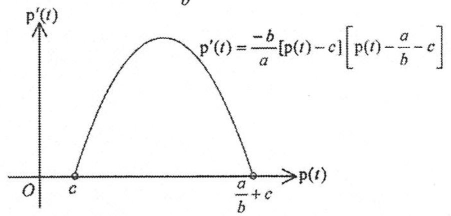
$e^{-t}=\dfrac{a}{p(t)-c}-b$ — 1A

$\therefore\ p'(t)=\dfrac{a\left[\dfrac{a}{p(t)-c}-b\right]}{\left[b+\left(\dfrac{a}{p(t)-c}-b\right)\right]^2}$ — 1M

$=\dfrac{a[p(t)-c]\{a-b[p(t)-c]\}}{a^2}$

$=\dfrac{-b}{a}[p(t)-c]\left[p(t)-\dfrac{a}{b}-c\right]$

Hence $\alpha=c$ and $\beta=\dfrac{a}{b}+c$. — 1A



1A

From the graph, we can see that $p'(t)$ is maximum when $p(t)$ is the mean of c and $\dfrac{a}{b}+c$, i.e. the mean of the *primordial population* and *ultimate population*. — 1 — Follow through

(5)

(a) (i)		Fair. Many candidates confused the maximum growth rate and the maximum population and hence could not determine the time required.
(ii) (iii)		Fair. Some candidates mistook $p(0)$ to be the primordial population.
(b)		Fair.
(c)		Poor. Most candidates did not understand the question.
(d)		Very poor. Most candidates could not go beyond expressing e^{-t} in terms of a, b, c and $p(t)$.

28. (2009 ASL-M&S Q9)

(a) (i) $R_6=\int_0^6 \ln(2t+1)\,dt$

$\approx\frac{1}{2}\{\ln(2\cdot0+1)+2[\ln(2\cdot1+1)+\ln(2\cdot2+1)+\ln(2\cdot3+1)+\ln(2\cdot4+1)$
$+\ln(2\cdot5+1)]+\ln(2\cdot6+1)\}$ — 1M

$=10.53155488$

The total amount of revenue in the first 6 weeks is 10.5316 million dollars. — 1A

(ii) Let $f(t)=\ln(2t+1)$

$f'(t)=\dfrac{2}{2t+1}$

$f''(t)=\dfrac{-4}{(2t+1)^2}$ — 1A

<0 for $0\le t\le 6$

$\therefore$ $f(t)$ is concave downward for $0\le t\le 6$.

Hence the estimate in (a)(i) is an under-estimate. — 1A — Follow through

(4)

(b) (i) $Q_1=\int_0^1\left[45t(1-t)+\dfrac{1.58}{t+1}\right]dt$ — 1A

$=\left[45\left(\dfrac{t^2}{2}-\dfrac{t^3}{3}\right)+1.58\ln|t+1|\right]_0^1$

$=\dfrac{15}{2}+1.58\ln 2$

≈ 8.595172545

The total amount of revenue in the first week is 8.5952 million dollars. — 1A

(ii) $Q_n=Q_1+\int_1^n\dfrac{30e^{-t}}{(3+2e^{-t})^2}dt$ — 1M

Let $u=3+2e^{-t}$

$du=-2e^{-t}dt$

$\therefore\ Q_n=Q_1+\int_{3+2e^{-1}}^{3+2e^{-n}}\dfrac{-15}{u^2}du$ — 1M — For $\dfrac{-15}{u^2}$

$=Q_1+\left[\dfrac{15}{u}\right]_{3+2e^{-1}}^{3+2e^{-n}}$ — 1A — For $\left[\dfrac{15}{u}\right]_{3+2e^{-1}}^{3+2e^{-n}}$

$=\dfrac{15}{2}+1.58\ln 2+\dfrac{15}{3+2e^{-n}}-\dfrac{15}{3+2e^{-1}}$

Hence the total amount of revenue in the first n weeks is

$\left(\dfrac{15}{2}+1.58\ln 2+\dfrac{15}{3+2e^{-n}}-\dfrac{15}{3+2e^{-1}}\right)$ million dollars, where $n>1$. — 1A — Accept $4.5799+\dfrac{15}{3+2e^{-n}}$

(6)

(c) For $n>6$, $R_n=R_6+\int_6^n 0\,dt\approx 10.5316$ (by (a)(i)) — 1M — For $\int_6^n 0\,dt=0$

When $n\to\infty$, $e^{-n}\to 0$ and so $Q_n\to 4.5799+\dfrac{15}{3+0}=9.5799$ — 1M — For $e^{-n}\to 0$

Therefore, over a long period of time, plan A produces approximately 10.5316 million dollars and plan B produces 9.5799 million dollars of revenue. — 1A

Moreover, the revenue of plan A is even an under-estimate. — 1M

Hence, plan A will produce more revenue over a long period of time. — 1A — Follow through

(5)

(a) (i)		Good.
(ii)		Poor. The poor performance was rather unexpected since applying the concept of concave and convex curves should be quite standard.
(b) (i)		Good.
(ii)		Poor. The problem might look unfamiliar. Many candidates did not realize that the lower and upper limits of the integral should be 1 and n, and Q_1 should be added to the integral to get Q_n.
(c)		Very poor. Most candidates got the wrong conclusion due to mistakes made in the previous parts.

29. (2005 ASL-M&S Q8)

(a) $r(t)=\alpha t e^{-\beta t}$	
$\frac{r(t)}{t}=\alpha e^{-\beta t}$	
$\ln\frac{r(t)}{t}=\ln\alpha-\beta t$	1A
	---------(1)
(b) $\because\ \ln\alpha=2.3$	
$\therefore\ \alpha\approx 10$ (correct to 1 significant figure)	1A
Also, we have $\beta\approx 0.5$ (correct to 1 significant figure).	1A
$r(t)=10te^{-0.5t}$	
$\frac{dr(t)}{dt}=10t(-0.5e^{-0.5t})+10e^{-0.5t}$	
$=10e^{-0.5t}-5te^{-0.5t}$	1A
$=(10-5t)e^{-0.5t}$	
$\frac{dr(t)}{dt}\begin{cases}>0 & \text{if } 0\le t<2\\ =0 & \text{if } t=2\\ <0 & \text{if } t>2\end{cases}$	1M for testing+1A
So, $r(t)$ attains its greatest value when $t=2$.	
Hence, greatest value of $r(t)$ is $(10)(2)e^{-0.5(2)}\approx 7.357588823$.	
Thus, the greatest rate of change is 7 ppm per hour.	1A

$\frac{dr(t)}{dt}=10t(-0.5e^{-0.5t})+10e^{-0.5t}$	
$=10e^{-0.5t}-5te^{-0.5t}$	1A
$=(10-5t)e^{-0.5t}$	
$\frac{d^2r(t)}{dt^2}=-5e^{-0.5t}+(-5+2.5t)e^{-0.5t}=(2.5t-10)e^{-0.5t}$	
$\frac{dr(t)}{dt}=0$ when $t=2$ only and $\left.\frac{d^2r(t)}{dt^2}\right\|_{t=2}=-5e^{-1}<0$	1M for testing+1A
So, $r(t)$ attains its greatest value when $t=2$.	
Hence, greatest value of $r(t)$ is $(10)(2)e^{-0.5(2)}\approx 7.357588823$.	
Thus, the greatest rate of change is 7 ppm per hour.	1A

---------(6)

(c) (i) $\frac{d}{dt}\left((t+\frac{1}{\beta})e^{-\beta t}\right)$	
$=\frac{d}{dt}\left((t+2)e^{-0.5t}\right)$	
$=e^{-0.5t}-0.5(t+2)e^{-0.5t}$	1M for product rule or chain rule
$=-0.5te^{-0.5t}$	1M accept $-\beta te^{-\beta t}$

The required amount	
$=\int_0^T r(t)dt$	
$=\int_0^T 10te^{-0.5t}\,dt$	1M
$=\left[-20(t+2)e^{-0.5t}\right]_0^T$	1M+1A
$=\left(40-20(T+2)e^{-0.5T}\right)$ ppm	1A

Note that	
$\int r(t)dt$	
$=\int 10te^{-0.5t}dt$	
$=-20(t+2)e^{-0.5t}+C$	1M+1A
Let $A(t)$ ppm be the amount of soot reduced when the petrol additive has been used for t hours.	
Then, we have $A(t)=-20(t+2)e^{-0.5t}+C$.	1M
Since $A(0)=0$, we have $C=40$.	
So, we have $A(t)=\left(40-20(t+2)e^{-0.5t}\right)$.	
Note that $A(0)=0$.	
Thus, the required amount $=A(T)=\left(40-20(T+2)e^{-0.5T}\right)$ ppm	1A

Note that	
$\int r(t)dt$	
$=\int 10te^{-0.5t}dt$	
$=-20(t+2)e^{-0.5t}+C$	1M+1A
Let $A(t)$ ppm be the amount of soot reduced when the petrol additive has been used for t hours.	
Then, we have $A(t)=-20(t+2)e^{-0.5t}+C$.	1M
The required amount	
$=A(T)-A(0)$	
$=\left(-20(T+2)e^{-0.5T}+C\right)-\left(-40+C\right)$	
$=\left(40-20(T+2)e^{-0.5T}\right)$ ppm	1A

(ii) The required amount	
$=\lim_{T\to\infty}\left(40-20(T+2)e^{-0.5T}\right)$	
$=40-20\lim_{T\to\infty}Te^{-0.5T}-40\lim_{T\to\infty}e^{-0.5T}$	
$=40-20(0)-40(0)$	1M for $\lim_{T\to\infty}e^{-0.5T}=0$ and can be absorbed
$=40$ ppm	1A
	---------(8)

(a)		Very Good.
(b)		Good. Some candidates were not able to show that the stationary point is a maximum point.
(c) (i)		Fair. The first part was done well but the later part was less satisfactory. Only some candidates were able to work out the total amount of soot reduced.
(ii)		Poor. Many candidates were not able to complete this part because they failed to solve (c)(i).

30. (2003 ASL-M&S Q9)

(a) $\because$ $P(0) = 5.9$

$\therefore$ $a + \frac{1}{5}(0 - 0 - 8) = 5.9$

So, $a = 7.5$ 1A

$P(t) = 7.5 + \frac{1}{5}(t^2 - 8t - 8)e^{-kt}$

$\because$ $P(8) - P(4) = 1.83$

$\therefore$ $-1.6e^{-8k} + 4.8e^{-4k} = 1.83$ 1M+1A

$160(e^{-4k})^2 - 480e^{-4k} + 183 = 0$ 1M can be absorbed

$e^{-4k} \approx 2.551784198$ or $e^{-4k} \approx 0.448215801$

$k \approx -0.2341982$ or $k \approx 0.200620116$

$\because$ $k > 0$

$\therefore$ $k \approx 0.2$ (correct to 1 decimal place) 1A

---------(5)

(b) $P(t) = \frac{15}{2} + \frac{1}{5}(t^2 - 8t - 8)e^{-0.2t}$

(i) $\frac{dP(t)}{dt} = \frac{-1}{25}\left[(t^2 - 8t - 8) - 5(2t - 8)\right]e^{-0.2t}$ 1M for Product Rule or Chain Rule

$= \frac{-1}{25}(t^2 - 18t + 32)e^{-0.2t}$ 1A independent of the obtained value of a

$= \frac{-1}{25}(t - 2)(t - 16)e^{-0.2t}$

For $\frac{dP(t)}{dt} = 0$, we have $t = 2$ or $t = 16$.

$$\frac{dP(t)}{dt}\begin{cases} < 0 & \text{if } 0 \le t < 2 \\ = 0 & \text{if } t = 2 \\ > 0 & \text{if } 2 < t < 16 \end{cases}$$

So, the minimum pH value occurred at $t = 2$. 1M+1A

$$\frac{dP(t)}{dt}\begin{cases} > 0 & \text{if } 2 < t < 16 \\ = 0 & \text{if } t = 16 \\ < 0 & \text{if } t > 16 \end{cases}$$

So, the maximum pH value occurred at $t = 16$. 1M+1A accept max at $t = 0$ and at $t = 16$

$\frac{dP(t)}{dt} = \frac{-1}{25}\left[(t^2 - 8t - 8) - 5(2t - 8)\right]e^{-0.2t}$ 1M for Product Rule or Chain Rule

$= \frac{-1}{25}(t^2 - 18t + 32)e^{-0.2t}$ 1A independent of the obtained value of a

$= \frac{-1}{25}(t - 2)(t - 16)e^{-0.2t}$

For $\frac{dP(t)}{dt} = 0$, we have $t = 2$ or $t = 16$.

$\frac{d^2P(t)}{dt^2} = \frac{1}{125}[t^2 - 18t + 32 - 5(2t - 18)]e^{-0.2t}$

$= \frac{1}{125}(t^2 - 28t + 122)e^{-0.2t}$

$\left.\frac{d^2P(t)}{dt^2}\right|_{t=2} \approx 0.375379225 > 0$

So, the minimum pH value occurred at $t = 2$. 1M+1A

$\left.\frac{d^2P(t)}{dt^2}\right|_{t=16} \approx -0.022826834 < 0$

So, the maximum pH value occurred at $t = 16$. 1M+1A accept max at $t = 0$ and at $t = 16$

(ii) $\frac{d^2P}{dt^2} = \frac{1}{125}[t^2 - 18t + 32 - 5(2t - 18)]e^{-0.2t}$

$= \frac{1}{125}(t^2 - 28t + 122)e^{-0.2t}$ 1A

$\because$ $\frac{d^2P}{dt^2} = \frac{1}{125}\left(t - (14 - \sqrt{74})\right)\left(t - (14 + \sqrt{74})\right)e^{-0.2t}$

$5 < 14 - \sqrt{74} < 6$ and $22 < 14 + \sqrt{74} < 23$

$\therefore$ $\frac{d^2P}{dt^2} > 0$ for all $t \ge 23$. 1

---------(8)

(c) The required pH value

$= \lim_{t\to\infty}\left(7.5 + \frac{1}{5}(t^2 - 8t - 8)e^{-0.2t}\right)$

$= 7.5 + \frac{1}{5}\lim_{t\to\infty}\left(t^2e^{-0.2t}\right) - \frac{8}{5}\lim_{t\to\infty}\left(te^{-0.2t}\right) - \frac{8}{5}\lim_{t\to\infty}e^{-0.2t}$

$= 7.5 + \frac{1}{5}(0) - \frac{8}{5}(0) - \frac{8}{5}(0)$ $\left(\because \lim_{t\to\infty}(te^{-0.2t}) = \left(\lim_{t\to\infty}\frac{1}{t}\right)\left(\lim_{t\to\infty}t^2e^{-0.2t}\right) = (0)(0) = 0\right)$ 1A for $\lim_{t\to\infty}(te^{-0.2t}) = 0$ (can be absorbed)

$= 7.5$ 1M accept the required pH value $= a$

---------(2)

(a)		Good. Some candidates were unable to transform the equation '$-1.6e^{-8k} + 4.8e^{-4k} = 1.83$' into a quadratic equation.
(b)		Good. Most candidates were able to differentiate functions involving 'exp' function.
(c)		Satisfactory. Some candidates had difficulty in finding the limit.

31. (2001 ASL-M&S Q9)

Solution	Marks
(a) (i) $\ln P'(t) = -kt + \ln\dfrac{0.04ak}{1-a}$	1A
From the graph, $-k \approx \dfrac{-8-(-3.5)}{18-0}$, $k \approx 0.25$	1A a–1 for more than 2 d.p.
$\ln\dfrac{0.04ak}{1-a} \approx -3.5$, $a \approx 0.7512 \approx 0.75$	1A a–1 for more than 2 d.p.
$P'(t) \approx 0.03e^{-0.25t}$	
$P(t) \approx -0.12e^{-0.25t} + c$ for some constant c	1M
Since $P(0) = 0.09$, $\therefore$ $c \approx 0.21$	1A
Hence $P(t) \approx -0.12e^{-0.25t} + 0.21$	
(ii) $\mu = P(3) \approx 0.1533$	1A $\mu \in [0.1530, 0.1533]$
(iii) Stabilized PPI in town $A = \lim\limits_{t\to\infty} P(t) = 0.21$	1M+1A
	----------(8)
(b) (i) Suppose $b = 0.09$.	
(I) $Q'(t) = 0.24(3t+4)^{-\frac{3}{2}}$	
$Q(t) = \dfrac{1}{3}(0.24)(-2)(3t+4)^{-\frac{1}{2}} + c$ for some constant c	1A
$= -0.16(3t+4)^{-\frac{1}{2}} + c$	
Since $Q(0) = 0.09$, $\therefore$ $c = 0.17$	1A
If $Q(t) = \mu \approx 0.1533$	
$-0.16(3t+4)^{-\frac{1}{2}} + 0.17 \approx 0.1533$	1M
$(3t+4)^{\frac{1}{2}} \approx \dfrac{0.16}{0.0167}$	
Since $3t+4>0$	
$\therefore$ $t \approx 29.3$	$t \in [28.2, 29.3]$
i.e. the PPI will reach the value of μ .	1
Since $Q(0) = 0.09$, $\lim\limits_{t\to\infty} Q(t) = 0.17$ and Q is continuous and strictly increasing ($Q'(t) > 0$) ,	$\boxed{1M}$
$\therefore$ Q can reach any value between 0.09 and 0.17 including $\mu \approx 0.1533$.	$\boxed{1}$
(II) Stabilized PPI in town $B = \lim\limits_{t\to\infty} Q(t) = 0.17$	
$\therefore$ The stabilized PPI will be reduced by 0.04 .	1A
(ii) $0.05 < b$ $\boxed{(\le 1)}$.	1A
Otherwise, $Q'(t) \le 0$ and the PPI will not increase. It follows that the epidemic will not break out.	1
	----------(7)

32. (2000 ASL-M&S Q9)

Solution	Marks
(a) (i) $f(x) = 16 + 4xe^{-0.25x}$	
$f'(x) = 4e^{-0.25x}(1-0.25x)$	1M attempting to find f' 1A
$\begin{cases} >0 & \text{if } 0<x<4 \\ =0 & \text{if } x=4 \\ <0 & \text{if } x>4 \end{cases}$	accept considering $f''(x) = e^{-0.25x}(0.25x-2)$
$\therefore$ $f(x) \le f(4)$ for $x>0$.	1 follow through
(ii) see table below	1A correct to 1 d.p.

x	0	1	2	3	4	5	6
f(x)	16 (16)	19.1152 (19.1)	20.8522 (20.9)	21.6684 (21.7)	21.8861 (21.9)	21.7301 (21.7)	21.3551 (21.4)

Solution	Marks
$\int_0^6 f(x)\,dx$	
$\approx \dfrac{1}{2}[16 + 21.3551 + 2(19.1152 + 20.8522 + 21.6684 + 21.8861 + 21.7301)]$	1M
≈ 124	
$\therefore$ The expected increase in profit is 124 hundred thousand dollars.	1A a–1 for r.t. 124 pp–1 for wrong/missing unit
(b) (i) $g(x) = 16 + \dfrac{6x}{\sqrt{1+8x}}$	
$g'(x) = \dfrac{6\sqrt{1+8x} - \dfrac{6x\cdot 8}{2\sqrt{1+8x}}}{1+8x}$	1A
$= \dfrac{6(1+4x)}{(1+8x)^{\frac{3}{2}}}$	
>0 for $x>0$.	1
$\therefore$ $g(x)$ is strictly increasing for $x>0$.	
$\because$ $\lim\limits_{x\to\infty}\left(16 + \dfrac{6x}{\sqrt{1+8x}}\right) = \lim\limits_{x\to\infty}\left(16 + \dfrac{6\sqrt{x}}{\sqrt{\dfrac{1}{x}+8}}\right)$	
$\therefore$ $g(x) \to \infty$ as $x \to \infty$	1A

(ii) Let $u=\sqrt{1+8x}$, then $u^2=1+8x$, $2u\mathrm{d}u=8\mathrm{d}x$

$\int_0^6 g(x)\mathrm{d}x=\int_0^6\left(16+\frac{6x}{\sqrt{1+8x}}\right)\mathrm{d}x$ (or $\int_0^6 16\mathrm{d}x+\int_0^6\frac{6x}{\sqrt{1+8x}}\mathrm{d}x$)

$=\int_1^7\left(16+\frac{6(u^2-1)}{8u}\right)\frac{1}{4}u\mathrm{d}u$ (or $\left[16x\right]_0^6+\int_1^7\frac{6(u^2-1)}{8u}\frac{1}{4}u\mathrm{d}u$) — 1A integrand; 1A limits

$=\int_1^7\left(\frac{3}{16}u^2+4u-\frac{3}{16}\right)\mathrm{d}u$ (or $96+\int_1^7\left(\frac{3}{16}u^2-\frac{3}{16}\right)\mathrm{d}u$)

$=\left[\frac{1}{16}u^3+2u^2-\frac{3}{16}u\right]_1^7$ (or $96+\left[\frac{1}{16}u^3-\frac{3}{16}u\right]_1^7$) — 1A ignore limits

$=116\frac{1}{4}$

≈ 116

$\therefore$ The expected increase in profit is 116 hundred thousand dollars. — 1A $a-1$ for r.t. 116; $pp-1$ for wrong/missing unit

(c) From (a)(i), $f(x)\le f(4)$ (≈ 21.8861) for $x>0$.
i.e. $f(x)$ is bounded above by $f(4)$.
From (b)(i), $g(x)$ increases to infinity as x increases to infinity. — 1M

$\because$ $f(x)>0$ and $g(x)>0$ for $x>0$,
the area under the graph of $g(x)$ will be greater than that of $f(x)$ as x increases indefinitely.

$\therefore$ Plan G will eventually result in a bigger profit. — 1A

33. (2000 ASL-M&S Q11)

(a) $\begin{cases}\ln 55=a-e^{1-2k}\\ \ln 98=a-e^{1-4k}\end{cases}$

Eliminating a , we have

$e^{1-4k}-e^{1-2k}+\ln 98-\ln 55=0$

$e^{-4k}-e^{-2k}+\frac{1}{e}\ln\frac{98}{55}=0$ — 1

$(e^{-2k})^2-e^{-2k}+\frac{1}{e}\ln\frac{98}{55}=0$ — 1M quadratic equation

$e^{-2k}=\frac{1\pm\sqrt{1-\frac{4}{e}\ln\frac{98}{55}}}{2}$

≈ 0.30635 or 0.69365

≈ 0.306 or 0.694 — 1A r.t. 0.306 , 0.694

$\begin{cases}k\approx 0.5915\\ a\approx 4.8401\end{cases}$ or $\begin{cases}k\approx 0.1829\\ a\approx 5.8929\end{cases}$

$\begin{cases}k\approx 0.59\\ a\approx 4.84\end{cases}$ or $\begin{cases}k\approx 0.18\\ a\approx 5.89\text{ (or 5.90)}\end{cases}$. (2 d.p.) — 1A $a-1$ for more than 2 d.p.

(b) Using $\begin{cases}k\approx 0.59\\ a\approx 4.84\end{cases}$, $\ln N(7)\approx 4.80$, — 1M r.t. 4.80

$N(7)\approx 121$. — r.t. 121

Using $\begin{cases}k\approx 0.18\\ a\approx 5.89\end{cases}$, $\ln N(7)\approx 5.12$, — r.t. 5.12 – 5.14

$N(7)\approx 167$. (or comparing $\ln 170\approx 5.1358$) — r.t. 167 – 170

$\therefore$ $\begin{cases}k\approx 0.18\\ a\approx 5.89\end{cases}$ will make the model fit for the known data. — 1A follow through

$\because$ $N(t)=e^{\ln N(t)}\approx e^{5.89-e^{1-0.18t}}$ — 1M

$\therefore$ $N(t)\to e^{5.89}\approx 361$ as $t\to\infty$

The total possible catch of coral fish in that area since January 1, 1992 is 361 thousand tonnes. — 1A r.t. 361 – 365; $pp-1$ for wrong/missing unit

Solution	Marks	Remarks
(c) (i) $\because \ln N(t) = a - e^{1-kt}$		
$\therefore \dfrac{N'(t)}{N(t)} = ke^{1-kt}$		
$N'(t) = k\,N(t)e^{1-kt}$	1	
Alternatively, $N(t) = e^{a-e^{1-kt}}$		
$N'(t) = -e^{1-kt}(-k)e^{a-e^{1-kt}} = ke^{1-kt}\,N(t)$	1	
(ii) $N''(t) = k[N'(t)e^{1-kt} - k\,N(t)e^{1-kt}]$		
$= k^2\,N(t)e^{1-kt}(e^{1-kt} - 1)$	1A	
$\begin{cases} >0 & \text{when } t < \frac{1}{k} \\ =0 & \text{when } t = \frac{1}{k} \\ <0 & \text{when } t > \frac{1}{k} \end{cases}$	1M	
$\therefore$ $N'(t)$ is maximum at $t = \dfrac{1}{k}$	1A	
≈ 5.56		$t \in [5.47, 5.56]$
The maximum rate of change of the total catch of coral fish in that area since January 1, 1992 occurred in 1997.	1A	
$\ln N(6) \approx 4.97$, $N(6) \approx 143.6$		$\ln N(6) \in [4.97, 4.99]$, $N(6) \in [143.6, 146.3]$
$\ln N(5) \approx 4.78$, $N(5) \approx 119.7$		$\ln N(5) \in [4.78, 4.80]$, $N(5) \in [119.7, 122.0]$
$\therefore$ The volume of fish caught in 1997		
$= [N(6) - N(5)]$ thousand tonnes	1M	
≈ 24 thousand tonnes	1A	*pp*−1 for wrong/missing unit

34. (1997 ASL-M&S Q8)

Solution	Marks	Remarks
(a) $\because N(0) = 16$		
$\therefore \dfrac{40}{1+b} = 16$	1M	
$b = 1.5$	1A	
$\because N(7) = 17.4$		
$\therefore \dfrac{40}{1+1.5e^{-7r}} = 17.4$	1M	
$e^{-7r} = \dfrac{1}{1.5}\left(\dfrac{40}{17.4} - 1\right)$		
$r = \dfrac{1}{-7}\ln\left[\dfrac{1}{1.5}\left(\dfrac{40}{17.4} - 1\right)\right]$	1M	
≈ 0.02	1A	
(b) $N(t) = \dfrac{40}{1+be^{-rt}}$ (or $\dfrac{40}{1+1.5e^{-0.02t}}$)		
$N'(t) = \dfrac{-40(-bre^{-rt})}{(1+be^{-rt})^2}$ (or $\dfrac{-40(-1.5)(0.02)e^{-0.02t}}{(1+1.5e^{-0.02t})^2}$)	1M+1A	
$= \dfrac{40bre^{-0.02t}}{(1+be^{-rt})^2}$ (or $\dfrac{1.2e^{-0.02t}}{(1+1.5e^{-0.02t})^2}$)		
> 0		
$\therefore$ $N(t)$ is increasing.	1	
(c) $\because \lim\limits_{t\to\infty} e^{-rt} = 0$		
$\therefore N_a = \lim\limits_{t\to\infty} \dfrac{40}{1+be^{-rt}}$ (or $\lim\limits_{t\to\infty} \dfrac{40}{1+1.5e^{-0.02t}}$)	1M	
$= 40$	1A	
(d) (i) $N''(t)$		
$= \dfrac{[(1+1.5e^{-0.02t})(1.2) - 1.2e^{-0.02t}(2)(1.5)](1+1.5e^{-0.02t})(-0.02)e^{-0.02t}}{(1+1.5e^{-0.02t})^4}$	1M	
$= \dfrac{0.012e^{-0.02t}(3e^{-0.02t} - 2)}{(1+1.5e^{-0.02t})^3}$	1A	
(ii) From (i), $N''(t) \begin{cases} >0 & \text{when } t < t_0 \\ =0 & \text{when } t = t_0 \\ <0 & \text{when } t > t_0 \end{cases}$	1M	For Solving $N''(t) = 0$
where $t_0 = -\dfrac{1}{0.02}\ln\dfrac{2}{3} \approx 20.2733$		
$\therefore$ The rate of increase is the greatest when $t = t_0 \approx 20.2733$	1M	For checking maximum
$\because N'(20) \approx 0.199999$		
$N'(21) \approx 0.199989$		
$\therefore$ The company should start to advertise on the 20th day after the first week.	1A	

Out of syllabus

35. (2012 DSE-MATH-M1 Q4)

Solution	Marks	Remarks
(a) $y=\sqrt[3]{\dfrac{3x-1}{x-2}}$		
$\ln y=\dfrac{1}{3}\ln(3x-1)-\dfrac{1}{3}\ln(x-2)$	1A	
$\therefore\ \dfrac{1}{y}\cdot\dfrac{dy}{dx}=\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}$	1A	
(b) By (a), $\dfrac{dy}{dx}=\left[\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}\right]\sqrt[3]{\dfrac{3x-1}{x-2}}$	1A	
$\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left[\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}\right]\sqrt[3]{\dfrac{3x-1}{x-2}}+\left[\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}\right]\dfrac{d}{dx}\left(\sqrt[3]{\dfrac{3x-1}{x-2}}\right)$		
$=\left[\dfrac{-3}{(3x-1)^2}+\dfrac{1}{3(x-2)^2}\right]\sqrt[3]{\dfrac{3x-1}{x-2}}+\left[\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}\right]^2\sqrt[3]{\dfrac{3x-1}{x-2}}$ by (a)	1M	For using (a)
When $x=3$, $\dfrac{d^2y}{dx^2}=\left\{\dfrac{-3}{(3\cdot3-1)^2}+\dfrac{1}{3(3-2)^2}+\left[\dfrac{1}{3\cdot3-1}-\dfrac{1}{3(3-2)}\right]^2\right\}\sqrt[3]{\dfrac{3\cdot3-1}{3-2}}$	1M	
<u>Alternative Solution</u>		
When $x=3$, $y=2$ and so $\dfrac{dy}{dx}=\dfrac{-5}{12}$.	1A	For both y and $\dfrac{dy}{dx}$
By (a), $\dfrac{1}{y}\cdot\dfrac{d^2y}{dx^2}-\dfrac{1}{y^2}\cdot\dfrac{dy}{dx}\cdot\dfrac{dy}{dx}=\dfrac{-3}{(3x-1)^2}+\dfrac{1}{3(x-2)^2}$	1M	For chain rule
When $x=3$, $\dfrac{1}{2}\cdot\dfrac{d^2y}{dx^2}-\dfrac{1}{2^2}\cdot\dfrac{-5}{12}\cdot\dfrac{-5}{12}=\dfrac{-3}{(3\cdot3-1)^2}+\dfrac{1}{3(3-2)^2}$	1M	
i.e. $\dfrac{d^2y}{dx^2}=\dfrac{95}{144}$	1A	OR 0.6597
	(6)	

(a)	Good. Some candidates wrote $\sqrt[3]{\dfrac{3x-1}{x-2}}=\left(\dfrac{3x-1}{x-2}\right)^{3/2}$. Some did not use logarithmic differentiation.
(b)	Fair. Some candidates did not use the result in (a). Some wrote $\dfrac{d^2y}{dx^2}=\left(\dfrac{dy}{dx}\right)^2$, $\dfrac{d}{dx}\left(\dfrac{1}{y}\cdot\dfrac{dy}{dx}\right)=-y^{-2}\dfrac{d^2y}{dx^2}$ or $\dfrac{d}{dx}\left(\dfrac{1}{y}\cdot\dfrac{dy}{dx}\right)=\dfrac{1}{y}\cdot\dfrac{d^2y}{dx^2}-\dfrac{1}{y^2}\dfrac{dy}{dx}$.

36. (SAMPLE DSE-MATH-M1 Q6)

Solution	Marks	Remarks
(a) $\ln u=\dfrac{1}{2}\ln(2x+3)-\dfrac{1}{2}\ln(x+1)-\dfrac{1}{2}\ln(x+2)$	1A	
Differentiate both sides with respect to x, we have $\dfrac{1}{u}\dfrac{du}{dx}=\dfrac{1}{2x+3}-\dfrac{1}{2(x+1)}-\dfrac{1}{2(x+2)}$		
$\dfrac{du}{dx}=u\left[\dfrac{1}{2x+3}-\dfrac{1}{2(x+1)}-\dfrac{1}{2(x+2)}\right]$	1A	
(b) $u=3^y$ gives $y=\dfrac{\ln u}{\ln 3}$		
$\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$		
$=\dfrac{1}{u\ln 3}\cdot u\left[\dfrac{1}{2x+3}-\dfrac{1}{2(x+1)}-\dfrac{1}{2(x+2)}\right]$	1M+1M	1M for chain rule
$=\dfrac{1}{\ln 3}\left[\dfrac{1}{2x+3}-\dfrac{1}{2(x+1)}-\dfrac{1}{2(x+2)}\right]$	1A	
	(5)	

37. (2012 ASL-M&S Q2)

Solution	Marks	Remarks
(a) $y=e^{t^2+4t+4}$ and $x=\ln(2t+4)$		
$\dfrac{dy}{dt}=e^{t^2+4t+4}(2t+4)$ and $\dfrac{dx}{dt}=\dfrac{1}{t+2}$	1A	For both
$\dfrac{dy}{dx}=\dfrac{dy}{dt}\div\dfrac{dx}{dt}$		
$=2e^{t^2+4t+4}(t+2)\cdot(t+2)$	1M	
<u>Alternative Solution</u>		
$\ln y=(t+2)^2$ and $x=\ln 2+\ln(t+2)$		OR $\cdots$ and $t+2=\dfrac{1}{2}e^x$
$\therefore\ x=\ln 2+\dfrac{1}{2}\ln(\ln y)$	1A	OR $\ln y=\dfrac{1}{4}e^{2x}$
$\dfrac{dx}{dy}=\dfrac{1}{2}\cdot\dfrac{1}{\ln y}\cdot\dfrac{1}{y}$	1M	OR $\dfrac{1}{y}\cdot\dfrac{dy}{dx}=\dfrac{1}{4}e^{2x}\cdot 2$
$\dfrac{dy}{dx}=2y\ln y$	1A	
(b) $\dfrac{d^2y}{dx^2}=\left(2y\cdot\dfrac{1}{y}+2\ln y\right)\dfrac{dy}{dx}$ $=4y\ln y(1+\ln y)$	1M	For chain rule
When $x=0$, $t=\dfrac{-3}{2}$ and so $y=e^{\frac{1}{4}}$.	1A	
$\therefore\ \dfrac{d^2y}{dx^2}=4e^{\frac{1}{4}}\left(\dfrac{1}{4}\right)\left(1+\dfrac{1}{4}\right)$		
$=\dfrac{5}{4}e^{\frac{1}{4}}$	1A	OR 1.6050
	(6)	

Fair.
Candidates did not perform well in chain rule.

38. (2009 ASL-M&S Q3)

(a) $x = y^4 - y$		
$\frac{dx}{dy} = 4y^3 - 1$	1M	For finding $\frac{dx}{dy}$
$\therefore \frac{dy}{dx} = \frac{1}{4y^3 - 1}$	1M+1A	1M for $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$
Alternative Solution		
$1 = 4y^3 \frac{dy}{dx} - \frac{dy}{dx}$	1M+1M	1M for finding $\frac{dy}{dx}$ 1M for chain rule
$\therefore \frac{dy}{dx} = \frac{1}{4y^3 - 1}$	1A	
(b) $\therefore \frac{1}{4y^3 - 1} = \frac{1}{3}$	1M	
$y = 1$		
$\therefore x = 1^4 - 1 = 0$	1A	
Hence the required equation of the tangent is		
$y - 1 = \frac{1}{3}(x - 0)$	1M	
i.e. $x - 3y + 3 = 0$	1A	
	(7)	

Good. Most candidates had good knowledge in differentiation of inverse function and were able to find the equation of tangent.

39. (2008 ASL-M&S Q2)

(a) $y^3 - uy = 1$		
$3y^2 \frac{dy}{du} - \left(u\frac{dy}{du} + y\right) = 0$	1M	
$\frac{dy}{du} = \frac{y}{3y^2 - u}$	1A	
Alternative Solution		
$u = y^2 - \frac{1}{y}$		
$\frac{du}{dy} = 2y + \frac{1}{y^2}$	1M	
$\frac{dy}{du} = \frac{y^2}{2y^3 + 1}$	1A	
(b) $u = 2^{x^2}$		
$\ln u = x^2 \ln 2$	1M	
$\frac{1}{u}\frac{du}{dx} = 2x \ln 2$	1M	
$\frac{du}{dx} = 2^{x^2} \cdot 2x \ln 2$	1A	
Alternative Solution		
$u = 2^{x^2} = e^{x^2 \ln 2}$	1M	
$\frac{du}{dx} = e^{x^2 \ln 2} \cdot 2x \ln 2$	1M	
$= 2^{x^2} \cdot 2x \ln 2$	1A	
(c) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$		
$= \frac{y}{3y^2 - u} \cdot 2^{x^2} \cdot 2x \ln 2$	1M	
$= \frac{2^{x^2} \cdot 2xy \ln 2}{3y^2 - 2^{x^2}}$	1A	OR $\frac{2^{x^2+1} xy \ln 2}{3y^2 - 2^{x^2}}$
	(7)	

Good. Some candidates were not familiar with the use of the logarithmic function in differentiation.

40. (2005 ASL-M&S Q3)

Solution	Marks
(a) $\ln w = \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+2) - \frac{1}{2}\ln(2x+1)$	1A
Differentiate both sides w.r.t. x, we have	
$\frac{1}{w}\frac{dw}{dx} = \frac{3}{2(x-1)} - \frac{1}{2(x+2)} - \frac{1}{2x+1}$	1M
$\frac{dw}{dx} = w\left(\frac{3}{2(x-1)} - \frac{1}{2(x+2)} - \frac{1}{2x+1}\right)$	1A
$\boxed{\frac{dw}{dx} = \sqrt{\frac{(x-1)^3}{(x+2)(2x+1)}}\left(\frac{3}{2(x-1)} - \frac{1}{2(x+2)} - \frac{1}{2x+1}\right)}$	
$\boxed{\frac{dw}{dx} = \frac{w(2x^2+14x+11)}{2(x-1)(x+2)(2x+1)}}$	
(b) $w = 2^y$	
$\ln w = y\ln 2$	1M for taking log on both sides and can be absorbed
$y = \frac{\ln w}{\ln 2}$	
$\frac{dy}{dw} = \frac{1}{w\ln 2}$	1A
$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx}$	
$\frac{dy}{dx} = \left(\frac{1}{w\ln 2}\right)\left[w\left(\frac{3}{2(x-1)} - \frac{1}{2(x+2)} - \frac{1}{2x+1}\right)\right]$	1M for Chain Rule
$\frac{dy}{dx} = \frac{1}{\ln 2}\left(\frac{3}{2(x-1)} - \frac{1}{2(x+2)} - \frac{1}{2x+1}\right)$	1A accept $\frac{(2x^2+14x+11)}{2(x-1)(x+2)(2x+1)\ln 2}$
$w = 2^y$	
$\ln w = y\ln 2$	1M for taking log on both sides and can be absorbed
Differentiate both sides w.r.t. y, we have	
$\frac{1}{w}\frac{dw}{dy} = \ln 2$	
$\frac{dw}{dy} = w\ln 2$	
$\frac{dy}{dw} = \frac{1}{w\ln 2}$	1A
$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx}$	
$\frac{dy}{dx} = \left(\frac{1}{w\ln 2}\right)\left[w\left(\frac{3}{2(x-1)} - \frac{1}{2(x+2)} - \frac{1}{2x+1}\right)\right]$	1M for Chain Rule
$\frac{dy}{dx} = \frac{1}{\ln 2}\left(\frac{3}{2(x-1)} - \frac{1}{2(x+2)} - \frac{1}{2x+1}\right)$	1A accept $\frac{(2x^2+14x+11)}{2(x-1)(x+2)(2x+1)\ln 2}$
	---------(7)

Good. Some candidates failed to apply the chain rule.

41. (2002 ASL-M&S Q1)

Solution	Marks
$\frac{dx}{dt} = \frac{10}{t^3} - 6e^{-3t}$	1M+1A
$\frac{dy}{dt} = -\frac{20}{t^3} + 2e^{2t}$	(1M for $(e^{at})' = ae^{at}$)
$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{1}{\frac{dx}{dt}}\right) = \frac{-\frac{20}{t^3}+2e^{2t}}{\frac{10}{t^3}-6e^{-3t}}$	1M for Chain Rule and Inverse Function Rule
For $\frac{dy}{dx} = -2$	
$\frac{-\frac{20}{t^3}+2e^{2t}}{\frac{10}{t^3}-6e^{-3t}} = -2$	1M
$e^{5t} = 6$	
$t = \frac{1}{5}\ln 6 (\approx 0.3584)$	1A $a-1$ for r.t. 0.358
	---------(5)

42. (2000 ASL-M&S Q1)

Solution	Marks
$\ln(xy) = \frac{x}{y}$	
$\ln x + \ln y = \frac{x}{y}$	
$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = \frac{y - x\frac{dy}{dx}}{y^2}$ (or $\frac{x\frac{dy}{dx}+y}{xy} = \frac{y-x\frac{dy}{dx}}{y^2}$)	1M differentaition of $\ln x$ 1M chain rule 1M quotient/product rule
$y^2 + xy\frac{dy}{dx} = xy - x^2\frac{dy}{dx}$	
$\frac{dy}{dx} = \frac{xy - y^2}{xy + x^2}$	1 at least one step
Alternatively,	
$xy = e^{\frac{x}{y}}$	
$x\frac{dy}{dx} + y = e^{\frac{x}{y}}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right)$	1M differentaition of e^x 1M chain rule 1M quotient rule
$x\frac{dy}{dx} + y = xy\left(\frac{y - x\frac{dy}{dx}}{y^2}\right)$	
$xy\frac{dy}{dx} + y^2 = xy - x^2\frac{dy}{dx}$	
$\frac{dy}{dx} = \frac{xy - y^2}{xy + x^2}$	1
	----(4)

43. (1999 ASL-M&S Q1)

Solution	Marks	Remarks
(a) When $x=1$, $e^y=\dfrac{2^3}{2}=4$		
$y=\ln 4$ (or $y=2\ln 2$)	1A	
(or $y=1.3863$)		$a-1$ for r.t. 1.386
(b) $\because$ $e^{xy}=\dfrac{x(x+1)^3}{x^2+1}$		
$xy=\ln\left(\dfrac{x(x+1)^3}{x^2+1}\right)$	1A	taking log on both sides (one side must correct)
$xy=\ln x+3\ln(x+1)-\ln(x^2+1)$		
$x\dfrac{dy}{dx}+y=\dfrac{1}{x}+\dfrac{3}{x+1}-\dfrac{2x}{x^2+1}$	1M+1M	1M for product rule 1M for differentiating log
When $x=1$, $\dfrac{dy}{dx}+\ln 4=1+\dfrac{3}{2}-1$		
$\dfrac{dy}{dx}=\dfrac{3}{2}-\ln 4$ (or 0.1137)	1A	$a-1$ for r.t. 0.114
Alternatively,		
$e^{xy}(x\dfrac{dy}{dx}+y)=\dfrac{(x^2+1)[(x+1)^3+3(x+1)^2x]-x(x+1)^3(2x)}{(x^2+1)^2}$	1M+1M+1A	1M for differentiating e^{xy} 1M for product/quotient rule
When $x=1$, $e^{xy}(\dfrac{dy}{dx}+\ln 4)=6$		
$\dfrac{dy}{dx}=\dfrac{3}{2}-\ln 4$	1A	
	(5)	

44. (1995 ASL-M&S Q2)

Solution	Marks	Remarks
(a) $e^x+e^y=xy$		
$e^x+e^y\dfrac{dy}{dx}=y+x\dfrac{dy}{dx}$	1A + 1A	
$\dfrac{dy}{dx}=\dfrac{e^x-y}{x-e^y}$	1A	
(b) $y=\dfrac{(x-2)^{\frac{1}{2}}(x+3)^{\frac{1}{2}}}{(x+1)^{\frac{3}{2}}}$		
$\ln y=\frac{1}{2}\ln(x-2)+\frac{1}{2}\ln(x+3)-\frac{3}{2}\ln(x+1)$		
$\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{1}{2(x-2)}+\dfrac{1}{2(x+3)}-\dfrac{3}{2(x+1)}$	1M + 1A	1M for taking log. on both sides and applying: $\ln ab=\ln a+\ln b$ $n\ln a=\ln a^n$ $\dfrac{d}{dx}\ln y=\dfrac{1}{y}\dfrac{dy}{dx}$
$\dfrac{dy}{dx}=\dfrac{y}{2}\left(\dfrac{1}{x-2}+\dfrac{1}{x+3}-\dfrac{3}{x+1}\right)$	1A	
(or $\dfrac{1}{2(x+1)}\sqrt{\dfrac{(x-2)(x+3)}{x+1}}\left(\dfrac{1}{x-2}+\dfrac{1}{x+3}-\dfrac{3}{x+1}\right)$		
or $\dfrac{y(19-x^2)}{2(x-2)(x+3)(x+1)}$)		
	(6)	