2. Exponential and Logarithmic Functions

Learning Unit	Learning Objective
Foundation Knowledge Area	
2. Exponential and logarithmic functions	2.1 recognise the definition of the number e and the exponential series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
	2.2 recognise exponential functions and logarithmic functions
	2.3 use exponential functions and logarithmic functions to solve problems
	2.4 transform $y = kx^n$ and $y = ka^x$ to linear relations, where a, n and k are real numbers, $a > 0$ and $a \neq 1$

Section A

1. (a) Expand e^{-18x} in ascending powers of x as far as the term in x^2 .

   (b) Let n be a positive integer. If the coefficient of x^2 in the expansion of $e^{-18x}(1+4x)^n$ is -38 . Find n .

   (6 marks) (2019 DSE-MATH-M1 Q6)

2. Let k be a constant.

   (a) Expand $e^{kx} + e^{2x}$ in ascending powers of x as far as the term in x^2 .

   (b) If the coefficient of x and the coefficient of x^2 in the expansion of $(1-3x)^8(e^{kx} + e^{2x} - 1)$ are equal, find k .

   (6 marks) (2018 DSE-MATH-M1 Q6)

3. Let k be a constant.

(a) Expand $e^{kx}+e^{2x}$ in ascending powers of x as far as the term in x^2.

(b) If the coefficient of x and the coefficient of x^2 in the expansion of $(1-3x)^8(e^{kx}+e^{2x}-1)$ are equal, find k.

(6 marks) (2018 DSE-MATH-M1 Q6)

4. (a) Expand $(1+e^{3x})^2$ in ascending powers of x as far as the term in x^2.

(b) Find the coefficient of x^2 in the expansion of $(5-x)^4(1+e^{3x})^2$.

(6 marks) (2017 DSE-MATH-M1 Q5)

5. Let k be a constant.

(a) Expand e^{kx} in ascending powers of x as far as the term in x^2.

(b) If the coefficient of x in the expansion of $(1+2x)^7 e^{kx}$ is 8, find the coefficient of x^2.

(5 marks) (2016 DSE-MATH-M1 Q5)

6. (a) Expand e^{-4x} in ascending powers of x as far as the term in x^2.

(b) Find the coefficient of x^2 in the expansion of $\dfrac{(2+x)^5}{e^{4x}}$.

(5 marks) (2015 DSE-MATH-M1 Q5)

7. After launching an advertisement for x weeks, the number y (in thousand) of members of a club can be modeled by

$$y=\frac{8(1-ae^{-bx})}{1+ae^{-bx}},\text{ where } a \text{ and } b \text{ are positive integers and } x\ge 0\ .$$

The values of y when $x=2,4,6,8,10$ were recorded as follows:

x	2	4	6	8	10
y	5.97	6.26	6.75	7.11	7.37

(a) Let $u=ae^{-bx}$.

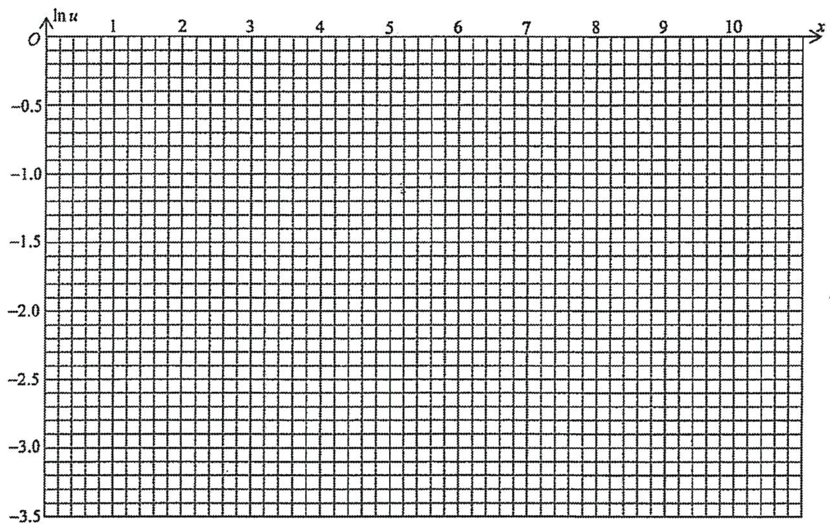
(i) Express $\ln u$ as a linear function of x.

(ii) Find u in terms of y.

(b) It is known that one of the values of y in the above table is incorrect.

(i) Using the graph paper on page 9 to determine which value of y is incorrect.

(ii) By removing the incorrect value of y, estimate the values of a and b. Correct your answers to 2 decimal places.



(7 marks)

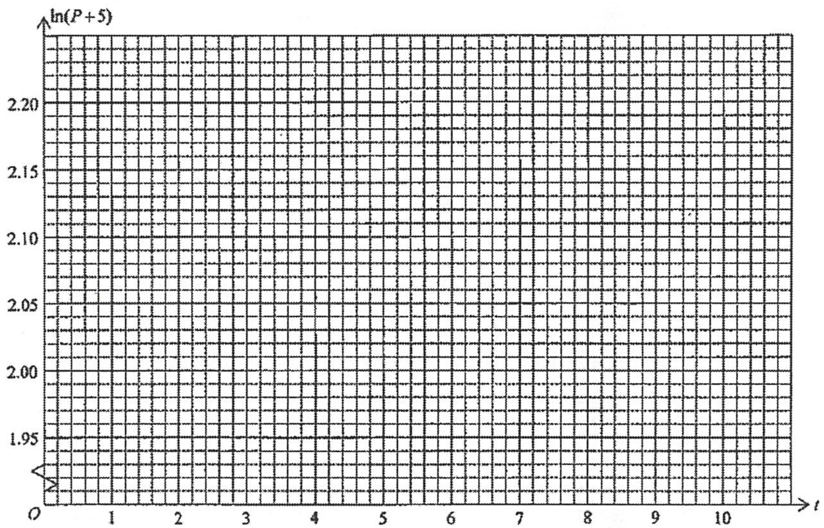
(2013 DSE-MATH-M1 Q4)

8. The population P (in millions) of a city can be modelled by $P = ae^{\frac{kt}{40}} - 5$ where a and k are constants and t is the number of years since the beginning of a certain year. The population of the city is recorded as follows.

t	2	4	6	8	10
P	2.36	2.81	3.23	3.55	4.01

(a) Express $\ln(P+5)$ as a linear function of t.

(b) Using the graph paper below, estimate the values of a and k. Correct your answers to the nearest integers.



(5 marks)

(2012 DSE-MATH-M1 Q3)

9. The number $N(t)$ of fish, which are infected by a certain disease in a pool, can be modelled by

$$N(t) = \frac{500}{1 + ae^{-kt}},$$

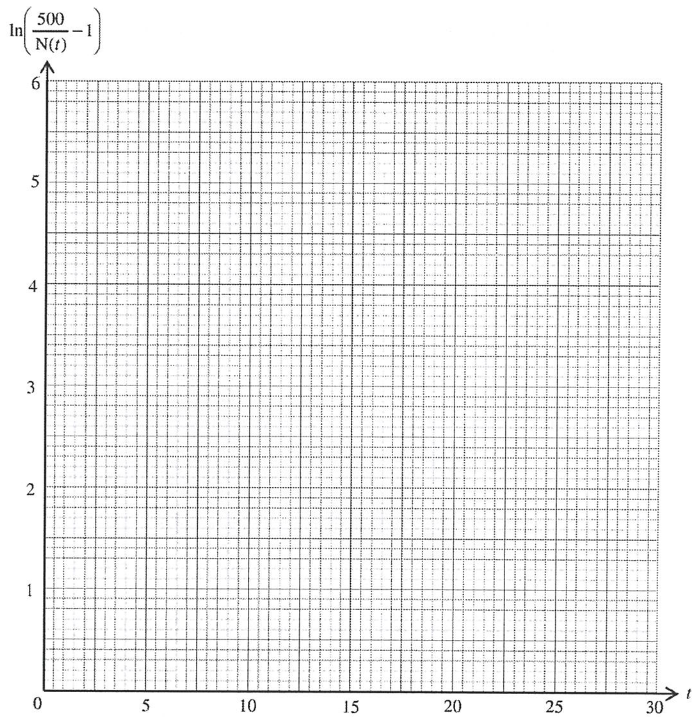
where a, k are positive constants and t is the number of days elapsed since the outbreak of the disease.

t	5	10	15	20
$N(t)$	13	34	83	175

(a) Express $\ln\left(\frac{500}{N(t)} - 1\right)$ as a linear function of t.

(b) Using the graph paper on page 10, estimate graphically the values of a and k (correct your answers to 1 decimal place).

(c) How many days after the outbreak of the disease will the number of fish infected by the disease reach 270?



(6 marks)

(SAMPLE DSE-MATH-M1 Q10)

10. After adding a chemical into a bottle of solution, the temperature S(t) of the surface of the bottle can be modeled by

$$S(t) = 2(t+1)^2 e^{-\lambda t} + 15 \ ,$$

where S(t) is measured in $°C$, $t\ (\geq 0)$ is the time measured in seconds after the chemical has been added and λ is a positive constant. It is given that $S(9) = S(19)$.

(a) Find the exact value of λ .

(b) Will the temperature of the surface of the bottle get higher than 90 $°C$? Explain your answer.

(6 marks) (2006 ASL-M&S Q2)

11. Let $y = \dfrac{1-e^{4x}}{1+e^{8x}}$.

(a) Find the value of $\dfrac{dy}{dx}$ when $x = 0$.

(b) Let $(z^2+1)e^{3z} = e^{\alpha+\beta x}$, where α and β are constants.

(i) Express $\ln(z^2+1)+3z$ as a liner function of x .

(ii) It is given that the graph of the linear function obtained in (b)(i) passes through the origin and the slope of the graph is 2. Find the values of α and β .

(iii) Using the values of α and β obtained in (b)(ii), find the value of $\dfrac{dy}{dz}$ when $z = 0$.

(7 marks) (2007 ASL-M&S Q3)

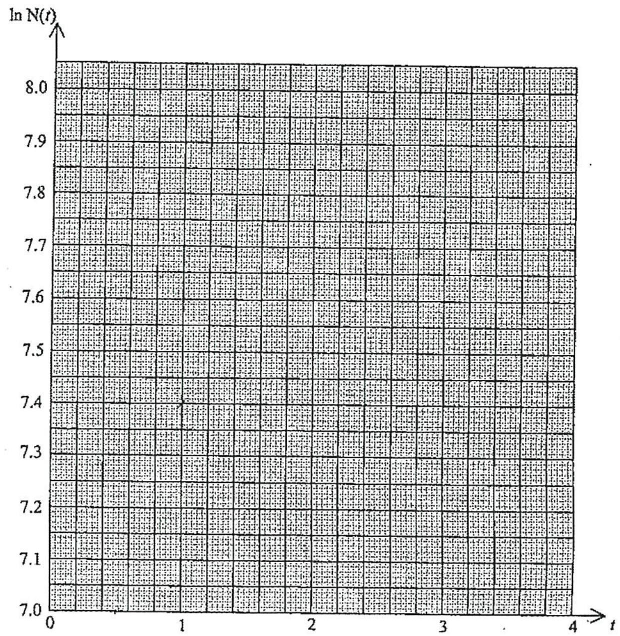
12. A researcher modeled the number of bacteria N(t) in a sample t hours after the beginning of his observation by $N(t) = 900a^{kt}$, where a (>0) and k are constants. He observed and recorded the following data:

t (in hours)	0.5	1.0	2.0	3.0
$N(t)$	1100	1630	2010	2980

The researcher made one mistake when writing down the data for N(t) .

Express $\ln N(t)$ as a linear function of t and use a graph paper to determine which one of the data was incorrect, and estimate the value of N(2.5) correct to 3 significant figures.

(4 marks) (2002 ASL-M&S Q3)



Section B

13. In an experiment, the temperature (in $°C$) of a certain liquid can be modelled by

$$S = \frac{200}{1+a2^{bt}} \ ,$$

where a and b are constants and t is the number of hours elapsed since the start of the experiment.

(a) Express $\ln\left(\dfrac{200}{S}-1\right)$ as a linear function of t .

(2 marks)

(b) It is found that the intercepts on the vertical axis and the horizontal axis of the graph of the linear function obtained in (a) are $\ln 4$ and 4 respectively.

(i) Find a and b .

(ii) Find $\dfrac{dS}{dt}$ and $\dfrac{d^2S}{dt^2}$.

(iii) Describe how S and $\dfrac{dS}{dt}$ vary during the first 48 hours after the start of the experiment. Explain your answer.

(11 marks)

(2015 DSE-MATH-M1 Q12)

14. Let y be the amount (in suitable units) of suspended particulate in a laboratory. It is given that

$$(E): \quad y=\frac{340}{2+e^{-t}-2e^{-2t}} \qquad (t\geq 0)\ ,$$

where t is the time (in hours) which has elapsed since an experiment started.

(a) Will the value of y exceed 171 in the long run? Justify your answer.

(2 marks)

(b) Find the greatest value and least value of y.

(6 marks)

(c) (i) Rewrite (E) as a quadratic equation in e^{-t}.

(ii) It is known that the amounts of suspended particulate are the same at the time $t=\alpha$ and $t=3-\alpha$. Given that $0\leq\alpha<3-\alpha$, find α.

(4 marks)

(2014 DSE-MATH-M1 Q11)

15. A researcher models the rate of change of the population size of a kind of insects in a forest by

$$P'(t)=kte^{\frac{a}{20}t}\ ,$$

where $P(t)$, in thousands, is the population size, t (≥ 0) is the time measured in weeks since the start of the research, and a, k are integers.

The following table shows some values of t and $P'(t)$.

t	1	2	3	4
$P'(t)$	22.83	43.43	61.97	78.60

(a) Express $\ln\frac{P'(t)}{t}$ as a linear function of t.

(1 mark)

(b) By plotting a suitable straight line on the graph paper on next page, estimate the integers a and k.

(5 marks)

(c) Suppose that $P(0)=30$. Using the estimates in (b),

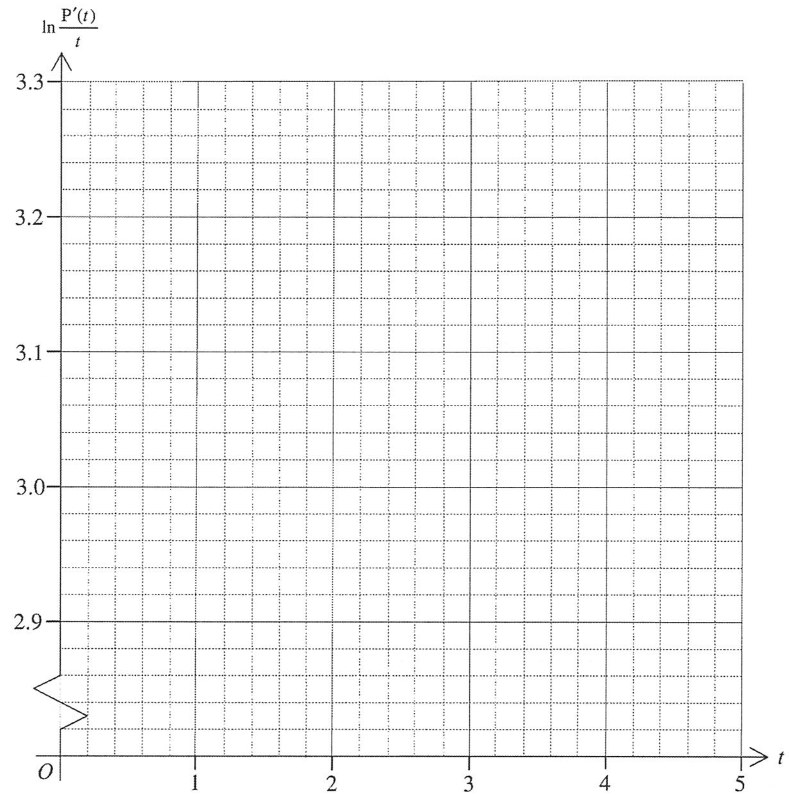
(i) find the value of t such that the rate of change of the population size of the insect is the greatest;

(ii) find $\frac{d}{dt}\left(te^{\frac{a}{20}t}\right)$ and hence, or otherwise, find $P(t)$;

(iii) estimate the population size after a very long time.

[Hint: You may use the fact that $\lim_{t\to\infty}\frac{t}{e^{mt}}=0$ for any positive constant m.]

(9 marks)



(PP DSE-MATH-M1 Q11)

16. A researcher studies the growth of the population size and the electricity consumption of a certain city. Suppose that the population size P (in hundred thousand) of the city can be modelled by

$$P=\frac{ke^{-\lambda t}}{t^2}\ ,\ 0<t<6\ ,$$

where k and λ are constants and t is the time in years elapsed since the start of the research.

(a) (i) Express $\ln P+2\ln t$ as a linear function of t.

(ii) Given that the intercepts on the horizontal and vertical axes of the graph of the linear function in (a)(i) above are -1.15 and 2.3 respectively, find the values of k and λ correct to the nearest integer.

Hence find the minimum population size correct to the nearest hundred thousand.

(6 marks)

(b) The annual electricity consumption E (in thousand terajoules per year) of the city can be modelled by

$$\frac{dE}{dt}=hte^{ht}-1.2e^{ht}+4.214\ ,\ t\geq 0\ ,$$

where h is a non-zero constant and t is the time in years elapsed since the start of the research. It is known that the population size and the rate of change of annual electricity consumption both attain minimum at the same time t_0, and when $t=0$, $E=1$.

(i) Find the value of h.

(ii) By considering $\frac{d}{dt}\left(te^{ht}\right)$, find $\int te^{ht}dt$.

Hence find the annual electricity consumption of the city at t_0 correct to the nearest thousand terajoules per year.

(iii) A green campaign is launched to save the annual electricity consumption immediately after t_0. The new annual electricity consumption F (in thousand terajoules per year) of the city can then be modelled by

$$F=\frac{6}{1-5e^{rt}+3e^{2rt}}+2\ ,\ t\geq t_0 .$$

If the new annual electricity consumption is the same as the original annual electricity consumption at $t=t_0$, find the value of r.

(9 marks)

(2011 ASL-M&S Q9)

17. A researcher models the population size R, in hundreds, of a certain species of fish in a lake by

$$R=kt^{1.2}e^{\frac{\lambda t}{20}}\quad (0\leq t\leq 30),$$

where t is the number of months elapsed since the beginning of the study and k and λ are constants.

(a) (i) Express $\ln R-1.2\ln t$ as a linear function of t.

(ii) It is given that the graph of $\ln R-1.2\ln t$ against t has intercept 2.89 on the vertical axis and slope -0.05. Find the values of k and λ correct to the nearest integer.

(iii) Using the approximate integral values of k and λ obtained in (a)(ii), find the maximum population size of the species of fish correct to the nearest hundreds. When will this take place?

(7 marks)

(b) In order to stimulate the growth of this species of fish, more food is added immediately when the population size of the fish attains 240 hundreds. The population size of the species of fish can then be modelled by

$$Q=L-20(6e^{-t}+t^3)\quad (0\leq t\leq 2),$$

where Q is the population size (in hundreds) of the species of fish, t is the number of months elapsed since more food has been added and L is a constant.

(i) Find the value of L.

(ii) Expand e^{-t} in ascending powers of t as far as the term in t^3.

Hence, find a quadratic polynomial which approximates Q.

(iii) Using the result obtained in (b)(ii), check whether the species of fish will reach a population size of 300 hundreds.

(iv) Do you think that the conclusion in (b)(iii) is still valid if terms up to and including t^7 in the expansion of e^{-t} in (b)(ii) are used? Explain your answer briefly.

(8 marks)

(2009 ASL-M&S Q8)

18. A biologist studied the population of fruit fly A under limited food supply. Let t be the number of days since the beginning of the experiment and $N'(t)$ be the number of fruit fly A at time t. The biologist modelled the rate of change of the number of fruit fly A by

$$N'(t)=\frac{20}{1+he^{-kt}}\quad(t\geq 0)\ ,$$

where h and k are positive constants.

(a) (i) Express $\ln\left(\frac{20}{N'(t)}-1\right)$ as a linear function of t.

(ii) It is given that the intercepts on the vertical axis and the horizontal axis of the graph of the linear function in (i) are 1.5 and 7.6 respectively. Find the values of h and k.

(4 marks)

(b) Take $h=4.5$ and $k=0.2$, and assume that $N(0)=50$.

(i) Let $v=h+e^{kt}$, find $\frac{dv}{dt}$.

Hence, or otherwise, find $N(t)$.

(ii) The population of fruit fly B can be modelled by

$$M(t)=21\left(t+\frac{h}{k}e^{-kt}\right)+b\ ,$$

where b is a constant. It is known that $M(20)=N(20)$.

(1) Find the value of b.

(2) The biologist claims that the number of fruit fly A will be smaller than that of fruit fly B for $t>20$. Do you agree? Explain your answer.

[Hint: Consider the difference between the rates of change of the two populations.]

(11 marks)

(2008 ASL-M&S Q8)

19. After upgrading the production line of a cloth factory, two engineers, John and Mary, model the rate of change of the amount of cloth production in thousand metres respectively by

$$f(t)=25t^2(t+10)^{\frac{-1}{3}}\quad\text{and}\quad g(t)=28+ke^{ht^2},$$

where h and k are positive constants and $t\ (\geq 0)$ is the time measured in months since the upgrading of the production line.

(a) Using the substitution $u=t+10$, or otherwise, find the total amount of cloth production from $t=0$ to $t=3$ under John's model.

(5 mark)

(b) Express $\ln(g(t)-28)$ as a linear function of t^2.

(1 mark)

(c) Given that the slope and the intercept on the vertical axis of the graph of the linear function in (b) are measured to be 0.3 and 1.0 respectively, estimate the values of h and k correct to 1 decimal place.

(2 marks)

(d) Using the estimated values of h and k obtained in (c) correct to 1 decimal place.

(i) expand $g(t)$ in ascending powers of t as fact as t^6, and hence estimate the total amount of cloth production from $t=0$ to $t=3$ under Mary's model;

(ii) determine whether the estimate in (d)(i) is an over-estimate or an under-estimate;

(iii) determine whether the total amount of cloth production from $t=0$ to $t=3$ under Mary's model is greater than that under John's model.

(7 marks)

(2006 ASL-M&S Q9)

20. A researcher studied the soot reduction effect of a petrol additive on soot emission of a car. Let t be the number of hours elapsed after the petrol additive has been used and $r(t)$, measured in ppm per hour, be the rate of change of the amount of soot reduced. The researcher suggested that $r(t)$ can be modeled by $r(t)=\alpha te^{-\beta t}$, where α and β are positive constants.

(a) Express $\ln\frac{r(t)}{t}$ as a linear function of t.

(1 mark)

(b) It is given that the slope and the intercept on the vertical axis of the graph of the linear function obtained in (a) are –0.50 and 2.3 respectively. Find the values of α and β correct to 1 significant figure.

Hence find the greatest rate of change of the amount of soot reduced after the petrol additive has been used. Give your answer correct to 1 significant figure.

(6 marks)

(c) Using the values of α and β obtained in (b) correct to 1 significant figure,

(i) find $\frac{d}{dt}\left((t+\frac{1}{\beta})e^{-\beta t}\right)$ and hence find, in terms of T, the total amount of soot reduced when the petrol additive has been used for T hours;

(ii) estimate the total amount of soot reduced when the petrol additive has been used for a very long time.

[Note: Candidates may use $\lim_{T\to\infty}(Te^{-\beta T})=0$ without proof.]

(8 marks)

(2005 ASL-M&S Q8)

21. A researcher modeled the relationship between the atmospheric pressure y (in cmHg) and the altitude x (in km) above sea-level by

$$\frac{dy}{dx} = -\alpha\beta^{-x} \qquad (x \geq 0) ,$$

where α and β are positive constants.

(a) It is known that $\ln\left(-\frac{dy}{dx}\right)$ can be expressed as a linear function of x. The slope of the graph of the linear function is -0.125.

(i) Find the value of β correct to 3 decimal places.

(ii) The researcher found that the atmospheric pressures at sea-level (i.e. $x = 0$) and at an altitude of 2 km above sea-level were 76 cmHg and 59.2 cmHg respectively. If $\beta^{-x} = e^{-\lambda x}$ for all $x \geq 0$, find the value of λ.

Hence or otherwise, find the value of α correct to 1 decimal place.

(8 marks)

(b) A balloon filled with helium gas is released from a point on a mountain. The altitude of the point is h km above sea-level. The balloon bursts when it reaches an altitude of $2h$ km above sea-level. The difference in the atmospheric pressures between the two altitudes is 13 cmHg. It is also known that the atmospheric pressure at the top of the mountain is 25.2 cmHg. Using the values of α and β obtained in (a),

(i) find the altitude of the mountain above sea-level correct to the nearest 0.1 km.

(ii) find the value(s) of h correct to 1 decimal place.

(7 marks)

(2004 ASL-M&S Q9)

22. The spread of an epidemic in a town can be measured by the value of PPI (the proportion of population infected). The value of PPI will increase when the epidemic breaks out and will stabilize when it dies out.

The spread of the epidemic in town A last year could be modelled by the equation

$P'(t) = \frac{0.04ake^{-kt}}{1-a}$, where a, $k > 0$ and $P(t)$ was the PPI t days after the outbreak of the epidemic. The figure shows the graph of $\ln P'(t)$ against t, which was plotted based on some observed data obtained last year. The initial value of PPI is 0.09 (i.e. $P(0) = 0.09$).

(a) (i) Express $\ln P'(t)$ as a linear function of t and use the figure to estimate the values of a and k correct to 2 decimal places.

Hence find $P(t)$.

(ii) Let μ be the PPI 3 days after the outbreak of the epidemic. Find μ.

(iii) Find the stabilized PPI.

(8 marks)

(b) In another town B, the health department took precautions so as to reduce the PPI of the epidemic. It is predicted that the rate of spread of the epidemic will follow the equation

$Q'(t) = 6(b-0.05)(3t+4)^{\frac{-3}{2}}$, where $Q(t)$ is the PPI t days after the outbreak of the epidemic in town B and b is the initial value of PPI.

(i) Suppose $b = 0.09$.

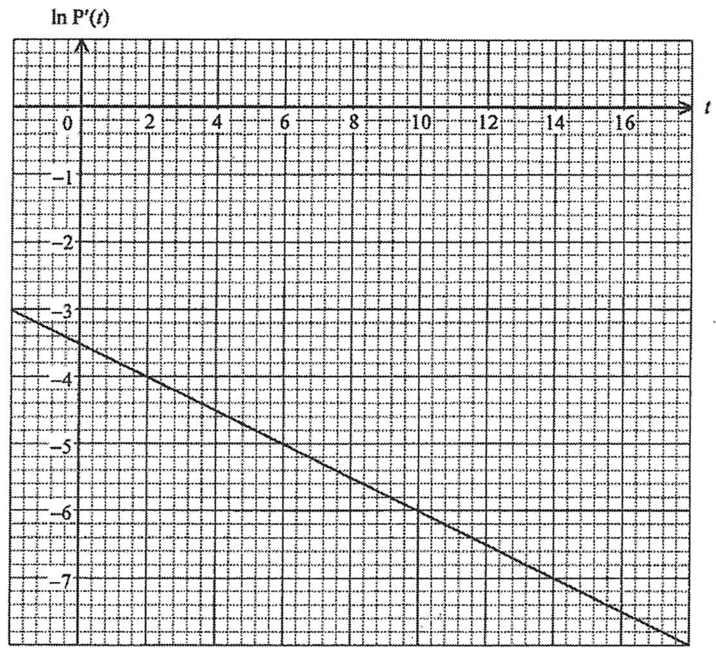
(I) Determine whether the PPI in town B will reach the value of μ in (a)(ii).

(II) How much is the stabilized PPI reduced in town B as compared with that in town A?

(ii) Find the range of possible values of b if the epidemic breaks out in town B. Explain your answer briefly.

(7 marks)

The graph of $\ln P'(t)$ against t



(2001 ASL-M&S Q9)

23. A researcher studied the growth of a certain kind of bacteria. 100 000 such bacteria were put into a beaker for cultivation. Let t be the number of days elapsed after the cultivation has started and $r(t)$, in thousands per day, be the growth rate of the bacteria. The researcher obtained the following data:

t	1	2	3	4
$r(t)$	7.9	12.3	15.3	17.5

(a) The researcher suggested that $r(t)$ can be modelled by $r(t) = at^b$, where a and b are positive constants.

(i) Express $\ln r(t)$ as a linear function of $\ln t$.

(ii) Using the graph paper, estimate graphically the value of $r(5)$ to 1 decimal place without finding the values of a and b .

(5 marks)

(b) The researcher later observed that $r(5)$ was 18.5 and considered the model in (a) unsuitable. After reviewing some literature, he used the model $r(t) = 20 - pe^{-qt}$, where p and q are positive constants.

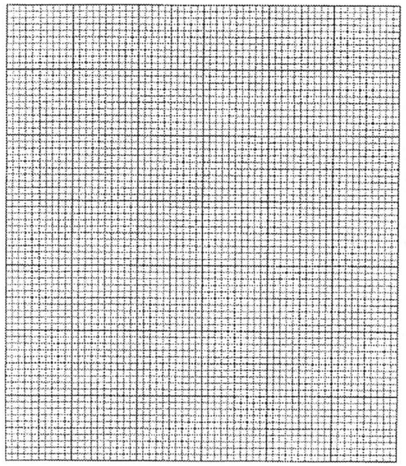
(i) Express $\ln[20 - r(t)]$ as a linear function of t .

(ii) Using the graph paper on next page, estimate graphically the values of p and q to 3 significant figures.

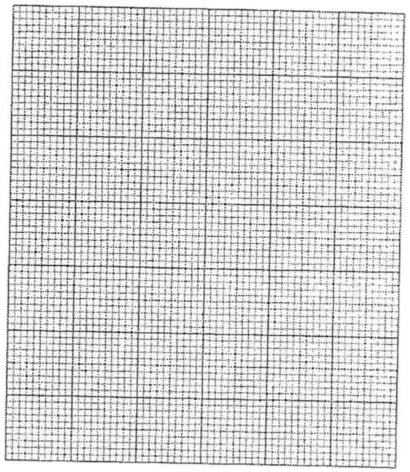
(iii) Estimate the total number of bacteria, to the nearest thousand, after 15 days of cultivation.

(10 marks)

Graph paper for part (a)(ii)



Graph paper for part (b)(ii)



(2000 ASL-M&S Q10)

24. An ecologist studies the birds at Mai Po Nature Reserve. Only 21% of the birds are "residents", i.e. found throughout the year. The remaining birds are migrants. The ecologist suggests that the number $N(t)$ of a certain species of migrants can be modelled by the function

$$N(t) = \frac{3000}{1 + ae^{-bt}} ,$$

where a , b are positive constants and t is the number of days elapsed since the first one of that species of migrants was found at Mai Po in that year.

(a) This year, the ecologist obtained the following data:

t	5	10	15	20
$N(t)$	250	870	1940	2670

(i) Express $\ln(\frac{3000}{N(t)} - 1)$ as a linear function of t .

(ii) Use the graph paper on next page to estimate graphically the values of a and b correct to 1 decimal place.

(5 marks)

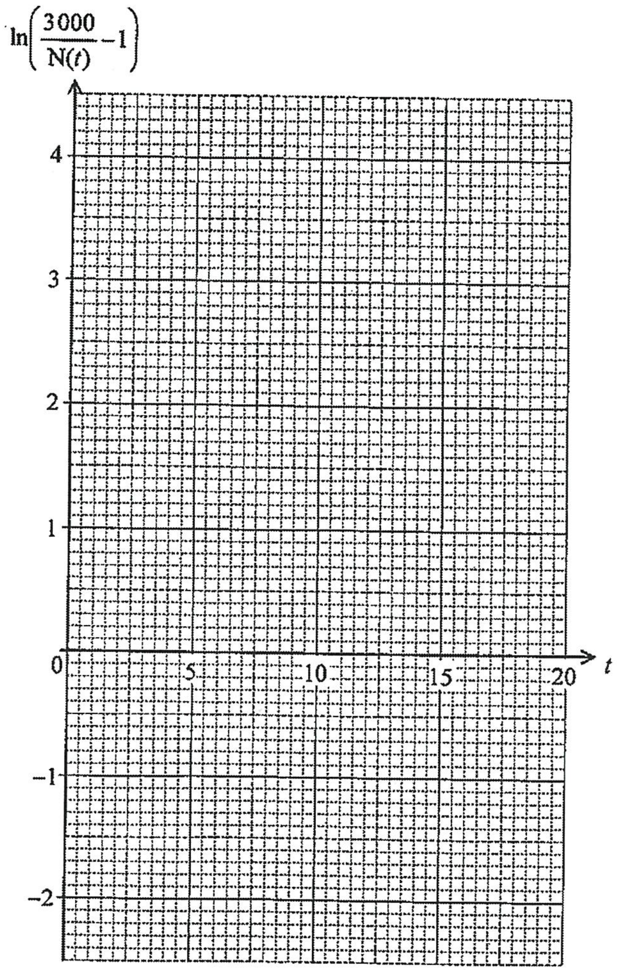
(b) Basing on previous observations, the migrants of that species start to leave Mai Po when the rate of change of $N(t)$ is equal to one hundredth of $N(t)$. Once they start to leave, the original model will not be valid and no more migrants will arrive. It is known that the migrants will leave at the rate $r(s)$ per day where $r(s) = 60\sqrt{s}$ and s is the number of days elapsed since they started to leave Mai Po. Using the values of a and b obtained in (a)(ii),

(i) find $N'(t)$, and show that $N(t)$ is increasing;

(ii) find the greatest number of the migrants which can be found at Mai Po this year;

(iii) find the number of days in which the migrants can be found at Mai Po this year.

(10 marks)



Q9)

25. A forest fire has started in a country. An official of the Department of Environmental Protection wants to estimate the number of trees destroyed in the fire when the fire is out of control. Let t be the number of days after the fire has started and $r(t)$, in hundred trees per day, be the rate of trees destroyed. The official obtained the following data:

t	2	3	4	5	6	7
$r(t)$	6.4	15.7	29.5	48.3	72.2	101.2

(a) It is suggested that $r(t)$ can be modelled by either one of the following functions

(I): $r(t) = \alpha t^{\beta}$ or

(II): $r(t) = \gamma e^{\lambda t}$,

where α , β , γ and λ are constants.

(i) Express $\ln r(t)$ in terms of $\ln t$ and t in (I) and (II) respectively.

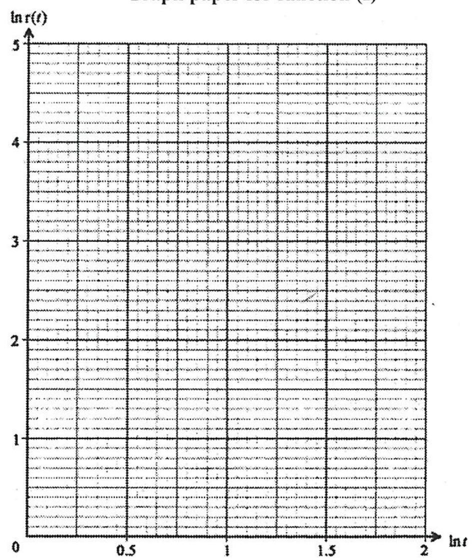
(ii) Use the graph papers to determine which function can better describe $r(t)$. Hence estimate graphically the two unknown constants in that function. Give your answers correct to 1 decimal place.

(10 marks)

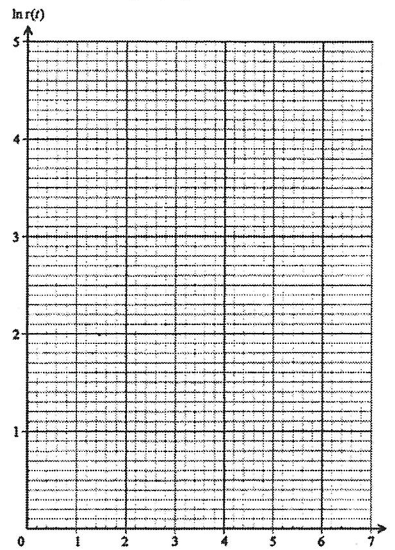
(b) Assume the fire is out of control and the function in (a) which describes $r(t)$ better is used. Estimate the total number, correct to the nearest hundred, of trees destroyed in the first 14 days of the fire. How many days more will it take for the total number of trees destroyed to be doubled?

(5 marks)

Graph paper for function (I)



Graph paper for function (II)



(1998 ASL-M&S Q10)

26. A stall sells clams only. The relationship between the selling price $\$x$ of each clam and the number $N(x)$ of clams sold per day can be modelled by

$$\ln N(x) = bx + \ln a$$

where a and b are constants. This relationship is represented by the straight line shown in the figure.

(a) Use the graph in the figure to estimate the values of a and b correct to 1 significant figure.

(3 marks)

(b) Suppose the daily running cost of the stall is \$ 5 000 and the cost of each clam is \$ 2. Using the values of a and b estimated in (a),

(i) express the daily profit of selling $N(x)$ clams in terms of x , and

(ii) determine the selling price of each clam so that the daily profit of selling $N(x)$ clams will attain its maximum. What is then the number of clams sold per day? Give the answer correct to the nearest integer.

(7 marks)

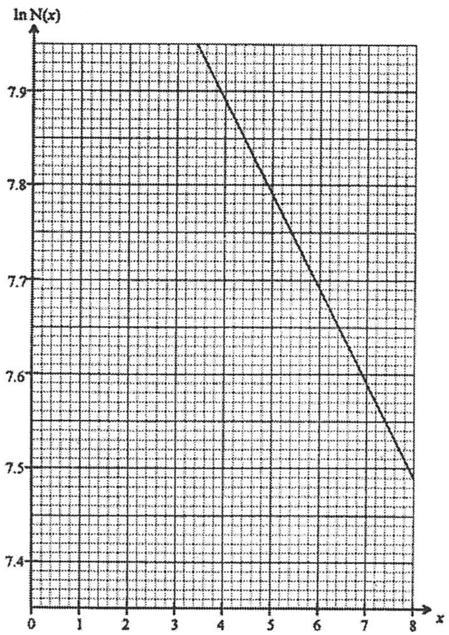
(c) The stall has been running a promotion programme every day from April 15, 1997. The number $M(n)$ of clams sold on the n-th day of the programme is given by

$$M(n) = 1500 + 1000(1 - e^{-0.1n}) \ .$$

The stall will stop running the programme once the increase in the number of clams sold between two consecutive days falls below 15 . Determine how many days the programme should be run. Give the answer correct to the nearest integer.

(5 marks)

(1997 ASL-M&S Q9)



27. A textile factory plans to install a weaving machine on 1st January 1995 to increase its production of cloth. The monthly output x (in km) of the machine, after t months, can be modelled by the function

$$x = 100e^{-0.01t} - 65e^{-0.02t} - 35 \ .$$

(a) (i) In which month and year will the machine cease producing any more cloth?

(ii) Estimate the total amount of cloth, to the nearest km, produced during the lifespan of the machine.

(5 marks)

(b) Suppose the cost of producing 1 km of cloth is US\$300; the monthly maintenance fee of the machine is US\$300 and the selling price of 1 km of cloth is US\$800. In which month and year will the greatest monthly profit be obtained? Find also the profit, to the nearest US\$, in that month.

(6 marks)

(c) The machine is regarded as 'inefficient' when the monthly profit falls below US\$500 and it should then be discarded. Find the month and year when the machine should be discarded. Explain your answer briefly.

(4 marks)

(1994 ASL-M&S Q9)

Questions involve other topics

28. The chickens in a farm are infected by a certain bird flu. The number of chickens (in thousand) in the farm is modelled by

$$N=\frac{27}{2+\alpha te^{\beta t}}\ ,$$

where $t\ (\geq 0)$ is the number of days elapsed since the start of the spread of the bird flu and α and β are constants.

(a) Express $\ln\left(\frac{27-2N}{Nt}\right)$ as a linear function of t.

(2 marks)

(b) It is given that the slope and the intercept on the horizontal axis of the graph of the linear function obtained in (a) are -0.1 and $10\ln 0.03$ respectively.

(i) Find α and β .

(ii) Will the number of chickens in the farm be less than 12 thousand on a certain day after the start of the spread of the bird flu? Explain your answer.

(iii) Describe how the rate of change of the number of chickens in the farm varies during the first 20 days after the start of the spread of the bird flu. Explain your answer.

(10 marks)

(2016 DSE-MATH-M1 Q12)

29. Let $y=\frac{1-e^{4x}}{1+e^{8x}}$.

(a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x=0$.

(b) Let $(z^2+1)e^{3z}=e^{\alpha+\beta x}$, where α and β are constants.

(i) Express $\ln(z^2+1)+3z$ as a liner function of x .

(ii) It is given that the graph of the linear function obtained in (b)(i) passes through the origin and the slope of the graph is 2. Find the values of α and β .

(iii) Using the values of α and β obtained in (b)(ii), find the value of $\frac{\mathrm{d}y}{\mathrm{d}z}$ when $z=0$.

(7 marks) (2007 ASL-M&S Q3)

30. The population of a kind of bacterium $\mathrm{p}(t)$ at time t (in days elapsed since 9 am on 16/4/2010, and can be positive or negative) is modelled by

$$\mathrm{p}(t)=\frac{a}{b+e^{-t}}+c\ ,\quad -\infty<t<\infty$$

where a , b and c are positive constants. Define the *primordial population* be the population of the bacterium long time ago and the *ultimate population* be the population of the bacterium after a long time.

(a) Find, in terms of a , b and c ,

(i) the time when the growth rate attains the maximum value;

(ii) the *primordial population*;

(iii) the *ultimate population*.

(5 marks)

(b) A scientist studies the population of the bacterium by plotting a linear graph of $\ln[\mathrm{p}(t)-c]$ against $\ln(b+e^{-t})$ and the graph shows the intercept on the vertical axis to be $\ln 8000$. If at 9 am on 16/4/2010 the population and the growth rate of the bacterium are 6000 and 2000 per day respectively, find the values of a , b and c .

(3 marks)

(c) Another scientist claims that the population of the bacterium at the time of maximum growth rate is the mean of the primordial population and ultimate population. Do you agree? Explain your answer.

(2 marks)

(d) By expressing e^{-t} in terms of a , b , c and $\mathrm{p}(t)$, express $\mathrm{p}'(t)$ in the form of $\frac{-b}{a}[\mathrm{p}(t)-\alpha][\mathrm{p}(t)-\beta]$, where $\alpha<\beta$.

Hence express α and β in terms of a , b and c .

Sketch $\mathrm{p}'(t)$ against $\mathrm{p}(t)$ for $\alpha<\mathrm{p}(t)<\beta$ and hence verify your answer in (c).

(5 marks)

(2010 ASL-M&S Q9)

31. A merchant sells compact discs (CDs). A market researcher suggests that if each CD is sold for $\$x$, the number $N(x)$ of CDs sold per week can be modeled by

$$N(x) = ae^{-bx}$$

where a and b are constants.

The merchant wants to determine the values of a and b based on the following results obtained from a survey:

x	20	30	40	50
$N(x)$	450	301	202	136

(a) (i) Express $\ln N(x)$ as a linear function of x .

(ii) Use a graph paper to estimate graphically the values of a and b correct to 2 decimal places.

(7 marks)

(b) Suppose the merchant wishes to sell 400 CDs in the next week. Use the values of a and b estimated in (a) to determine the price of each CD. Give your answer correct to 1 decimal place.

(2 marks)

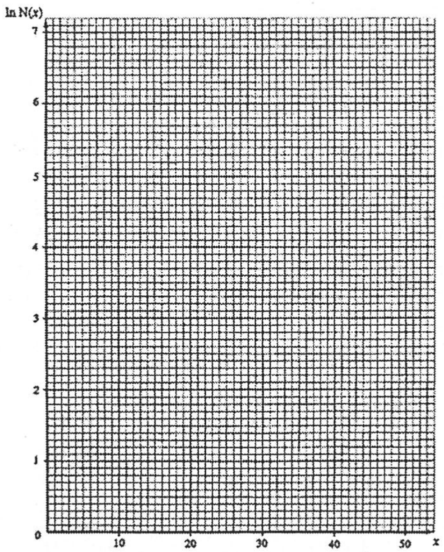
(c) It is known that the merchant obtains CDs at a cost of \$10 each. Let $G(x)$ dollars denote the weekly profit. Using the values of a and b estimated in (a),

(i) express $G(x)$ in terms of x.

(ii) find $G'(x)$ and hence determine the selling price for each CD in order to maximize the profit.

(6 marks)

(1995 ASL-M&S Q8)



32. A biologist studied the population of fruit fly A under limited food supply. Let t be the number of days since the beginning of the experiment and $N'(t)$ be the number of fruit fly A at time t. The biologist modelled the rate of change of the number of fruit fly A by

$$N'(t) = \frac{20}{1+he^{-kt}} \qquad (t \geq 0)$$

where h and k are positive constants.

(a) (i) Express $\ln\left(\frac{20}{N'(t)}-1\right)$ as a linear function of t .

(ii) It is given that the intercepts on the vertical axis and the horizontal axis of the graph of the linear function in (i) are 1.5 and 7.6 respectively. Find the values of h and k .

(4 marks)

(b) Take $h = 4.5$ and $k = 0.2$, and assume that $N(0) = 50$.

(i) Let $v = h + e^{kt}$, find $\frac{dv}{dt}$.

Hence, or otherwise, find $N(t)$.

(ii) The population of fruit fly B can be modelled by

$$M(t) = 21\left(t + \frac{h}{k}e^{-kt}\right) + b \ ,$$

where b is a constant. It is known that $M(20) = N(20)$.

(1) Find the value of b .

(2) The biologist claims that the number of fruit fly A will be smaller than that of fruit fly B for $t > 20$. Do you agree? Explain your answer.

[Hint: Consider the difference between the rates of change of the two populations.]

(11 marks)

(2008 ASL-M&S Q8)

33. In a certain country, the daily rate of change of the amount of oil production P, in million barrels per day, can be modelled by

$$\frac{dP}{dt}=\frac{k-3t}{1+ae^{-bt}}$$

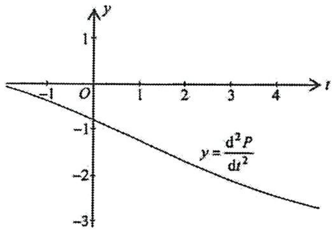
where $t\ (\geq 0)$ is the time measured in days. When $\ln\left(\frac{k-3t}{\frac{dP}{dt}}-1\right)$ is plotted against t, the graph is a straight line with slope -0.3 and the intercept on the horizontal axis 0.32. Moreover, P attains its maximum when $t=3$.

(a) Find the values of a, b and k.

(5 marks)

(b) (i) Using trapezoidal rule with 6 subintervals, estimate the total amount of oil production from $t=0$ to $t=3$.

(ii)



The figure shows the graph of $y=\frac{d^2P}{dt^2}$. Using the graph, determine whether the estimation in (i) is an under-estimate or an over-estimate.

(4 marks)

(c) The daily rate of change of the demand for oil D, in million barrels per day, can be modelled by

$$\frac{dD}{dt}=1.63^{2-0.1t}$$

where $t\ (\geq 0)$ is the time measured in days.

(i) Let $y=\alpha^{\beta x}$, where α, β ($\alpha>0$, $\alpha\neq 1$ and $\beta\neq 0$) are constants. Find $\frac{dy}{dx}$ in terms of x.

(ii) Find the demand of oil from $t=0$ to $t=3$.

(iii) Does the overall oil production meet the overall demand of oil from $t=0$ to $t=3$? Explain your answer.

(6 marks)

(part (c)(i) is out of syllabus) (2013 ASL-M&S Q8)

34. A textile factory has bought two new dyeing machines P and Q. The two machines start to operate at the same time and will emit sewage into a lake near the factory. The manager of the factory estimates the amount of sewage emitted (in tonnes) by the two machines and finds that the rates of emission of sewage by the two machines P and Q can be respectively modelled by

$$p'(t)=4.5+2t(1+6t)^{\frac{-2}{3}} \text{ and}$$

$$q'(t)=3+\ln(2t+1),$$

where $t\ (\geq 0)$ is the number of months that the machines have been in operation.

(a) By using a suitable substitution, find the total amount of sewage emitted by machine P in the first year of operation.

(4 marks)

(b) (i) By using the trapezoidal rule with 5 sub-intervals, estimate the total amount of sewage emitted by machine Q in the first year of operation.

(ii) The manager thinks that the amount of sewage emitted by machine Q will be less than that emitted by machine P in the first year of operation. Do you agree? Explain your answer.

(5 marks)

(c) The manager studies the relationship between the environmental protection tax R (in million dollars) paid by the factory and the amount of sewage x (in tonnes) emitted by the factory. He uses the following model:

$$R=16-ae^{-bx},$$

where a and b are constants.

(i) Express $\ln(16-R)$ as a linear function of x.

(ii) Given that the graph of the linear function in (c)(i) passes through the point $(-10,1)$ and the x-intercept of the graph is 90, find the values of a and b.

(iii) In addition to the sewage emitted by the machines P and Q, the other operations of the factory emit 80 tonnes of sewage annually. Using the model suggested by the manager and the values of a and b found in (c)(ii), estimate the tax paid by the factory in the first year of the operation of machines P and Q.

(6 marks)

(2012 ASL-M&S Q8)

35. According to the past production record, an oil company manager modelled the rate of change of the amount of oil production in thousand barrels by

$$f(t)=5+2^{-kt+h} ,$$

where h and k are positive constants and $t(\geq 0)$ is the time measured in months.

(a) Express $\ln(f(t)-5)$ as a linear function of t .

(1 marks)

(b) Given that the slope and the intercept on the vertical axis of the graph of the linear function in (a) are –0.35 and 1.39 respectively, find the values of h and k correct to 1 decimal place.

(2 marks)

(c) The manager decides to start a production improvement plan and predicts the rate of change of the amount of oil production in thousand barrels by

$$g(t)=5+\ln(t+1)+2^{-kt+h} ,$$

where h and k are the values obtained in (b) correct to 1 decimal place, and $t(\geq 0)$ is the time measured in months from the start of the plan.

Using the trapezoidal rule with 5 sub-intervals, estimate the total amount of oil production in thousand barrels from $t=2$ to $t=12$.

(2 marks)

(d) It is known that $g(t)$ in (c) satisfies

$$\frac{d^2g(t)}{dt^2}=p(t)-q(t), \text{ where } q(t)=\frac{1}{(t+1)^2} .$$

(i) If $2^t=e^{at}$ for all $t\geq 0$, find a .

(ii) Find $p(t)$.

(iii) It is known that there is no intersection between the curve $y=p(t)$ and the curve $y=q(t)$, where $2\leq t\leq 12$. Determine whether the estimate in (c) is an over-estimate or under -estimate.

(10 marks)

(2003 ASL-M&S Q8)

36. The monthly cost $C(t)$ at time t of operating a certain machine in a factory can be modelled by

$$C(t)=ae^{bt}-1 \qquad (0<t\leq 36),$$

where t is in month and $C(t)$ is in thousand dollars.

Table 2 shows the values of $C(t)$ when $t=1, 2, 3, 4$.

Table

t	1	2	3	4
$C(t)$	1.21	1.44	1.70	1.98

(a) (i) Express $\ln[C(t)+1]$ as a linear function of t .

(ii) Use the table and a graph paper to estimate graphically the values of a and b correct to 1 decimal place.

(iii) Using the values of a and b found in (a)(ii), estimate the monthly cost of operating this machine when $t=36$.

(8 marks)

(b) The monthly income $P(t)$ generated by this machine at time t can be modelled by

$$P(t)=439-e^{0.2t} \qquad (0<t\leq 36) ,$$

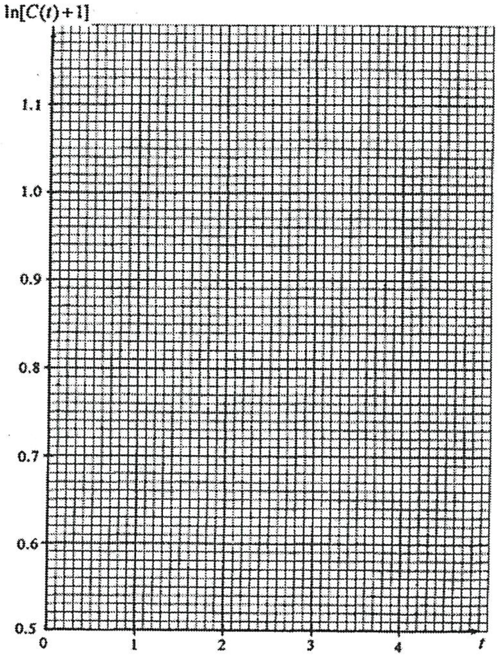
where t is in month and $P(t)$ is in thousand dollars.

The factory will stop using this machine when the monthly cost of operation exceeds the monthly income.

(i) Find the value of t when the factory stops using this machine. Give the answer correct to the nearest integer.

(ii) What is the total profit generated by this machine? Give the answer correct to the nearest thousand dollars.

(7 marks)



(1996 ASL-M&S Q10)

37. A chemical plant discharges pollutant to be a lake at an unknown rate of $r(t)$ units per month, where t is the number of months that the plant has been in operation.
Suppose that $r(0) = 0$.
The government measured $r(t)$ once every two months and reported the following figures:

t	2	4	6	8
$r(t)$	11	32	59	90

(a) Use the trapezoidal rule to estimate the total amount of pollutant which entered the lake in the first 8 months of the plant's operation.

(2 mark)

(b) An environmental scientist suggests that

$$r(t) = at^b,$$

where a and b are constants.

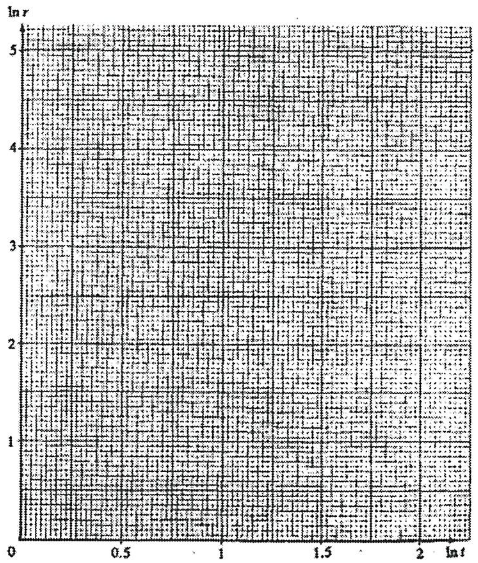
(i) Use a graph paper to estimate graphically the values of a and b correct to 1 decimal place.

(ii) Based on this scientist's model, estimate the total amount of pollutant, correct to 1 decimal place, which entered the lake in the first 8 months of the plant's operation.

(8 mark)

(c) It is known that no life can survive when 1000 units of pollutant have entered the lake. Adopting the scientist's model in (b), how long does it take for the pollutant from the plant to destroy all life in the lake? Give your answer correct to the nearest month.

(5 mark)



(1994 ASL-M&S Q10)

2021 DSE Q6

(a) Expand e^{-6x} in ascending powers of x as far as the term in x^4.

(b) Find the constant k such that the coefficient of x^4 in the expansion of $e^{-6x}(1-kx^2)^5$ is -26.

(5 marks)

Out of Syllabus

1. (a) Expand e^{-2x} in ascending powers of x as far as the term in x^3.

   (b) Using (a), expand $\frac{(1+x)^{\frac{1}{2}}}{e^{2x}}$ in ascending powers of x as far as the term in x^3.

   State the range of values of x for which the expansion is valid.

   (6 marks) (1999 ASL-M&S Q2)

2. Exponential and Logarithmic Functions

1. (2019 DSE-MATH-M1 Q6)

Solution	Marks
(a) e^{-18x}	
$=1+(-18x)+\dfrac{(-18x)^2}{2!}+\cdots$	1M
$=1-18x+162x^2+\cdots$	1A
(b) $(1+4x)^n$	
$=1+C_1^n(4x)+C_2^n(4x)^2+\cdots+C_n^n(4x)^n$	1M
$=1+4C_1^n x+16C_2^n x^2+\cdots+4^n x^n$	
$16C_2^n-72C_1^n+162=-38$	1M
$16\left(\dfrac{n(n-1)}{2}\right)-72n+162=-38$	
$n^2-10n+25=0$	1M
$n=5$	1A
	----------(6)

2. (2018 DSE-MATH-M1 Q6)

Solution	Marks	Remarks
(a) $e^{kx}+e^{2x}$		
$=\left(1+kx+\dfrac{(kx)^2}{2!}+\cdots\right)+\left(1+2x+\dfrac{(2x)^2}{2!}+\cdots\right)$	1M	for expanding e^{kx} or e^{2x}
$=2+(k+2)x+\dfrac{(k^2+4)}{2}x^2+\cdots$	1A	
(b) $(1-3x)^8$		
$=1+C_1^8(-3x)+C_2^8(-3x)^2+\cdots$	1M	
$=1-24x+252x^2+\cdots$		
$e^{kx}+e^{2x}-1$		
$=1+(k+2)x+\dfrac{(k^2+4)}{2}x^2+\cdots$		
$(1)(k+2)+(-24)(1)=(1)\left(\dfrac{k^2+4}{2}\right)+(-24)(k+2)+(252)(1)$	1M+1M	
$k^2-50k+456=0$		
$k=12$ or $k=38$	1A	
	----------(6)	

3. (2017 DSE-MATH-M1 Q5)

Solution	Marks	Remarks
(a) $(1+e^{3x})^2$		
$=1+2e^{3x}+e^{6x}$	1M	
$=1+2\left(1+3x+\dfrac{(3x)^2}{2!}+\cdots\right)+\left(1+6x+\dfrac{(6x)^2}{2!}+\cdots\right)$	1M	for expanding e^{3x} or e^{6x}
$=4+12x+27x^2+\cdots$	1A	
$(1+e^{3x})^2$		
$=\left(1+1+3x+\dfrac{(3x)^2}{2!}+\cdots\right)^2$	1M	for expanding e^{3x}
$=(2)(2)+(2)(2)(3x)+(3x)(3x)+(2)(2)\left(\dfrac{9x^2}{2}\right)+\cdots$	1M	
$=4+12x+27x^2+\cdots$	1A	
(b) $(5-x)^4$		
$=5^4-C_1^4(5^3)x+C_2^4(5^2)x^2-C_3^4(5)x^3+x^4$	1M	
$=625-500x+150x^2-20x^3+x^4$		
The required coefficient		
$=(625)(27)+(-500)(12)+(150)(4)$	1M	withhold 1M if the step is skipped
$=11475$	1A	
	----------(6)	

(a)	Very good. Most candidates were able to expand $(1+e^{3x})^2$.
(b)	Very good. Most candidates were able to find the coefficient of x^2 .

4. (2016 DSE-MATH-M1 Q5)

(a) e^{kx}

$= 1 + kx + \frac{(kx)^2}{2!} + \cdots$

$= 1 + kx + \frac{k^2x^2}{2} + \cdots$ 1A

(b) $(1+2x)^7 e^{kx}$

$= \left(1 + C_1^7(2x) + C_2^7(2x)^2 + \cdots + (2x)^7\right)\left(1 + kx + \frac{k^2x^2}{2} + \cdots\right)$ 1M

$= \left(1 + 14x + 84x^2 + \cdots + (2x)^7\right)\left(1 + kx + \frac{k^2x^2}{2} + \cdots\right)$

$\therefore\ 14 + k = 8$ 1M

$k = -6$

The coefficient of x^2

$= (1)\left(\frac{(-6)^2}{2}\right) + 14(-6) + (84)(1)$ 1M

$= 18$ 1A

----------(5)

(a)	Very good. A very high proportion of the candidates were able to expand e^{kx} while some candidates were unable to simplify the coefficient of x^2 .
(b)	Very good. More than 70% of the candidates were able to find the coefficient of x^2 while a small number of candidates made careless mistakes in expanding $(1+2x)^7$.

5. (2015 DSE-MATH-M1 Q5)

(a) e^{-4x}

$= 1 + (-4x) + \frac{(-4x)^2}{2!} + \cdots$ 1M

$= 1 - 4x + 8x^2 - \cdots$ 1A

(b) $(2+x)^5$

$= 2^5 + C_1^5(2^4)x + C_2^5(2^3)x^2 + \cdots + x^5$ 1M

$= 32 + 80x + 80x^2 + \cdots + x^5$

The required coefficient

$= (1)(80) + (-4)(80) + (8)(32)$ 1M

$= 16$ 1A

----------(5)

(a)	Very good. Most candidates were able to expand e^{-4x} while a few candidates failed to show working steps.
(b)	Very good. Most candidates were able to find the coefficient of x^2 while a few candidates made a carless mistake in expanding $(2+x)^5$ as $2^5 + C_1^5(2^4)x + C_2^5(2^3)x^2 + \cdots + x^5$.

6. (2013 DSE-MATH-M1 Q4)

(a) (i) $u = ae^{-bx}$

$\ln u = \ln a - bx$ 1A

(ii) $y = \frac{8(1-ae^{-bx})}{1+ae^{-bx}}$

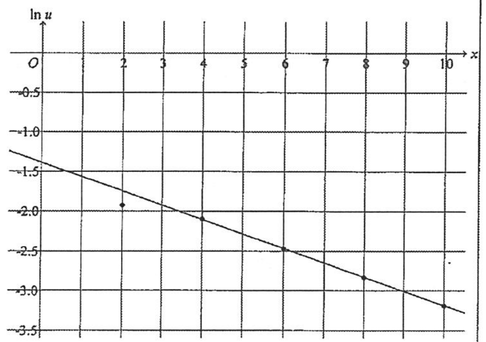
$= \frac{8-8u}{1+u}$

$u = \frac{8-y}{8+y}$ 1A

(b) (i) By (a), $\ln\frac{8-y}{8+y} = \ln a - bx$

x	2	4	6	8	10
$\ln\frac{8-y}{8+y}$	−1.93	−2.10	−2.47	−2.83	−3.19

1A For any two pairs of values



1A

From the graph, we see that the value $y = 5.97$ is incorrect. 1A

(ii) The y-intercept $= \ln a \approx -1.4$ 1M (For either one)

$\therefore\ a \approx 0.25$

The slope $= -b \approx \frac{-3.19-(-2.10)}{10-4}$

$\therefore\ b \approx 0.18$ 1A For both a and b

(7)

(a)	(i) (ii)	Excellent. Very good. A few candidates found $\frac{dy}{du}$ or $\ln y$ which was not required.
(b)	(i)	Satisfactory. Many candidates used values of $\ln u$ with only one decimal place to plot graphs, which made it difficult to determine which value of y should be incorrect. Some others made mistakes in plotting the graph although using more accurate values of $\ln u$.
	(ii)	Fair. Some candidates used correct algebraic method but with values of $\ln u$ not accurate enough. Some others used graphs plotted in (i), but were not able to get the value of the ($\ln u$)-intercept of the straight line accurate enough.

7. (2012 DSE-MATH-M1 Q3)

(a) $P = ae^{\frac{kt}{40}} - 5$

$\ln(P+5) = \frac{k}{40}t + \ln a$ 1A

(b)

t	2	4	6	8	10
P	2.36	2.81	3.23	3.55	4.01
$\ln(P+5)$	2.00	2.06	2.11	2.15	2.20

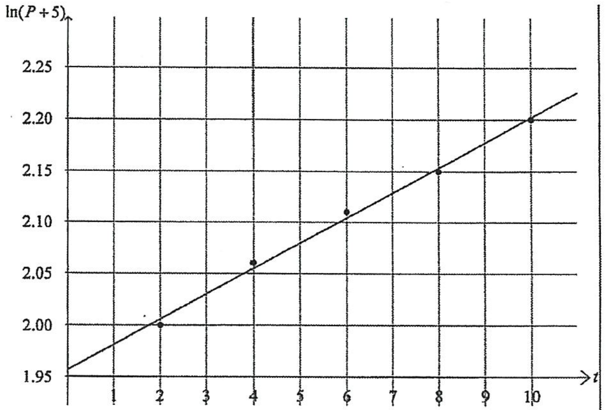
1M

From the graph on the next page, $\ln a \approx 1.96$ 1M (Either one)

$a \approx 7$

$\frac{k}{40} \approx \frac{2.21 - 1.96}{10 - 0}$ (Either one)

$k \approx 1$ 1A For both a and k



1A

(5)

(a)	Very good.
(b)	Very good. Candidates performed well in plotting graphs, but a small number of them did not use the plotting to estimate the values of a and k.

8. (SAMPLE DSE-MATH-M1 Q10)

(a) $N(t) = \frac{500}{1 + ae^{-kt}}$

$\ln\left(\frac{500}{N(t)} - 1\right) = \ln(ae^{-kt})$

$= -kt + \ln a$ 1A

(b)

t	5	10	15	20
$\ln\left(\frac{500}{N(t)} - 1\right)$	3.6	2.6	1.6	0.6

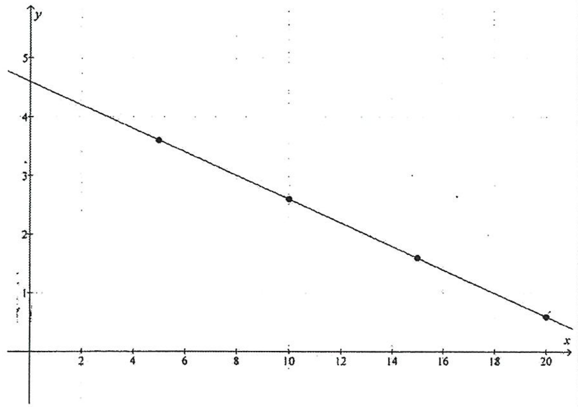
From the graph below, $\ln a \approx 4.6$ Accept $4.5 \le \ln a \le 4.7$

$a \approx 99.48431564$

≈ 99.5 (correct to 1 d.p.) 1A Accept $90.0 \le a \le 110.0$

$-k = \frac{0.6 - 3.6}{20 - 5}$ 1M

$k = 0.2$ 1A



1A

(c) $270 = \frac{500}{1 + 99.48431564e^{-0.2t}}$

$t = 23.80171325$ 1A Accept $23.3 \le t \le 24.3$

24 days after the outbreak of the disease, the number of fish infected by the disease will reach 270 .

(6)

9. (2006 ASL-M&S Q2)

(a) $S(9)=S(19)$

$2(10^2)e^{-9\lambda}+15=2(20^2)e^{-19\lambda}+15$

$e^{10\lambda}=4$

$\lambda=\dfrac{\ln 4}{10}$ 1A

Thus, we have $\lambda=\dfrac{\ln 2}{5}$.

(b) $S(t)=2(t+1)^2e^{-\lambda t}+15$

$\dfrac{dS(t)}{dt}=2\left(2(t+1)e^{-\lambda t}-\lambda(t+1)^2e^{-\lambda t}\right)$ 1A

$=2(t+1)(2-\lambda-\lambda t)e^{-\lambda t}$

$\dfrac{dS(t)}{dt}=0$ when $t=\dfrac{2-\lambda}{\lambda}=\dfrac{2-\frac{\ln 2}{5}}{\frac{\ln 2}{5}}=\dfrac{10-\ln 2}{\ln 2}=T\approx 13.42695041$ 1M

$\dfrac{dS(t)}{dt}\begin{cases}>0 & \text{if } 0\le t<T\\ =0 & \text{if } t=T\\ <0 & \text{if } t>T\end{cases}$ 1M for testing + 1A

Therefore, $S(t)$ attains its greatest value when $t=T$.

The greatest value of $S(t)$

$=2\left(\dfrac{10-\ln 2}{\ln 2}+1\right)^2 e^{\frac{\ln 2}{5}-2}+15$

≈ 79.71368176

<90

Thus, the temperature will not get higher than 90°C . 1A f.t.

$S(t)=2(t+1)^2e^{-\lambda t}+15$

$\dfrac{dS(t)}{dt}=2\left(2(t+1)e^{-\lambda t}-\lambda(t+1)^2e^{-\lambda t}\right)$ 1A

$=2(t+1)(2-\lambda-\lambda t)e^{-\lambda t}$

$\dfrac{d^2S(t)}{dt^2}=2\left(2e^{-\lambda t}-2\lambda(t+1)e^{-\lambda t}-2\lambda(t+1)e^{-\lambda t}+\lambda^2(t+1)^2e^{-\lambda t}\right)$

$=2\left((\lambda^2-4\lambda+2)+(2\lambda^2-4\lambda)t+\lambda^2t^2\right)e^{-\lambda t}$

$\dfrac{dS(t)}{dt}=0$ when $t=\dfrac{2-\lambda}{\lambda}=\dfrac{2-\frac{\ln 2}{5}}{\frac{\ln 2}{5}}=\dfrac{10-\ln 2}{\ln 2}=T\approx 13.42695041$ 1M

$\left.\dfrac{d^2S(t)}{dt^2}\right|_{t=T}=-4e^{-\lambda T}<0$ 1M for testing + 1A

Note that there is only one local maximum.

So, $S(t)$ attains its greatest value when $t=T$.

The greatest value of $S(t)$

$=2\left(\dfrac{10-\ln 2}{\ln 2}+1\right)^2 e^{\frac{\ln 2}{5}-2}+15$

≈ 79.71368176

<90

Thus, the temperature will not get higher than 90°C . 1A f.t.

----------(6)

Fair. Many candidates did not get marks in (a) because they did not give the 'exact value' as required.

10. (2007 ASL-M&S Q3)

(a) $y=\dfrac{1-e^{4x}}{1+e^{8x}}$

$\dfrac{dy}{dx}=\dfrac{(1+e^{8x})(-4e^{4x})-(1-e^{4x})(8e^{8x})}{(1+e^{8x})^2}$ 1M for quotient rule or product rule

When $x=0$, we have $\dfrac{dy}{dx}=-2$. 1A

(b) (i) Since $(z^2+1)e^{3z}=e^{\alpha+\beta x}$, we have $\ln(z^2+1)+3z=\alpha+\beta x$. 1A

(ii) Since the graph of the linear function passes through the origin and the slope of the graph is 2 , we have $\alpha=0$ and $\beta=2$. 1A for both correct

(iii) $\ln(z^2+1)+3z=2x$

$\dfrac{2z}{z^2+1}+3=2\dfrac{dx}{dz}$

Therefore, we have $\left.\dfrac{dx}{dz}\right|_{z=0}=\dfrac{3}{2}$. 1A

Note that $x=0$ when $z=0$.

Also note that $\left.\dfrac{dy}{dx}\right|_{z=0}=-2$.

$\left.\dfrac{dy}{dz}\right|_{z=0}$

$=\left(\left.\dfrac{dy}{dx}\right|_{z=0}\right)\left(\left.\dfrac{dx}{dz}\right|_{z=0}\right)$

$=(-2)\left(\dfrac{3}{2}\right)$ 1M for chain rule

$=-3$ 1A

$y=\dfrac{1-e^{6z+2\ln(z^2+1)}}{1+e^{12z+4\ln(z^2+1)}}$ 1A

$y=\dfrac{1-(z^2+1)^2e^{6z}}{1+(z^2+1)^4e^{12z}}$

$\dfrac{dy}{dz}=\dfrac{\left(1+(z^2+1)^4e^{12z}\right)\left(-6(z^2+1)^2e^{6z}-2(z^2+1)(2z)e^{6z}\right)-\left(1-(z^2+1)^2e^{6z}\right)\left(12(z^2+1)^4e^{12z}+4(z^2+1)^3(2z)e^{12z}\right)}{\left(1+(z^2+1)^4e^{12z}\right)^2}$ 1M for quotient rule or product rule

$\left.\dfrac{dy}{dz}\right|_{z=0}=-3$ 1A

----------(7)

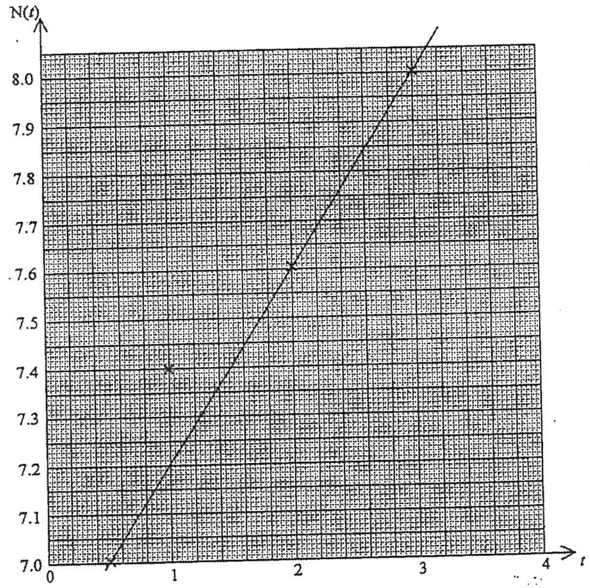
Good. Most candidates could handle quotient rule and product rule. It is more efficient to apply $\dfrac{dy}{dz}=\dfrac{dy}{dx}\cdot\dfrac{dx}{dz}$ but some candidates went through the tedious way by expressing the function of y in terms of z .

11. (2002 ASL-M&S Q3)

$N(t) = 900a^{kt}$

$\ln N(t) = (k \ln a)t + \ln 900$ — 1A

t	0.5	1.0	2.0	3.0
$N(t)$	1100	1630	2010	2980
$\ln N(t)$	7.0031	7.3963	7.6059	7.9997



1M

1A

At $t = 1.0$, $N(t) = 1630$ is incorrect,

$\ln N(2.5) \approx 7.8$

$\therefore$ $N(2.5) \approx 2440$ — 1A a–1 for more than 3 s.f.

(Accept : $N(2.5) \in [2420, 2470]$)

----------(4)

DSE Mathematics Module 1 — 2. Exponential and Logarithmic Functions

Section B

12. (2015 DSE-MATH-M1 Q12)

(a) $S = \dfrac{200}{1 + a2^{bt}}$

$\dfrac{200}{S} - 1 = a2^{bt}$ — 1M

$\ln\left(\dfrac{200}{S} - 1\right) = (b \ln 2)t + \ln a$ — 1A

----------(2)

(b) (i) $\ln a = \ln 4$

$a = 4$ — 1A

$b \ln 2 = \dfrac{0 - \ln 4}{4 - 0}$

$b = -0.5$ — 1A

(ii) $\dfrac{dS}{dt}$

$= \dfrac{-200(4)2^{-0.5t}(-0.5)\ln 2}{(1 + 4(2^{-0.5t}))^2}$ — 1M for $\dfrac{d}{dt}2^{bt}$

$= \dfrac{(400 \ln 2)2^{-0.5t}}{(1 + 4(2^{-0.5t}))^2}$ — 1A

$\dfrac{d^2S}{dt^2}$

$= \dfrac{-200(\ln 2)^2(1 + 4(2^{-0.5t}))^2 2^{-0.5t} + 1600(\ln 2)^2(1 + 4(2^{-0.5t}))2^{-t}}{(1 + 4(2^{-0.5t}))^4}$ — 1M for quotient rule

$= \dfrac{-200(\ln 2)^2 2^{-0.5t}(1 - 4(2^{-0.5t}))}{(1 + 4(2^{-0.5t}))^3}$ — 1A

(iii) Note that $\dfrac{dS}{dt} > 0$ for $0 \le t \le 48$. — 1M

Therefore, S increases for $0 \le t \le 48$. — 1A f.t.

For $\dfrac{d}{dt}\left(\dfrac{dS}{dt}\right) = 0$, we have $4(2^{-0.5t}) = 1$.

Hence, we have $\dfrac{d}{dt}\left(\dfrac{dS}{dt}\right) = 0$ when $t = 4$.

t	$0 \le t < 4$	$t = 4$	$4 < t \le 48$
$\dfrac{d}{dt}\left(\dfrac{dS}{dt}\right)$	+	0	–

1M+1A

Thus, $\dfrac{dS}{dt}$ increases for $0 \le t \le 4$ and

$\dfrac{dS}{dt}$ decreases for $4 \le t \le 48$. — 1A f.t.

--------(11)

(a)	Very good. Most candidates were able to express $\ln\left(\dfrac{200}{S} - 1\right)$ as a linear function of t.
(b) (i)	Very good. A few candidates failed to use the slope of the linear function as a means to find the value of the unknown b.
(ii)	Fair. Many candidates failed to differentiate $2^{-0.5t}$ with respect to t correctly when finding the required derivatives $\dfrac{dS}{dt}$ and $\dfrac{d^2S}{dt^2}$.
(iii)	Poor. Only a few candidates were able to make use of the sign of $\dfrac{dS}{dt}$ to discuss the behaviour of S . Most candidates failed to determine the change of the sign of $\dfrac{d^2S}{dt^2}$ correctly.

13. (2014 DSE-MATH-M1 Q11)

(a) $\lim_{t\to\infty}\frac{340}{2+e^{-t}-2e^{-2t}}=\frac{340}{2+0-2\cdot 0}$ — 1M

$=170$

Hence the value of y will not exceed 171 in the long run. — 1A

(2)

(b) $\frac{dy}{dt}=340[-(2+e^{-t}-2e^{-2t})^{-2}](-e^{-t}+4e^{-2t})$ — 1A

$=\frac{340(e^{-t}-4e^{-2t})}{(2+e^{-t}-2e^{-2t})^2}$

$\therefore\ \frac{dy}{dt}=0$ when $e^{-t}-4e^{-2t}=0$ — 1M (For $\frac{dy}{dt}=0$)

i.e. $t=\ln 4$ — 1A

t	$0\le t<\ln 4$	$t=\ln 4$	$t>\ln 4$
$\frac{dy}{dt}$	-ve	0	+ve

— 1M

Hence y is minimum when $t=\ln 4$.

The minimum value of $y=\frac{340}{2+e^{-\ln 4}-2e^{-2\ln 4}}$

$=160$ — 1A

When $t=0$, $y=\frac{340}{2+e^{0}-2e^{0}}$

$=340$

As the graph of y is continuous, and by (a), the greatest value of y is 340 and the least value of y is 160. — 1A

(6)

(c) (i) $y=\frac{340}{2+e^{-t}-2e^{-2t}}$

$2y+ye^{-t}-2ye^{-2t}=340$

$2y(e^{-t})^2-ye^{-t}+340-2y=0$ — 1A

(ii) Since $e^{-\alpha}$ and $e^{\alpha-3}$ are roots of the equation in (i),

$\frac{340-2y}{2y}=e^{-\alpha}e^{\alpha-3}$ — 1M

$340-2y=2ye^{-3}$

Hence the equation becomes $2y(e^{-t})^2-ye^{-t}+2ye^{-3}=0$

i.e. $2(e^{-t})^2-e^{-t}+2e^{-3}=0$ — 1A

$\therefore\ e^{-\alpha}=\frac{1+\sqrt{1-16e^{-3}}}{4}$ or $\frac{1-\sqrt{1-16e^{-3}}}{4}$ (rejected as $e^{-\alpha}$ is the greater root)

i.e. $\alpha=-\ln\frac{1+\sqrt{1-16e^{-3}}}{4}$ — 1A (OR $\ln\frac{1-\sqrt{1-16e^{-3}}}{4}+3$ OR 1.0140)

(4)

(a)	Good. Some candidates thought that $\lim_{t\to+\infty}e^{-t}=1$.
(b)	Fair. Quite a lot of candidates failed to consider both the value of y at $t=0$ and the limit found in (a).
(c)	Very poor. Most candidates wrote wrongly the equation required in (i).

14. (PP DSE-MATH-M1 Q11)

(a) $P'(t)=kte^{\frac{a}{20}t}$

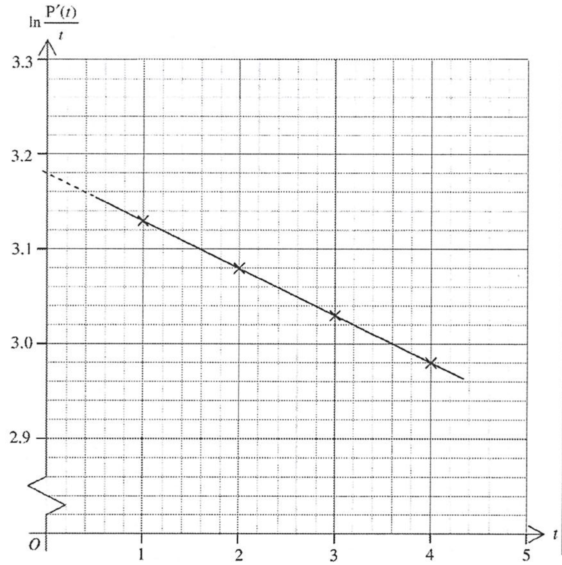
$\ln\frac{P'(t)}{t}=\frac{a}{20}t+\ln k$ — 1A

(1)

(b)

t	1	2	3	4
$P'(t)$	22.83	43.43	61.97	78.60
$\ln\frac{P'(t)}{t}$	3.13	3.08	3.03	2.98

— 1A



— 1A

From the graph, $\frac{a}{20}\approx\frac{2.98-3.13}{4-1}$ — 1M

$a\approx-1$ — 1A

From the graph, $\ln k\approx 3.18$ — (Either one)

$k\approx 24$ — 1A

(5)

(c) (i) $\frac{d}{dt}P'(t)=\frac{d}{dt}\left(24te^{\frac{-t}{20}}\right)$

$=24e^{\frac{-t}{20}}\left(1-\frac{t}{20}\right)$ 1A

$\therefore\ \frac{d}{dt}P'(t)=0$ when $t=20$

t	<20	20	>20
$\frac{d}{dt}P'(t)$	+ve	0	-ve

1M

Alternative Solution

$\frac{d^2}{dt^2}P'(t)=24e^{\frac{-t}{20}}\left[\frac{-1}{20}\left(1-\frac{t}{20}\right)+\frac{-1}{20}\right]$

$=\frac{6}{5}e^{\frac{-t}{20}}\left(\frac{t}{20}-2\right)$

$\therefore\ \frac{d^2}{dt^2}P'(t)<0$ when $t=20$ } 1M

Hence the rate of change of the population size is greatest when $t=20$. 1A

(ii) $\frac{d}{dt}\left(te^{\frac{-t}{20}}\right)=e^{\frac{-t}{20}}-\frac{1}{20}te^{\frac{-t}{20}}$ 1A

$24te^{\frac{-t}{20}}=480e^{\frac{-t}{20}}-480\frac{d}{dt}\left(te^{\frac{-t}{20}}\right)$

$\int 24te^{\frac{-t}{20}}\,dt=-9600e^{\frac{-t}{20}}-480te^{\frac{-t}{20}}+C$ 1M

$P(t)=C-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}$ 1A

Since $P(0)=30$, we have

$C-480(0)e^0-9600e^0=30$ 1M

$C=9630$

$\therefore\ P(t)=9630-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}$ 1A

(iii) $\lim_{t\to\infty}P(t)=\lim_{t\to\infty}\left(9630-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}\right)$

$=9630$ 1A

$\therefore$ the population size after a very long time is estimated to be 9630 thousands.

(9)

(a)	良好。 大部分學生能正確表達線性函數。
(b)	良好。 部分學生看漏了精確度的要求。
(c) (i)	平平。 部分學生在取得第一導數並使該導數相等於零之後，沒有作出極大測試。
(ii)	平平。 大部分學生不懂如何利用 $\frac{d}{dx}\left(te^{\frac{a}{20}t}\right)$ 的結果求 $P(t)$。
(iii)	甚差。 大部分學生因爲在前部分出錯而得出錯誤的結論。

15. (2011 ASL-M&S Q9)

(a) (i) $P=\frac{ke^{-\lambda t}}{t^2}$

$\ln P+2\ln t=\ln k-\lambda t$ 1A

(ii) Intercept of vertical axis $=2.3=\ln k$

$\therefore\ k\approx 10$ correct to the nearest integer 1A

slope $=\frac{2.3-0}{0-(-1.15)}=-\lambda$

$\therefore\ \lambda=-2$ 1A

$P=\frac{10e^{2t}}{t^2}$

$\frac{dP}{dt}=10\frac{t^2 2e^{2t}-e^{2t}2t}{t^4}$ 1A

$=\frac{20e^{2t}(t-1)}{t^3}$

t	$0<t<1$	$t=1$	$t>1$
$\frac{dP}{dt}$	-ve	0	+ ve

1M

Hence the minimum population size is attained when $t=1$.

$P=\frac{10e^{2(1)}}{(1)^2}$

$=74$ 1A

Hence the minimum population size is 74 hundred thousands.

(6)

(b) (i) $\frac{dE}{dt}=hte^{ht}-1.2e^{ht}+4.214$

$\frac{d^2E}{dt^2}=he^{ht}+h^2te^{ht}-1.2he^{ht}$ 1A

$=he^{ht}(ht-0.2)$

When $t=1$, $\frac{dE}{dt}$ is minimum and hence $\frac{d^2E}{dt^2}=0$.

Thus, $h=0.2$. 1A

(ii) $\frac{d}{dt}(te^{0.2t}) = 0.2te^{0.2t} + e^{0.2t}$ — 1A

$\therefore\ te^{0.2t} = 5\frac{d}{dt}(te^{0.2t}) - 5e^{0.2t}$

$\int te^{0.2t}dt = 5te^{0.2t} - 5\int e^{0.2t}dt$ — 1M

$= 5te^{0.2t} - 25e^{0.2t} + C_1$ — 1A

$\frac{dE}{dt} = 0.2te^{0.2t} - 1.2e^{0.2t} + 4.214$

$E = 0.2\int te^{0.2t}\,dt - 1.2\int e^{0.2t}\,dt + \int 4.214\,dt$

$= te^{0.2t} - 5e^{0.2t} - 6e^{0.2t} + 4.214t + C$ — 1M

$= te^{0.2t} - 11e^{0.2t} + 4.214t + C$

When $t = 0$, $E = 1$.

Hence $1 = 0 - 11 + 0 + C$ which gives $C = 12$.

i.e. $E = te^{0.2t} - 11e^{0.2t} + 4.214t + 12$

When $t = 1$, $E = e^{0.2} - 11e^{0.2} + 4.214 + 12$

≈ 4 — 1A

Thus the annual electricity consumption is 4 thousand terajoules per year.

(iii) $F = \frac{6}{1-5e^{rt}+3e^{2rt}} + 2$

$\frac{6}{1-5e^{r}+3e^{2r}} + 2 \approx 4$ — 1M

$3e^{2r} - 5e^{r} - 2 \approx 0$

$e^{r} \approx 2$ or $\frac{-1}{3}$ (rejected)

$r \approx \ln 2$ — 1A — OR 0.6931

(9)

(a)(i)(ii)		Good.
(b)(i)		Fair. Some candidates overlooked that the given condition was for the rate of change of annual electricity consumption. When considering the minimum rate of change, the second derivative $\frac{d^2E}{dt^2}$ should be considered.
(ii)		Poor. Many were unable to proceed after (b)(i).
(iii)		Poor. Many candidates did not attempt this part.

16. (2009 ASL-M&S Q8)

(a) (i) $R = kt^{1.2}e^{\frac{\lambda t}{20}}$

$\ln R = \ln k + 1.2\ln t + \frac{\lambda t}{20}$

$\ln R - 1.2\ln t = \frac{\lambda}{20}t + \ln k$ which is a linear function of t — 1A

(ii) intercept on the vertical axis $= \ln k = 2.89$

$k \approx 18$ (correct to the nearest integer) — 1A

slope $= \frac{\lambda}{20} = -0.05$

$\lambda = -1$ — 1A

(iii) $\therefore\ R = 18t^{1.2}e^{-0.05t}$

$\frac{dR}{dt} = 18[1.2t^{0.2}e^{-0.05t} + t^{1.2}e^{-0.05t}(-0.05)]$ — 1M

$= 0.9t^{0.2}e^{-0.05t}(24-t)$

	$0<t<24$	$t=24$	$24<t\le 30$
$\frac{dR}{dt}$	>0	0	<0

— 1M

Hence, R will attain maximum after 24 months. — 1A

$R = 18(24)^{1.2}e^{-0.05(24)}$

≈ 245.6815916

Hence, the maximum population size is 246 hundreds. — 1A

(7)

(b) (i) When $t = 0$, $L - 20(6e^{0} + 0^3) = Q = 240$

$\therefore\ L = 360$ — 1A

(ii) $e^{-t} = 1 + (-t) + \frac{(-t)^2}{2!} + \frac{(-t)^3}{3!} + \cdots$

$= 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots$ — 1A

$\therefore\ Q = 360 - 20\left[6\left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots\right) + t^3\right]$

$= 360 - 20(6 - 6t + 3t^2 + \cdots)$

$\approx 240 + 120t - 60t^2$ which is a quadratic polynomial — 1A

(iii) Let $300 = Q = 240 + 120t - 60t^2$ (by (b)(ii)) — 1M

i.e. $t^2 - 2t + 1 = 0$

Hence, when $t = 1$, the species of fish will reach a population size of 300. — 1A — Follow through

(iv) $Q = L - 20(6e^{-t} + t^3)$

$= 360 - 20\left[6\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \cdots\right) + t^3\right]$

$= 240 + 120t - 60t^2 - 120\left[\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right) + \left(\frac{t^6}{6!} - \frac{t^7}{7!}\right) + \cdots\right]$

$= 300 - 60(t-1)^2 - 120\left[\frac{t^4}{5!}(5-t) + \frac{t^6}{7!}(7-t) + \cdots\right]$ — 1M+1M — 1M for completing square; 1M for factorization

Since $0 \le t \le 2$, $Q < 300$ and so the conclusion in (b)(iii) is no more valid. — 1A — Follow through

(8)

(a) (i)		Very good.
(ii)		Very good. Some candidates overlooked the precision requirement.
(iii)		Fair. Some candidates still failed to provide the test for maximum after getting the first derivative and equated it to zero. Algebraic simplification should also be strengthened.
(b) (i)		Very good.
(ii)		Good. Familiarity with algebraic simplification would help.
(iii)		Good. Candidates had little difficulty in attempting this part.
(iv)		Very poor. Candidates seemed to be unfamiliar in grouping the terms meaningfully for decision making.

17. (2008 ASL-M&S Q8)

(a) (i) $N'(t)=\dfrac{20}{1+he^{-kt}}\quad (t\ge 0)$

$\ln\left[\dfrac{20}{N'(t)}-1\right]=-kt+\ln h$ — 1A

(ii) $\ln h=1.5$ — 1M

$h=e^{1.5}$

≈ 4.4817 (correct to 4 d.p.) — 1A

$-k=\dfrac{1.5-0}{0-7.6}$

(1M / 1A: Either One)

$k=\dfrac{15}{76}$

≈ 0.1974 (correct to 4 d.p.) — 1A

(4)

(b) (i) $v=4.5+e^{0.2t}$

$\dfrac{dv}{dt}=0.2e^{0.2t}$ — 1A

$N(t)=\displaystyle\int\frac{20}{1+4.5e^{-0.2t}}dt$

$=\displaystyle\int\frac{100}{e^{0.2t}+4.5}(0.2e^{0.2t})\,dt$

$=\displaystyle\int\frac{100}{v}dv$ — 1M

$=100\ln|v|+C$

$=100\ln(4.5+e^{0.2t})+C\quad (\because 4.5+e^{0.2t}>0)$ — 1A

Since $N(0)=50$, so $C=50-100\ln 5.5$ — 1M

i.e. $N(t)=100\ln\dfrac{4.5+e^{0.2t}}{5.5}+50$ — 1A

(ii) (1) $M(20)=N(20)$

$21\left[(20)+\dfrac{4.5}{0.2}e^{-0.2(20)}\right]+b=100\ln\dfrac{4.5+e^{0.2(20)}}{5.5}+50$

$b\approx -141.2090$ — 1A

(2) Consider $M'(t)-N'(t)$

$=21(1-4.5e^{-0.2t})-\dfrac{20}{1+4.5e^{-0.2t}}$ — 1A For the 1st term

$=\dfrac{1-425.25e^{-0.4t}}{1+4.5e^{-0.2t}}$ — 1A

$\therefore M'(t)-N'(t)>0$ when $e^{-0.4t}<\dfrac{1}{425.25}$ — 1M

i.e. $t>\dfrac{\ln 425.25}{0.4}\approx 15.1317$ — 1A

Since $M(20)=N(20)$ and $M(t)-N(t)$ is increasing when $t>20$,
so $M(t)>N(t)$ for $t>20$.
Hence the biologist's claim is correct. — 1A Follow through

(11)

(a) (i)		Very good.
(a) (ii)		Very good, though some careless mistakes were found.
(b) (i)		Fair. A number of candidates could not apply substitution to do integration.
(ii) (1)		Fair. Some candidates were hindered by failing to complete (b) (i).
(2)		Poor. Not too many candidates attempted and those who attempted could not make use of the given hint.

Marking 2.17

18. (2006 ASL-M&S Q9)

(a) Let $u=t+10$. Then, we have $\dfrac{du}{dt}=1$. — 1A can be absorbed

The total amount

$=\displaystyle\int_0^3 f(t)\,dt$

$=\displaystyle\int_0^3 25t^2(t+10)^{\frac{-1}{3}}\,dt$

$=\displaystyle\int_{10}^{13} 25(u-10)^2u^{\frac{-1}{3}}\,du$ — 1A

$=25\displaystyle\int_{10}^{13}(u^{\frac{5}{3}}-20u^{\frac{2}{3}}+100u^{\frac{-1}{3}})\,du$ — 1M

$=25\left[\dfrac{3}{8}u^{\frac{8}{3}}-12u^{\frac{5}{3}}+150u^{\frac{2}{3}}\right]_{10}^{13}$ — 1M

≈ 97.65521668

≈ 97.6552 thousand metres — 1A $a-1$ for r.t. 97.655

--------(5)

(b) $\ln(g(t)-28)=\ln k+ht^2$ — 1A

--------(1)

(c) $h\approx 0.3$ (correct to 1 decimal place) — 1A

$\ln k\approx 1.0$

$k\approx 2.718281828$

$k\approx 2.7$ (correct to 1 decimal place) — 1A

--------(2)

(d) (i) $g(t)\approx 28+2.7e^{0.3t^2}$

$=28+2.7\left(1+0.3t^2+\dfrac{(0.3t^2)^2}{2!}+\dfrac{(0.3t^2)^3}{3!}+\cdots\right)$ — 1M

$=30.7+0.81t^2+0.1215t^4+0.01215t^6+\cdots$ — 1A pp−1 for omitting '…'

The total amount

$=\displaystyle\int_0^3 g(t)\,dt$

$\approx\displaystyle\int_0^3(30.7+0.81t^2+0.1215t^4+0.01215t^6)\,dt$ — 1M

$=\left[30.7t+\dfrac{0.81t^3}{3}+\dfrac{0.1215t^5}{5}+\dfrac{0.01215t^7}{7}\right]_0^3$

≈ 109.0909071

≈ 109.0909 thousand metres — 1A $a-1$ for r.t. 109.091

(ii) $e^{0.3t^2}=1+0.3t^2+\dfrac{(0.3t^2)^2}{2!}+\dfrac{(0.3t^2)^3}{3!}+r(t)$ and $r(t)>0$ for all $t>0$.

Thus, the estimate in (d)(i) is an under-estimate. — 1A f.t.

(iii) Note that the estimate in (d)(i) is greater than the total amount of cloth production under John's model and that the estimate in (d)(i) is an under-estimate. — 1M for using (a), (d)(i) and (d)(ii)

Thus, the total amount of cloth production under Mary's model is greater than that under John's model. — 1A f.t.

--------(7)

(a)		Good. Some candidates could not handle change of variables in integration.
(b)		Very good.
(c)		Very good.
(d)(i)		Fair. Some candidates could not expand the exponential series.
(ii)		Fair. Many candidates could not state the fact that the neglected part of the exponential series expansion is positive and hence the estimate is an under-estimate.
(iii)		Not satisfactory. A few candidates could develop the logical analysis and arrive at a conclusion.

Marking 2.18

19. (2005 ASL-M&S Q8)

(a) $r(t)=\alpha t e^{-\beta t}$

$\frac{r(t)}{t}=\alpha e^{-\beta t}$

$\ln\frac{r(t)}{t}=\ln\alpha-\beta t$ — 1A

---------(1)

(b) $\because \ln\alpha=2.3$

$\therefore \alpha\approx 10$ (correct to 1 significant figure) — 1A

Also, we have $\beta\approx 0.5$ (correct to 1 significant figure). — 1A

$r(t)=10te^{-0.5t}$

$\frac{dr(t)}{dt}=10t(-0.5e^{-0.5t})+10e^{-0.5t}$

$=10e^{-0.5t}-5te^{-0.5t}$ — 1A

$=(10-5t)e^{-0.5t}$

$\frac{dr(t)}{dt}\begin{cases} >0 & \text{if } 0\le t<2 \\ =0 & \text{if } t=2 \\ <0 & \text{if } t>2 \end{cases}$ — 1M for testing + 1A

So, $r(t)$ attains its greatest value when $t=2$.

Hence, greatest value of $r(t)$ is $(10)(2)e^{-0.5(2)}\approx 7.357588823$.

Thus, the greatest rate of change is 7 ppm per hour. — 1A

$\frac{dr(t)}{dt}=10t(-0.5e^{-0.5t})+10e^{-0.5t}$	
$=10e^{-0.5t}-5te^{-0.5t}$	1A
$=(10-5t)e^{-0.5t}$	
$\frac{d^2r(t)}{dt^2}=-5e^{-0.5t}+(-5+2.5t)e^{-0.5t}=(2.5t-10)e^{-0.5t}$	
$\frac{dr(t)}{dt}=0$ when $t=2$ only and $\left.\frac{d^2r(t)}{dt^2}\right\vert_{t=2}=-5e^{-1}<0$	1M for testing + 1A
So, $r(t)$ attains its greatest value when $t=2$.	
Hence, greatest value of $r(t)$ is $(10)(2)e^{-0.5(2)}\approx 7.357588823$.	
Thus, the greatest rate of change is 7 ppm per hour.	1A

---------(6)

(c) (i) $\frac{d}{dt}\left((t+\frac{1}{\beta})e^{-\beta t}\right)$

$=\frac{d}{dt}\left((t+2)e^{-0.5t}\right)$

$=e^{-0.5t}-0.5(t+2)e^{-0.5t}$ — 1M for product rule or chain rule

$=-0.5te^{-0.5t}$ — 1M accept $-\beta te^{-\beta t}$

The required amount

$=\int_0^T r(t)dt$

$=\int_0^T 10te^{-0.5t}dt$ — 1M

$=\left[-20(t+2)e^{-0.5t}\right]_0^T$ — 1M+1A

$=\left(40-20(T+2)e^{-0.5T}\right)$ ppm — 1A

Note that	
$\int r(t)dt$	
$=\int 10te^{-0.5t}dt$	
$=-20(t+2)e^{-0.5t}+C$	1M+1A
Let $A(t)$ ppm be the amount of soot reduced when the petrol additive has been used for t hours.	
Then, we have $A(t)=-20(t+2)e^{-0.5t}+C$.	1M
Since $A(0)=0$, we have $C=40$.	
So, we have $A(t)=\left(40-20(t+2)e^{-0.5t}\right)$.	
Note that $A(0)=0$.	
Thus, the required amount $=A(T)=\left(40-20(T+2)e^{-0.5T}\right)$ ppm	1A

Note that	
$\int r(t)dt$	
$=\int 10te^{-0.5t}dt$	
$=-20(t+2)e^{-0.5t}+C$	1M+1A
Let $A(t)$ ppm be the amount of soot reduced when the petrol additive has been used for t hours.	
Then, we have $A(t)=-20(t+2)e^{-0.5t}+C$.	1M
The required amount	
$=A(T)-A(0)$	
$=\left(-20(T+2)e^{-0.5T}+C\right)-\left(-40+C\right)$	
$=\left(40-20(T+2)e^{-0.5T}\right)$ ppm	1A

(ii) The required amount

$=\lim_{T\to\infty}\left(40-20(T+2)e^{-0.5T}\right)$

$=40-20\lim_{T\to\infty}Te^{-0.5T}-40\lim_{T\to\infty}e^{-0.5T}$

$=40-20(0)-40(0)$ — 1M for $\lim_{T\to\infty}e^{-0.5T}=0$ and can be absorbed

$=40$ ppm — 1A

---------(8)

(a)		Very Good.
(b)		Good. Some candidates were not able to show that the stationary point is a maximum point.
(c)(i)		Fair. The first part was done well but the later part was less satisfactory. Only some candidates were able to work out the total amount of soot reduced.
(ii)		Poor. Many candidates were not able to complete this part because they failed to solve (c)(i).

20. (2004 ASL-M&S Q9)

Solution	Marks
(a) (i) $\frac{dy}{dx}=-\alpha\beta^{-x}$	
$-\frac{dy}{dx}=\alpha\beta^{-x}$	
$\ln(-\frac{dy}{dx})=\ln\alpha-(\ln\beta)x$	1A do not accept $\ln\alpha-\ln\beta x$
$-0.125=-\ln\beta$	
[redacted]	
$\beta\approx 1.133$ (correct to 3 decimal places)	1A
(ii) $\beta^{-x}=e^{-\lambda x}$ for all $x\geq 0$	
$\lambda=\ln\beta$	
$\lambda=0.125$	1A accept $\lambda\approx 0.1249$ a−1 for r.t. 0.125
$\frac{dy}{dx}=-\alpha\beta^{-x}$	
$\frac{dy}{dx}=-\alpha e^{-\lambda x}$	1M can be absorbed
$y=-\alpha\int e^{-\lambda x}dx$	1M for finding y by integration
$y=\frac{\alpha}{\lambda}e^{-\lambda x}+C$ [redacted]	1A for correct integration
Note that $y(0)=76$ and $y(2)=59.2$. Then, we have	
$\frac{\alpha}{\lambda}+C=76$ and $\frac{\alpha}{\lambda}e^{-2\lambda}+C=59.2$.	1M
So, we have $\frac{\alpha}{\lambda}(1-e^{-2\lambda})=16.8$. Hence, we have	
[redacted]	
$\alpha\approx 9.5$ (correct to 1 decimal place)	1A

Solution	Marks
$\beta^{-x}=e^{-\lambda x}$ for all $x\geq 0$	
$\lambda=\ln\beta$	
$\lambda=0.125$	1A accept $\lambda\approx 0.1249$ a−1 for r.t. 0.125
$\frac{dy}{dx}=-\alpha\beta^{-x}$	
$\frac{dy}{dx}=-\alpha e^{-\lambda x}$	1M can be absorbed
$[y]_0^2=-\alpha\int_0^2 e^{-\lambda x}dx$	1M for integrating from $x=0$ to $x=2$ on both sides
$y(2)-y(0)=\frac{\alpha}{\lambda}\left[e^{-\lambda x}\right]_0^2$	1A for correct integration
So, we have $\frac{\alpha}{\lambda}(1-e^{-2\lambda})=16.8$. Hence, we have	1M
[redacted]	
$\alpha\approx 9.5$ (correct to 1 decimal place)	1A
	----------(8)

Solution	Marks
(b) (i) By (a)(ii),	
[redacted]	
$C\approx 0.0504$	accept $C\in[-0.08,\ 0.06]$
So, $y\approx 75.94963597e^{-0.125x}+0.050364028$	accept $y\approx Be^{-0.125x}+C$ where $B\in[75.94,\ 76.08]$
When $y=25.2$, we have	
$25.2\approx 75.94963597e^{-0.125x}+0.050364028$	1M for leaving x only
[redacted]	
$x\approx 8.8$ (correct to 1 decimal place) Thus, the altitude of the mountain is 8.8 km above sea-level (correct to the nearest 0.1 km).	1A provided B and C both acceptable
(ii) $\frac{\alpha}{\lambda}e^{-\lambda h}-\frac{\alpha}{\lambda}e^{-2\lambda h}=13$	1M for using $y(h)-y(2h)=13$
$\frac{\alpha}{\lambda}(e^{-\lambda h})^2-\frac{\alpha}{\lambda}e^{-\lambda h}+13=0$	1M for transforming into a quadratic equation
$75.94963597(e^{-0.125h})^2-75.94963597e^{-0.125h}+13\approx 0$	
$e^{-0.125h}\approx 0.780773822$ or $e^{-0.125h}\approx 0.219226177$	
$h\approx 1.979758169$ or $h\approx 12.1412048$	1M for taking ln to find h
Note that $h\approx 12.1412048$ is rejected since $h>8.8$ is impossible. Thus, we have $h\approx 2.0$ (correct to 1 decimal place).	2A provided (b)(i) is correct
	----------(7)

(a)		Fair. Some candidates still forgot the integration constant.
(b)		Not satisfactory. Difficulties mainly arose from misunderstanding the question. Some candidates thought that $y(2h)-y(h)=13$.

21. (2001 ASL-M&S Q9)

(a) (i) $\ln P'(t) = -kt + \ln \dfrac{0.04ak}{1-a}$ — 1A

From the graph,

$-k \approx \dfrac{-8-(-3.5)}{18-0}$, $k \approx 0.25$ — 1A a−1 for more than 2 d.p.

$\ln \dfrac{0.04ak}{1-a} \approx -3.5$, $a \approx 0.7512 \approx 0.75$ — 1A a−1 for more than 2 d.p.

$P'(t) \approx 0.03e^{-0.25t}$

$P(t) \approx -0.12e^{-0.25t} + c$ for some constant c — 1M

Since $P(0) = 0.09$, $\therefore$ $c \approx 0.21$ — 1A

Hence $P(t) \approx -0.12e^{-0.25t} + 0.21$

(ii) $\mu = P(3) \approx 0.1533$ — 1A $\mu \in [0.1530, 0.1533]$

(iii) Stabilized PPI in town $A = \lim\limits_{t\to\infty} P(t) = 0.21$ — 1M+1A

---------(8)

(b) (i) Suppose $b = 0.09$.

(I) $Q'(t) = 0.24(3t+4)^{-\frac{3}{2}}$

$Q(t) = \dfrac{1}{3}(0.24)(-2)(3t+4)^{-\frac{1}{2}} + c$ for some constant c — 1A

$= -0.16(3t+4)^{-\frac{1}{2}} + c$

Since $Q(0) = 0.09$, $\therefore$ $c = 0.17$ — 1A

If $Q(t) = \mu \approx 0.1533$

$-0.16(3t+4)^{-\frac{1}{2}} + 0.17 \approx 0.1533$ — 1M

$(3t+4)^{\frac{1}{2}} \approx \dfrac{0.16}{0.0167}$

Since $3t+4>0$

$\therefore$ $t \approx 29.3$ — $t \in [28.2, 29.3]$

i.e. the PPI will reach the value of μ. — 1

Since $Q(0) = 0.09$, $\lim\limits_{t\to\infty} Q(t) = 0.17$ and Q is continuous and strictly increasing ($Q'(t) > 0$), $\therefore$ Q can reach any value between 0.09 and 0.17 including $\mu \approx 0.1533$.	1M 1

(II) Stabilized PPI in town $B = \lim\limits_{t\to\infty} Q(t) = 0.17$

$\therefore$ The stabilized PPI will be reduced by 0.04. — 1A

(ii) $0.05 < b \le 0.1$. — 1A

Otherwise, $Q'(t) \le 0$ and the PPI will not increase.

It follows that the epidemic will not break out. — 1

---------(7)

22. (2000 ASL-M&S Q10)

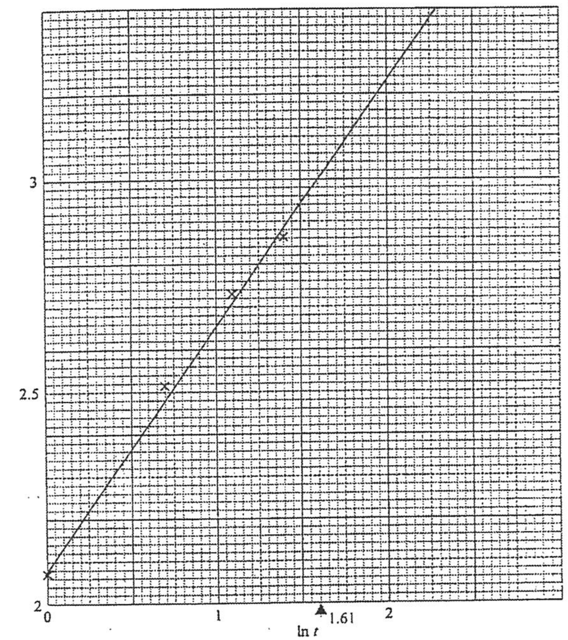
(a) (i) $r(t) = at^b$

$\ln r(t) = \ln a + b \ln t$ — 1A

(ii)

$\ln t$	0	0.69	1.10	1.39
$\ln r(t)$	2.07	2.51	2.73	2.86

Accept using common logarithm



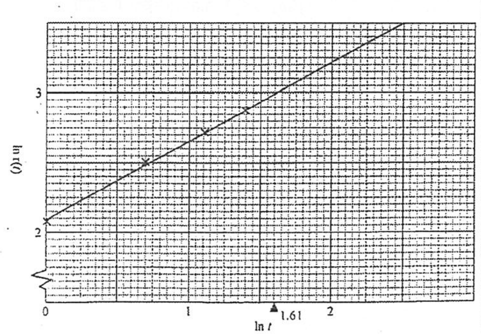
1M scale and labelling
1A points and line

When $t = 5$, $\ln t \approx 1.61$

$\therefore$ $\ln r(5) \approx 3$ from the graph. — 1M for either

$r(5) \approx 20.1$ — 1A $r(5) \in [19.1, 21.1]$, $\ln r(5) \in [2.95, 3.05]$

Alternative graph for 10(a)(ii)

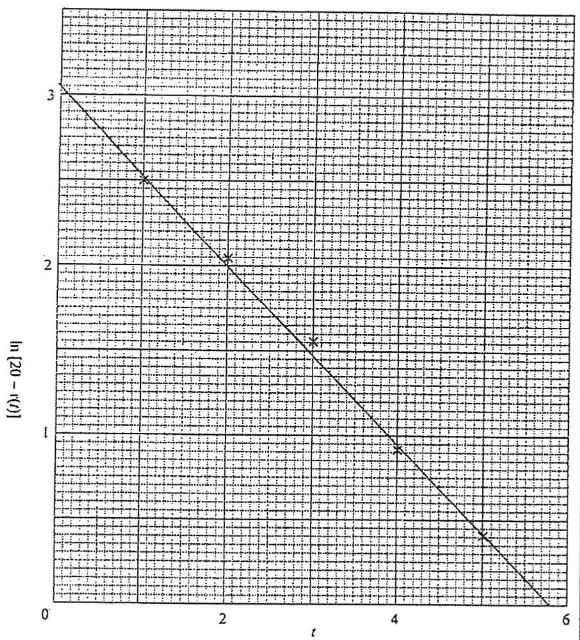


(b) (i) $r(t) = 20 - pe^{-qt}$

$\ln[20 - r(t)] = \ln p - qt$ — 1A

(ii)

t	1	2	3	4	5
$\ln[20-r(t)]$	2.49	2.04	1.55	0.92	0.41



1M scale and labelling
1A ignoring the point at $t = 5$

From the graph, $\ln p \approx 3.05$

$p \approx 21.1$ — 1A $p \in [20.1, 23.3]$, $\ln p \in [3.00, 3.15]$

$-q \approx \dfrac{0.41 - 3.05}{5} \approx -0.528$

$q \approx 0.528$ — 1A $q \in [0.518, 0.550]$

No marks for p or q if the graph is not correct.

(iii) The total number, in thousands, of bacteria after 15 days of cultivation

$= \int_0^{15} [20 - pe^{-qt}]dt + 100$ — 1M definite integral; 1M adding 100

$= \left[20t + \dfrac{p}{q}e^{-qt}\right]_0^{15} + 100$ — 1M for integration

$= 300 + \dfrac{p}{q}e^{-15q} - \dfrac{p}{q} + 100$

$\approx 260 + 100$ — 1A $\int_0^{15}[20 - pe^{-qt}]dt \in [255, 263]$

≈ 360 — 1A Ans. $\in [355, 363]$; $pp-1$ for wrong/missing unit

Alternatively,
Let N(t) thousand be the total number of bacteria after t days of cultivation. Then

$N(t) = \int [20 - pe^{-qt}]dt$ — 1M

$= 20t + \dfrac{p}{q}e^{-qt} + c$ — 1M for integration

$\because$ N(0) = 100

$\therefore$ $100 = \dfrac{p}{q} + c$

$c = 100 - \dfrac{p}{q} \approx 60.04$ — 1A $c \in [55.02, 63.46]$

Hence the total number, in thousands, of bacteria after 15 days of cultivation is

$N(15) = 20 \times 15 + \dfrac{p}{q}e^{-15q} + c \approx 360$ — 1M+1A N(15) $\in [355, 363]$

23. (1999 ASL-M&S Q9)

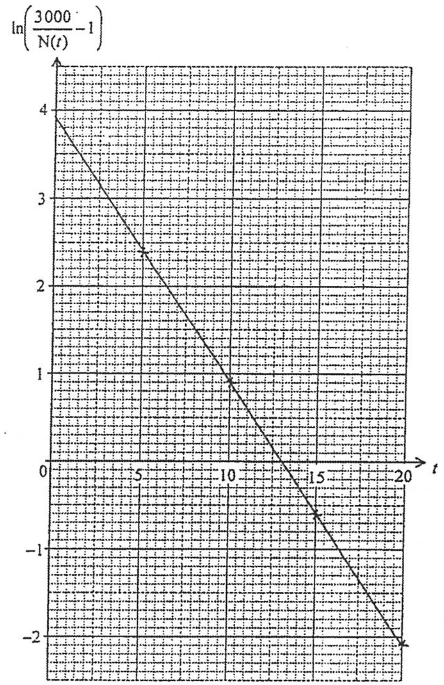
(a) (i) $N(t)=\dfrac{3000}{1+ae^{-bt}} \Leftrightarrow \dfrac{3000}{N(t)}-1=ae^{-bt}$

$\Leftrightarrow \ln\left(\dfrac{3000}{N(t)}-1\right)=-bt+\ln a$ — 1A — pp−1 for $-bt\ln e+\ln a$

(ii)

t	5	10	15	20
$\ln\left(\dfrac{3000}{N(t)}-1\right)$	2.40 (2.4)	0.90 (0.9)	−0.60 (−0.6)	−2.09 (−2.1)

1A — Correct to 1 d.p.



1A — the line must pass through all the 4 points

From the graph, $\ln a\approx 3.9$, — accept 3.85 – 3.95

$a\approx 49.4$ — 1A — accept 47.0 – 51.9

$b\approx -\dfrac{-2.09-2.40}{20-5}\approx 0.3$ — 1A

(b) (i) $N(t)=\dfrac{3000}{1+ae^{-bt}}$ (or $\dfrac{3000}{1+49.4e^{-0.3t}}$)

$N'(t)=\dfrac{3000abe^{-bt}}{\left(1+ae^{-bt}\right)^2}$ — 1M

$=\dfrac{3000(49.4)(0.3)e^{-0.3t}}{\left(1+49.4e^{-0.3t}\right)^2}$ (or $\dfrac{44460e^{-0.3t}}{\left(1+49.4e^{-0.3t}\right)^2}$) — 1A — accept $a\in[47.0, 51.9]$ and $3000ab\in[42300, 46710]$

$\therefore$ $N'(t)>0$ for all t

$N(t)$ is increasing — 1

(ii) If $N'(t)=\dfrac{1}{100}N(t)$

$\dfrac{3000abe^{-bt}}{\left(1+ae^{-bt}\right)^2}=\dfrac{1}{100}\cdot\dfrac{3000}{1+ae^{-bt}}$ — 1M — $a\in[47.0, 51.9]$, $b=0.3$

$e^{-bt}=\dfrac{1}{a(100b-1)}$ — $a\in[47.0, 51.9]$, $b=0.3$

$t=\dfrac{1}{0.3}\ln[a(100b-1)]$

OR

$\dfrac{44460e^{-0.3t}}{\left(1+49.4e^{-0.3t}\right)^2}=\dfrac{1}{100}\cdot\dfrac{3000}{1+49.4e^{-0.3t}}$ — 1M

$1482e^{-0.3t}=1+49.4e^{-0.3t}$

$t\approx 24.2242$ — 1A — $t\in[24.0581, 24.3887]$

$\because$ $N\left(\dfrac{1}{0.3}\ln[a(100b-1)]\right)=\dfrac{3000}{1+ae^{-b\left(\frac{1}{0.3}\ln[a(100b-1)]\right)}}=2900$ — 1M

OR

$N(24.2242)=\dfrac{3000}{1+49.4e^{-0.3(24.2242)}}\approx 2900$ — 1M

$\therefore$ The greatest number of migrants found at Mai Po is 2900. — 1A

(iii) Suppose all the migrants leave Mai Po in x days.

Then $\int_0^x 60\sqrt{s}\,ds=2900$ — 1M

$\left[40s^{\frac{3}{2}}\right]_0^x=2900$ — 1A — for integration (including limits)

$x\approx 17.3870$

$\therefore$ The number of days in which we can see the migrants is $24.2242+17.3870\approx 42$ — 1A — r.t. 42

24. (1998 ASL-M&S Q10)

(a) (i) (I) : $\ln r(t) = \ln\alpha + \beta\ln t$ 1A

(II) : $\ln r(t) = \ln\gamma + \lambda t$ 1A

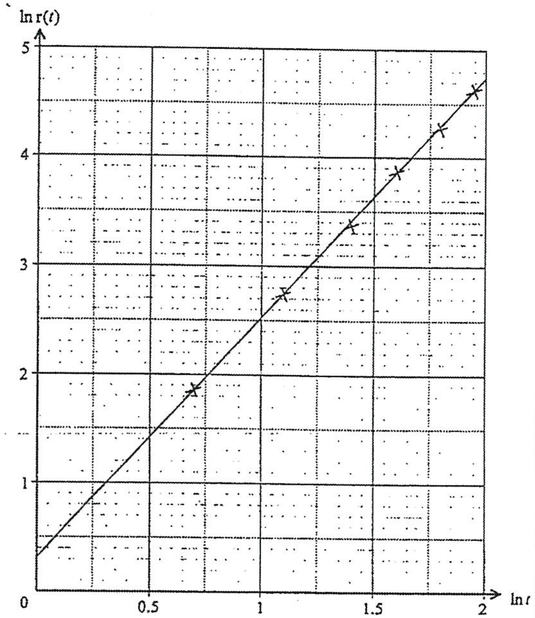
(ii)

t	2	3	4	5	6	7
$r(t)$	6.4	15.7	29.5	48.3	72.2	101.2
$\ln t$	0.69	1.10	1.39	1.61	1.79	1.95
$\ln r(t)$	1.86	2.75	3.38	3.88	4.28	4.62

1A Correct to 1 d.p.

1A Correct to 1 d.p.

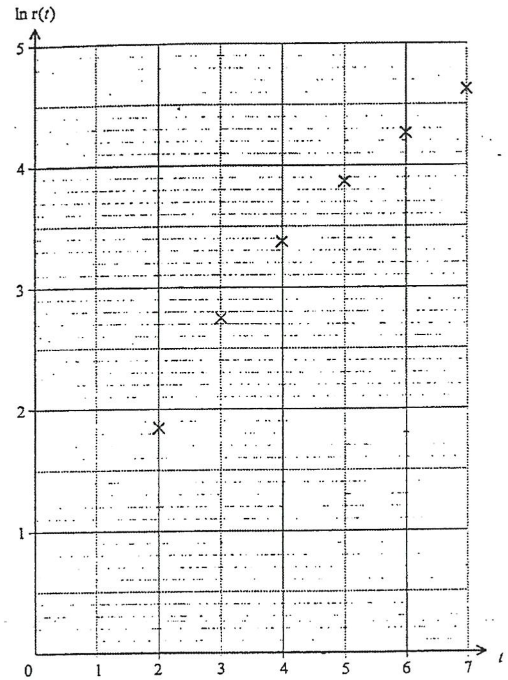
(I)



1A for any 2 points being correct

1A for all the 6 points being correct

(II)



1A for any 2 points being correct

1A for all the 6 points being correct

From the graphs, equation (I) would be a better model and

$\ln\alpha \approx 0.3$ — Accept 0.3 - 0.4

$\alpha \approx e^{0.3} \approx 1.3$ — 1A — Accept 1.3 - 1.5

$\beta \approx \dfrac{4.62-1.86}{1.95-0.69} \approx 2.2$ — 1A — Accept 2.0 – 2.4

(b) $\int_0^{14} \alpha t^{\beta}\,dt$ where $\alpha \approx 1.3$, $\beta \approx 2.2$ — 1M+1A — Accept $\alpha \in [1.3, 1.5]$, $\beta \in [2.0, 2.4]$

$= \dfrac{\alpha}{\beta+1}\left[t^{\beta+1}\right]_0^{14}$ $\left(\approx \dfrac{1.3}{3.2}\left[t^{3.2}\right]_0^{14}\right)$

≈ 1889

$\therefore$ 1889 hundred of trees would be destroyed in the first 14 days. — 1A — Accept 1498 - 3015

Consider $\int_0^k \alpha t^{\beta}\,dt = 1889 \times 2$

$\dfrac{1.3}{3.2}\left[t^{3.2}\right]_0^k = 3778$ — 1M

$k^{3.2} \approx 9299.69$

$k \approx e^{\frac{\ln 9299.69}{3.2}} \approx 17.3839$

$\therefore$ The total number of trees destroyed will be doubled in 4 days more. — 1A

25. (1997 ASL-M&S Q9)

(a) $b \approx \dfrac{7.49-7.95}{8-3.4}$

≈ -0.1 — 1A

Sub. (8, 7.49) into $\ln N(x) = -0.1x + \ln a$.

$7.49 \approx \ln a - 0.8$ — 1M

$a \approx 4000$ — 1A

(b) (i) $N(x) = ae^{bx} = 4000e^{-0.1x}$ — 1M

Daily profit (in dollars) of selling N(x) clams:

$P(x) = N(x)\cdot x - (2N(x)+5000)$ — 1A — for 2N(x)+5000

$= (x-2)N(x) - 5000$

$= 4000(x-2)e^{-0.1x} - 5000$ — 1A

(ii) $P'(x) = 4000[(x-2)(-0.1e^{-0.1x}) + e^{-0.1x}]$ — 1A

$= 400e^{-0.1x}(12-x)$

$$P'(x) \begin{cases} >0 & \text{if } 0<x<12 \\ =0 & \text{if } x=12 \\ <0 & \text{if } x>12 \end{cases}$$ — 1

$\therefore$ P(x) attains its maximum when $x = 12$.

Hence the selling price of each clam = \$12 — 1A

the number of clams sold per day $= N(12)$

$= 4000e^{-0.1(12)}$

≈ 1205 — 1A

(c) The difference between the numbers of clams sold on the n-th and $(n-1)$-th days after the launch of the promotion programme

$= M(n) - M(n-1)$

$= \left[1500 + 1000(1-e^{-0.1n})\right] - \left[1500 + 1000(1-e^{-0.1(n-1)})\right]$ — 1M

$= 1000(-e^{-0.1n} + e^{-0.1n}\cdot e^{0.1})$

$= 1000e^{-0.1n}(e^{0.1}-1)$ — 1A

If $M(n) - M(n-1) < 15$ — 1M

then $e^{-0.1n} < \dfrac{15}{1000(e^{0.1}-1)}$

$n > 19.475$ — 1M

$\therefore$ The promotion programme should run for 20 days. — 1A

26. (1994 ASL-M&S Q9)

(a) (i) The machine will cease producing cloth when $x=0$, — 1M

$100e^{-0.01t} - 65e^{-0.02t} - 35 = 0$

Put $y = e^{-0.01t}$,

$100y - 65y^2 - 35 = 0$

$13y^2 - 20y + 7 = 0$

$(y-1)(13y-7) = 0$

$y = 1$ or $\dfrac{7}{13}$

$\therefore$ $e^{-0.01t} = 1$ or $\dfrac{7}{13}$

$t = 0$ (rej.) or $t = \dfrac{\ln\frac{7}{13}}{-0.01}$

$= 61.9039$ — 1A — r.t. 62

It will cease producing cloth in February, 2000

(ii) The total amount of cloth produced during the lifespan of the machine

$= \int_0^{61.904} x\,dt$ — 1M — Accept $\int_0^{62} x\,dt$

$= \int_0^{61.904} (100e^{-0.01t} - 65e^{-0.02t} - 35)\,dt$

$= -10000e^{-0.01t} + \dfrac{65}{0.02}e^{-0.02t} - 35t\Big|_0^{61.904}$ — 1M

≈ 141 (km) — 1A

(b) Let P be the monthly profit, then

$P = 800x - 300x - 300$ — 1A

$= 500x - 300$

$= 500(100e^{-0.01t} - 65e^{-0.02t} - 35) - 300$

$= 50000e^{-0.01t} - 32500e^{-0.02t} - 17800$

$\dfrac{dP}{dt} = -500e^{-0.01t} + 650e^{-0.02t}$ — 1M

$\dfrac{dP}{dt} = 0$ when $650e^{-0.02t} = 500e^{-0.01t}$

or $t = t_0$ where $t_0 = \dfrac{1}{0.01}\ln\left(\dfrac{650}{500}\right) = 26.2364$ — 1A — r.t. 26

$\left.\dfrac{d^2P}{dt^2}\right|_{t=t_0} = (5e^{-0.01t} - 13e^{-0.02t})\big|_{t=t_0} = -3.85 < 0$ — 1M — For proving max.

Hence P is maximum when $t = t_0$

<u>Alternatively</u>

$x = 100e^{-0.01t} - 65e^{-0.02t} - 35$

$\dfrac{dx}{dt} = -e^{-0.01t} + 1.3e^{-0.02t}$ — 1M

$\dfrac{dx}{dt} = 0$ when $e^{-0.01t} = 1.3e^{-0.02t}$

or $t = t_0$ where $t_0 = \dfrac{\ln 1.3}{0.01} = 26.2364$ — 1A

$\because P = 800x - 300x - 300 = 500x - 300$ — 1A

$\therefore P$ is maximum when x is maximum

$\left.\dfrac{d^2x}{dt^2}\right|_{t=t_0} = (0.01e^{-0.01t} - 0.026e^{-0.02t})\big|_{t=t_0} = -0.0077 < 0$ — 1M

Hence P is maximum when $t = t_0$

$\because$ $P\|_{t=26} \approx 1431$ $P\|_{t=27} \approx 1430$ $\therefore$ The greatest monthly profit will be obtained when $t=26$, i.e. in February, 1997.	1	For checking P when $t=26, 27$
The greatest monthly profit is US$1431.	1A	Accept $P\|_{t=t_0} = 1431$, r.t. 1431
(c) If $P = 500$, then $500 = 50000e^{-0.01t} + 32500e^{-0.02t} - 17800$ $5 = 500e^{-0.01t} - 325e^{-0.02t} - 178$	1A	
Alternatively $500x - 300 = 500$ $x = 1.6$ $100e^{-0.01t} - 65e^{-0.02t} - 35 = 1.6$ $500e^{-0.01t} - 325e^{-0.02t} - 183 = 0$	1A	
Put $y = e^{-0.01t}$, $325y^2 - 500y + 183 = 0$ $(65y - 61)(5y - 3) = 0$ $y = \frac{61}{65}$ or $\frac{3}{5}$ $e^{-0.01t} = \frac{61}{65}$ or $\frac{3}{5}$ $t = \frac{1}{-0.01}\ln\frac{61}{65}$ or $\frac{1}{-0.01}\ln\frac{3}{5}$ ≈ 6.35 or 51.08	1A	
$\because$ P is increasing when $t \approx 6.35$ (OR The machine has not yet reached its production climax when $t \approx 6.35$)	1	
$\therefore$ The machine should be replaced when $t \approx 51.08$, i.e. in April, 1999.	1A	