1. Binomial Expansion

Learning Unit	Learning Objective
Foundation Knowledge Area	
1. Binomial expansion	1.1 recognise the expansion of $(a+b)^n$, where n is a positive integer

Section A

1. (a) Expand e^{-18x} in ascending powers of x as far as the term in x^2.

 (b) Let n be a positive integer. If the coefficient of x^2 in the expansion of $e^{-18x}(1+4x)^n$ is -38. Find n.

(6 marks) (2019 DSE-MATH-M1 Q6)

2. Let k be a constant.

 (a) Expand $e^{kx}+e^{2x}$ in ascending powers of x as far as the term in x^2.

 (b) If the coefficient of x and the coefficient of x^2 in the expansion of $(1-3x)^8(e^{kx}+e^{2x}-1)$ are equal, find k.

(6 marks) (2018 DSE-MATH-M1 Q6)

3. (a) Expand $(1+e^{3x})^2$ in ascending powers of x as far as the term in x^2.

 (b) Find the coefficient of x^2 in the expansion of $(5-x)^4(1+e^{3x})^2$.

(6 marks) (2017 DSE-MATH-M1 Q5)

4. Let k be a constant.

 (a) Expand e^{kx} in ascending powers of x as far as the term in x^2.

 (b) If the coefficient of x in the expansion of $(1+2x)^7e^{kx}$ is 8, find the coefficient of x^2.

(5 marks) (2016 DSE-MATH-M1 Q5)

5. (a) Expand e^{-4x} in ascending powers of x as far as the term in x^2.

 (b) Find the coefficient of x^2 in the expansion of $\dfrac{(2+x)^5}{e^{4x}}$.

(5 marks) (2015 DSE-MATH-M1 Q5)

6. The slope of the tangent to a curve S at any point (x,y) on S is given by $\dfrac{dy}{dx}=\left(2x-\dfrac{1}{x}\right)^3$, where $x>0$.
A point $P(1,5)$ lies on S.

 (a) Find the equation of the tangent to S at P.

 (b) (i) Expand $\left(2x-\dfrac{1}{x}\right)^3$.

 (ii) Find the equation of S for $x>0$.

(7 marks) (2014 DSE-MATH-M1 Q3)

7. (a) Expand $\left(u+\dfrac{1}{u}\right)^4$ in descending powers of u.

 (b) Express $(e^{ax}+e^{-ax})^4$ in ascending powers of x up to the term in x^2.

 (c) Suppose the coefficient of x^2 in the result of (b) is 2. Find all possible values of a.

(5 marks) (2013 DSE-MATH-M1 Q1)

8. Let n be a positive integer.

 (a) Expand $(1+3x)^n$ in ascending powers of x up to the term x^2.

 (b) It is given that the coefficient of x^2 in the expansion of $e^{-2x}(1+3x)^n$ is 62. Find the value of n.

(4 marks) (2012 DSE-MATH-M1 Q1)

9. (a) Expand $(2x+1)^3$.

 (b) Expand e^{-ax} in ascending powers of x as far as the term in x^2, where a is a constant.

 (c) If the coefficient of x^2 in the expansion of $\dfrac{(2x+1)^3}{e^{ax}}$ is -4, find the value(s) of a.

(5 marks) (PP DSE-MATH-M1 Q1)

10. Expand the following in ascending powers of x as far as the term in x^2:

(a) e^{-2x};

(b) $\dfrac{(1+2x)^6}{e^{2x}}$.

(4 marks) (SAMPLE DSE-MATH-M1 Q1)

Math & Stat

11. (a) (i) Expand $(x+y+z)^2$.

(ii) Find the coefficients of x^3y , x^3z , xy^3 , y^3z , xz^3 and yz^3 in the expansion of $(x+y+z)^4$.

(b) If a cup is randomly selected from a box containing red cups, blue cups and green cups, the probabilities of getting a red cup, a blue cup and a green cup are p, q and r respectively.
If 4 cups are randomly selected from the box one by one with replacement, find, in terms of p, q and r ,

(i) the probability that at least 2 cups of different colours are selected;

(ii) the probability that exactly 3 cups of the same colour are selected.

(7 marks)
(2004 ASL-M&S Q4)

Out of Syllabus

12. (a) Expand e^{-2x} in ascending powers of x as far as the term in x^3.

(b) Using (a), expand $\dfrac{(1+x)^{\frac{1}{2}}}{e^{2x}}$ in ascending powers of x as far as the term in x^3 .

State the range of values of x for which the expansion is valid.

(6 marks) (1999 ASL-M&S Q2)

13. (a) Prove that $\dfrac{1}{1+\sqrt{1-x}}=\dfrac{1}{x}\left(1-\sqrt{1-x}\right)$ for $x<1$ and $x\neq 0$.

(b) Let $I=\int_{\frac{1}{4}}^{\frac{1}{2}}\dfrac{1}{1+\sqrt{1-x}}dx$. By considering the expansion of $\dfrac{1}{1+\sqrt{1-x}}$ in ascending powers of x as far as the term in x^2, estimate the value of I .

(c) Let $J=\int_{-2}^{-1}\dfrac{1}{1+\sqrt{1-x}}dx$. Can we use the same method in (b) to estimate the value of J?
Explain your answer.

(7 marks)
(2011 ASL-M&S Q1)

Suggested modification

14. (a) Expand $\dfrac{(1+x)^{10}-1}{x}$ in ascending powers of x as far as the term in x^2.

(b) Let $I=\int_{0.1}^{0.2}\dfrac{(1+x)^{10}-1}{x}dx$. By using (a), estimate the value of I .

(c) Determine whether the estimate in (b) is an over-estimate or an under-estimate.

1. Binomial Expansion

1. (2019 DSE-MATH-M1 Q6)

(a) e^{-18x}		
$=1+(-18x)+\frac{(-18x)^2}{2!}+\cdots$	1M	
$=1-18x+162x^2+\cdots$	1A	
(b) $(1+4x)^n$		
$=1+C_1^n(4x)+C_2^n(4x)^2+\cdots+C_n^n(4x)^n$	1M	
$=1+4C_1^n x+16C_2^n x^2+\cdots+4^n x^n$		
$16C_2^n-72C_1^n+162=-38$	1M	
$16\left(\frac{n(n-1)}{2}\right)-72n+162=-38$		
$n^2-10n+25=0$	1M	
$n=5$	1A	
	----------(6)	

2. (2018 DSE-MATH-M1 Q6)

(a) $e^{kx}+e^{2x}$		
$=\left(1+kx+\frac{(kx)^2}{2!}+\cdots\right)+\left(1+2x+\frac{(2x)^2}{2!}+\cdots\right)$	1M	for expanding e^{kx} or e^{2x}
$=2+(k+2)x+\frac{(k^2+4)}{2}x^2+\cdots$	1A	
(b) $(1-3x)^8$		
$=1+C_1^8(-3x)+C_2^8(-3x)^2+\cdots$	1M	
$=1-24x+252x^2+\cdots$		
$e^{kx}+e^{2x}-1$		
$=1+(k+2)x+\frac{(k^2+4)}{2}x^2+\cdots$		
$(1)(k+2)+(-24)(1)=(1)\left(\frac{k^2+4}{2}\right)+(-24)(k+2)+(252)(1)$	1M+1M	
$k^2-50k+456=0$		
$k=12$ or $k=38$	1A	
	----------(6)	

3. (2017 DSE-MATH-M1 Q5)

(a) $(1+e^{3x})^2$		
$=1+2e^{3x}+e^{6x}$	1M	
$=1+2\left(1+3x+\frac{(3x)^2}{2!}+\cdots\right)+\left(1+6x+\frac{(6x)^2}{2!}+\cdots\right)$	1M	for expanding e^{3x} or e^{6x}
$=4+12x+27x^2+\cdots$	1A	
$(1+e^{3x})^2$		
$=\left(1+1+3x+\frac{(3x)^2}{2!}+\cdots\right)^2$	1M	for expanding e^{3x}
$=(2)(2)+(2)(2)(3x)+(3x)(3x)+(2)(2)\left(\frac{9x^2}{2}\right)+\cdots$	1M	
$=4+12x+27x^2+\cdots$	1A	
(b) $(5-x)^4$		
$=5^4-C_1^4(5^3)x+C_2^4(5^2)x^2-C_3^4(5)x^3+x^4$	1M	
$=625-500x+150x^2-20x^3+x^4$		
The required coefficient		
$=(625)(27)+(-500)(12)+(150)(4)$	1M	withhold 1M if the step is skipped
$=11475$	1A	
	----------(6)	

(a)	Very good. Most candidates were able to expand $(1+e^{3x})^2$.
(b)	Very good. Most candidates were able to find the coefficient of x^2 .

4. (2016 DSE-MATH-M1 Q5)

(a) e^{kx}

$= 1 + kx + \frac{(kx)^2}{2!} + \cdots$

$= 1 + kx + \frac{k^2x^2}{2} + \cdots$ 1A

(b) $(1+2x)^7 e^{kx}$

$= \left(1 + C_1^7(2x) + C_2^7(2x)^2 + \cdots + (2x)^7\right)\left(1 + kx + \frac{k^2x^2}{2} + \cdots\right)$ 1M

$= \left(1 + 14x + 84x^2 + \cdots + (2x)^7\right)\left(1 + kx + \frac{k^2x^2}{2} + \cdots\right)$

$\therefore 14 + k = 8$ 1M

$k = -6$

The coefficient of x^2

$= (1)\left(\frac{(-6)^2}{2}\right) + 14(-6) + (84)(1)$ 1M

$= 18$ 1A

----------(5)

(a)	Very good. A very high proportion of the candidates were able to expand e^{kx} while some candidates were unable to simplify the coefficient of x^2.
(b)	Very good. More than 70% of the candidates were able to find the coefficient of x^2 while a small number of candidates made careless mistakes in expanding $(1+2x)^7$.

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5. (2015 DSE-MATH-M1 Q5)

(a) e^{-4x}

$= 1 + (-4x) + \frac{(-4x)^2}{2!} + \cdots$ 1M

$= 1 - 4x + 8x^2 - \cdots$ 1A

(b) $(2+x)^5$

$= 2^5 + C_1^5(2^4)x + C_2^5(2^3)x^2 + \cdots + x^5$ 1M

$= 32 + 80x + 80x^2 + \cdots + x^5$

The required coefficient

$= (1)(80) + (-4)(80) + (8)(32)$ 1M

$= 16$ 1A

----------(5)

(a)	Very good. Most candidates were able to expand e^{-4x} while a few candidates failed to show working steps.
(b)	Very good. Most candidates were able to find the coefficient of x^2 while a few candidates made a carless mistake in expanding $(2+x)^5$ as $2^5 + C_1^5(2^4)x + C_2^5(2^3)x^2 + \cdots + x^5$.

6. (2014 DSE-MATH-M1 Q3)

(a) $\left.\frac{dy}{dx}\right|_{(1,5)} = \left(2\cdot 1 - \frac{1}{1}\right)^3$

$= 1$ 1A

Hence the equation of tangent is $y - 5 = 1(x-1)$.

i.e. $x - y + 4 = 0$ 1A

(b) (i) $\left(2x - \frac{1}{x}\right)^3 = (2x)^3 - 3(2x)^2\left(\frac{1}{x}\right) + 3(2x)\left(\frac{1}{x}\right)^2 - \left(\frac{1}{x}\right)^3$ 1M

$= 8x^3 - 12x + \frac{6}{x} - \frac{1}{x^3}$ 1A

(ii) $y = \int\left(2x - \frac{1}{x}\right)^3 dx$

$= \int\left(8x^3 - 12x + \frac{6}{x} - \frac{1}{x^3}\right)dx$ by (i)

$= 2x^4 - 6x^2 + 6\ln|x| + \frac{1}{2x^2} + C$ 1M

Since $P(1, 5)$ lies on S, $5 = 2(1)^4 - 6(1)^2 + 6\ln|1| + \frac{1}{2(1)^2} + C$. 1M

i.e. $C = \frac{17}{2}$

Hence the equation of S is $y = 2x^4 - 6x^2 + 6\ln x + \frac{1}{2x^2} + \frac{17}{2}$ for $x > 0$. 1A

(7)

(a)		Very good.
(b)	(i)	Excellent.
	(ii)	Satisfactory. Some candidates did not know $\int\frac{1}{x}dx = \ln|x| + C$, or wrote $\int\frac{1}{x^3}dx = -\frac{2}{x^2}$ or $\frac{1}{2x^2}$.

7. (2013 DSE-MATH-M1 Q1)

Solution	Marks
(a) $\left(u+\frac{1}{u}\right)^4 = u^4+4u^3\left(\frac{1}{u}\right)+6u^2\left(\frac{1}{u}\right)^2+4u\left(\frac{1}{u}\right)^3+\left(\frac{1}{u}\right)^4$	
$= u^4+4u^2+6+\frac{4}{u^2}+\frac{1}{u^4}$	1A
(b) $(e^{ax}+e^{-ax})^4$	
$=e^{4ax}+4e^{2ax}+6+4e^{-2ax}+e^{-4ax}$ by (a)	1M
$=\left[1+\frac{4ax}{1!}+\frac{(4ax)^2}{2!}+\cdots\right]+4\left[1+\frac{2ax}{1!}+\frac{(2ax)^2}{2!}+\cdots\right]+6$	
$+4\left[1+\frac{-2ax}{1!}+\frac{(-2ax)^2}{2!}+\cdots\right]+\left[1+\frac{-4ax}{1!}+\frac{(-4ax)^2}{2!}+\cdots\right]$	1M
$=1+4ax+8a^2x^2+4+8ax+8a^2x^2+6+4-8ax+8a^2x^2+1-4ax+8a^2x^2+\cdots$	
$=16+32a^2x^2+\cdots$	1A
(c) $32a^2=2$	
$a^2=\frac{1}{16}$	
$a=\pm\frac{1}{4}$	1A
	(5)

(a)	Excellent. A few candidates neglected the requirement 'in descending powers of u' when expanding $\left(u+\frac{1}{u}\right)^4$.
(b)	Satisfactory. Some candidates repeated steps in (a) because they did not make use of the fact that $e^{-ax}=\frac{1}{e^{ax}}$. Some candidates were not able to use power series of an exponential function, while some others expressed $(e^{ax}+e^{-ax})^4$ in powers of e^{2ax}.
(c)	Poor. Many candidates were not able to get the correct answer of (b), hence failed to get the answer for this part.

8. (2012 DSE-MATH-M1 Q1)

Solution	Marks	Remarks
(a) $(1+3x)^n=1+C_1^n(3x)+C_2^n(3x)^2+\cdots$		
$=1+3nx+\frac{9n(n-1)}{2}x^2+\cdots$	1A	
(b) $e^{-2x}(1+3x)^n=\left[1+(-2x)+\frac{(-2x)^2}{2!}+\cdots\right]\left[1+3nx+\frac{9n(n-1)}{2}x^2+\cdots\right]$	1A	For $1+(-2x)+\frac{(-2x)^2}{2!}+\cdots$
$=(1-2x+2x^2+\cdots)\left[1+3nx+\frac{9n(n-1)}{2}x^2+\cdots\right]$		
$\therefore\ 1\cdot\frac{9n(n-1)}{2}+(-2)(3n)+2\cdot 1=62$	1M	
$9n^2-21n-120=0$		
$n=5$ or $\frac{-8}{3}$ (rejected)	1A	
	(4)	

(a)	Very good. A minority of candidates, however, did not simplify the results obtained.
(b)	Very good. A minority of candidates, however, did not reject the negative root $\frac{-8}{3}$.

9. (PP DSE-MATH-M1 Q1)

Solution	Marks
(a) $(2x+1)^3=8x^3+12x^2+6x+1$	1A
(b) $e^{-ax}=1-ax+\frac{a^2x^2}{2}-\cdots$	1A
(c) $\frac{(2x+1)^3}{e^{ax}}=(8x^3+12x^2+6x+1)\left(1-ax+\frac{a^2x^2}{2}-\cdots\right)$	1M
The coefficient of $x^2=12(1)+6(-a)+(1)\frac{a^2}{2}$	1M
$\therefore\ \frac{a^2}{2}-6a+12=-4$	
$a^2-12a+32=0$	
$a=4$ or 8	1A
	(5)

(a)	甚佳。 很多學生熟識二項式展式。
(b)	甚佳。 少部分學生未能展開指數函數。
(c)	良好。 少部分學生未能正確利用(b) 的結果。

10. (SAMPLE DSE-MATH-M1 Q1)

Solution	Marks	Remarks
(a) $e^{-2x}=1+(-2x)+\frac{(-2x)^2}{2!}-\cdots$		
$=1-2x+2x^2+\cdots$	1A	
(b) $\frac{(1+2x)^6}{e^{2x}}=[1+6(2x)+15(2x)^2+\cdots]\cdot e^{-2x}$	1A	For $1+6(2x)+15(2x)^2+\cdots$
$=(1+12x+60x^2+\cdots)(1-2x+2x^2+\cdots)$	1M	For using (a)
$=1+10x+38x^2+\cdots$	1A	(pp-1) if dots were omitted in most cases
	(4)	

11. (2004 ASL-M&S Q4)

(a) (i) $(x+y+z)^2$	
$= x^2+y^2+z^2+2xy+2yz+2xz$	1A
(ii) Note that	
$(x+y+z)^4$	
$= (x^2+y^2+z^2+2xy+2yz+2xz)(x^2+y^2+z^2+2xy+2yz+2xz)$	
Thus, we have	
the coefficients of x^3y , x^3z , xy^3 , y^3z , xz^3 and yz^3	
$= (1)(2)+(2)(1)$	1M can be absorbed
$= 4$	1A
(b) (i) The required probability	
$= 1-p^4-q^4-r^4$	1M for complementary probability + 1A

The required probability	
$= (p+q+r)^4-p^4-q^4-r^4$	1M
$= 1-p^4-q^4-r^4$	1A

The required probability	
$= (p+q+r)^4-p^4-q^4-r^4$	1M
$= (p^2+q^2+r^2+2pq+2gr+2pr)(p^2+q^2+r^2+2pq+2qr+2pr)-p^4-q^4-r^4$	
$= p^4+q^4+r^4+4p^3q+4p^3r+4pq^3+4q^3r+4pr^3+4qr^3+$	
$6p^2q^2+6p^2r^2+6q^2r^2+12p^2qr+12pq^2r+12pqr^2-p^4-q^4-r^4$	
$=4p^3q+4p^3r+4pq^3+4q^3r+4pr^3+4qr^3+6p^2q^2+6p^2r^2+6q^2r^2+12p^2qr+12pq^2r+12pqr^2$	1A

(ii) The required probability	
$= 4(p^3q+p^3r+pq^3+q^3r+pr^3+qr^3)$	1A for $(p^3q+p^3r+pq^3+q^3r+pr^3+qr^3)$ + 1A for all being correct

The required probability	
$= 4p^3(1-p)+4q^3(1-q)+4r^3(1-r)$	1A for $(p^3(1-p)+q^3(1-q)+r^3(1-r))$ + 1A for all being correct

The required probability	
$= 1-(p^4+q^4+r^4+6p^2q^2+6p^2r^2+6q^2r^2+12p^2qr+12pq^2r+12pqr^2)$	1A for $(p^4+q^4+r^4+6p^2q^2+6p^2r^2+6q^2r^2$ $+12p^2qr+12pq^2r+12pqr^2)$ + 1A for all being correct
	----------(7)

Fair. Many candidates did not make use of the fact that $p+q+r=1$, which simplifies the expressions.