1 Estimation

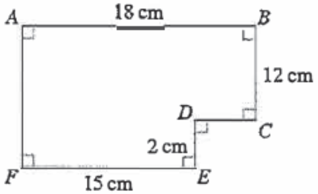
1.1 HKCEE MA 2006 – I – 11

In the figure, *ABCDEF* is a thin six-sided polygonal metal sheet, where all the measurements are correct to the nearest cm.

(a) Write down the maximum absolute error of the measurements.

(b) Find the least possible area of the metal sheet.

(c) The actual area of the metal sheet is $x\text{ cm}^2$. Find the range of values of x.



1.2 HKCEE MA 2007 I 10

(a) If the length of a piece of thin metal wire is measured as 5 cm correct to the nearest cm, find the least possible length of the metal wire.

(b) The length of a piece of thin metal wire is measured as 2.0 m correct to the nearest 0.1 m.

(i) Is it possible that the actual length of this metal wire exceeds 206 cm? Explain your answer.

(ii) Is it possible to cut this metal wire into 46 pieces of shorter metal wires, with each length measured as 5 cm correct to the nearest cm? Explain your answer.

1.3 HKCEE MA 2008 – I – 7

John wants to buy the following items in a supermarket:

Item	Unit price	Quantity needed
Biscuit	\$8.2 per pack	4 packs
Chocolate	\$16.3 per box	3 boxes
Soft drink	\$4.8 per can	2 cans

(a) By rounding up the unit price of each item to the nearest dollar, estimate the total amount that John should pay.

(b) If John has only \$100, does he have enough money to buy all the items needed? Use the result of (a) to explain your answer.

1.4 HKCEE MA 2009 – I – 4

Round off 405.504 to

(a) the nearest integer,

(b) 2 decimal places,

(c) 2 significant figures.

1.5 HKCEE MA 2010 I 8

Three students, Peter, John and Henry have \$16.8, \$24.3 and \$32.5 respectively.

(a) By rounding down the amount owned by each student to the nearest dollar, estimate the total amount they have.

(b) If the three students want to buy a football of price \$70, will they have enough money to buy the football? Use the result of (a) to explain your answer.

1.6 HKCEE MA 2011 – I – 4

(a) Round off 8 091.1908 to the nearest ten.

(b) Round up 8 091.1908 to 3 significant figures.

(c) Round down 8 091.1908 to 3 decimal places.

1.7 HKDSE MA 2013 – I – 8

A pack of sea salt is termed *regular* if its weight is measured as 100 g correct to the nearest g.

(a) Find the least possible weight of a *regular* pack of sea salt.

(b) Is it possible that the total weight of 32 *regular* packs of sea salt is measured as 3.1 kg correct to the nearest 0.1 kg? Explain your answer.

1.8 HKDSE MA 2014 – I – 3

(a) Round up 123.45 to 1 significant figure.

(b) Round off 123.45 to the nearest integer.

(c) Round down 123.45 to 1 decimal place.

1.9 HKDSE MA 2017 – I 9

A bottle is termed *standard* if its capacity is measured as 200 mL correct to the nearest 10 mL.

(a) Find the least possible capacity of a *standard* bottle.

(b) Someone claims that the total capacity of 120 *standard* bottles can be measured as 23.3 L correct to the nearest 0.1 L. Do you agree? Explain your answer.

1.10 HKDSE MA 2018 – I – 3

(a) Round up 265.473 to the nearest integer.

(b) Round down 265.473 to 1 decimal place.

(c) Round off 265.473 to 2 significant figures.

1.11 HKDSE MA 2020 – I – 3

(a) Round up 534.7698 to the nearest hundred.

(b) Round down 534.7698 to 2 decimal places.

(c) Round off 534.7698 to 2 significant figures.

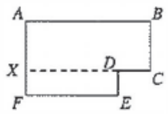
1 Estimation

1.1 HKCEE MA 2006 – I – 11

(a) Maximum absolute error $= 1\text{ cm} \div 2 = 0.5\text{ cm}$

(b) Least possible area of $ABCX = 17.5 \times 11.5 = 201.25\text{ cm}^2$
Least possible area of $DEFX = 1.5 \times 15.5 = 23.25\text{ cm}^2$
$\therefore$ Least possible area of sheet $= 224.5\text{ cm}^2$

(c) Upper limit of area $= 18.5 \times 12.5 + 2.5 \times 16.5 = 272.5\text{ cm}^2$
$\therefore 224.5 \le x < 272.5$



1.2 HKCEE MA 2007 – I – 10

(a) Least possible length $= 5 - 1 \div 2 = 4.5\text{ (cm)}$

(b) (i) Upper limit $= (2.0 + 0.1 \div 2)\text{ m} = 205\text{ cm} < 206\text{ cm}$
$\because$ No.

(ii) *Method 1*
Least possible total length of short wires
$4.5\text{ cm} \times 46 = 207\text{ cm} > 205\text{ cm}$
$\therefore$ No.

Method 2
Upper limit of length of one short wire
$= 205\text{ cm} \div 46 = 4.4565\text{ cm} < 4.5\text{ cm}$
$\because$ No.

1.3 HKCEE MA 2008 – I – 7

(a) Total amount $\approx \$(9 \times 4 + 17 \times 3 + 5 \times 2) = \$97$

(b) $\because$ Actual amount < Estimated amount < \$100
$\therefore$ Yes.

1.4 HKCEE MA 2009 – I – 4

(a) 406
(b) 405.50
(c) 410

1.5 HKCEE MA 2010 – I – 8

(a) Total amount $\approx \$(16 + 24 + 32) = \$72$

(b) $\because$ Actual amount > Estimated amount > \$70
Yes.

1.6 HKCEE MA 2011 – I – 4

(a) 8090
(b) 8100
(c) 8091.190

1.7 HKDSE MA 2013 – I – 8

(a) Least possible weight $= (100 - 1 \div 2)\text{ g} = 99.5\text{ g}$

(b) *Method 1*
Least possible total weight $= 99.5\text{ g} \times 32$
$= 3184\text{ g} = 3.2\text{ kg}$, nearest 0.1 kg
$\therefore$ No.

Method 2
Upper limit of weight of 1 pack $= \dfrac{3.1 + 0.1 \div 2}{32}\text{ kg}$
$= 98.43\text{ g} < 99.5\text{ g}$
$\therefore$ No.

1.8 HKDSE MA 2014 – I – 3

(a) 100
(b) 123
(c) 123.4

1.9 HKDSE MA 2017 – I – 9

(a) Least possible capacity $= (200 \quad 10 \div 5)\text{ mL} = 195\text{ mL}$

(b) *Method 1*
Least total capacity $= 195\text{ mL} \times 120 = 23.4\text{ L} > 23.35\text{ L}$
$\therefore$ No.

Method 2
Upper limit of capacity of 1 bottle $= \dfrac{23.3 + 0.1 \div 2}{120}\text{ L}$
$= 194.58\text{ mL} < 195\text{ mL}$
. No.

1.10 HKDSE MA 2018 – I – 3

(a) 266
(b) 265.4
(c) 270

1.11 HKDSE MA 2020 – I – 3

3a	600
b	534.76
c	530

2 Percentages

2A Basic percentages

2A.1 HKCEE MA 1989 – I – 1 (Also as 8A.4.)

(a) The monthly income of a man is increased from \$8000 to \$9000. Find the percentage increase.

(b) After the increase, the ratio of his savings to his expenditure is 3 : 7 for each month. How much does he save each month?

2A.2 HKCEE MA 2002 I 6

The radius of a circle is 8 cm. A new circle is formed by increasing the radius by 10%.

(a) Find the area of the new circle in terms of π.

(b) Find the percentage increase in the area of the circle.

2A.3 HKCEE MA 2006 – I – 6

The weight of Tom is 20% more than that of John. It is given that Tom weighs 60 kg.

(a) Find the weight of John.

(b) The weight of Susan is 20% less than that of Tom. Are Susan and John of the same weight? Explain your answer.

2A.4 HKCEE MA 2008 – I – 8

There are 625 boys in a school and the number of girls is 28% less than that of boys.

(a) Find the number of girls in the school.

(b) There are 860 local students in the school.

(i) Find the percentage of local students in the school.

(ii) It is given that 80% of the boys are local students. If x% of the girls are also local students, write down the value of x.

2A.5 HKCEE MA 2009 – I – 7

In a survey, there are 172 male interviewees. The number of female interviewees is 75% less than that of male interviewees. Find

(a) the number of female interviewees,

(b) the percentage of female interviewees in the survey.

2A.6 HKCEE MA 2010 – I – 7

Mary has 50 badges. The number of badges owned by Tom is 30% less than that owned by Mary.

(a) How many badges does Tom have?

(b) If Mary gives a certain number of her badges to Tom, will they have the same number of badges? Explain your answer.

2A.7 HKDSE MA 2012 – I – 4

The daily wage of Ada is 20% higher than that of Billy while the daily wage of Billy is 20% lower than that of Christine. It is given that the daily wage of Billy is \$480.

(a) Find the daily wage of Ada.

(b) Who has the highest daily wage? Explain your answer.

2A.8 HKDSE MA 2016 – I – 5

In a recreation club, there are 180 members and the number of male members is 40% more than the number of female members. Find the difference of the number of male members and the number of female members.

2A.9 HKDSE MA 2020 – I –

In a recruitment exercise, the number of male applicants is 28% more than the number of female applicants. The difference of the number of male applicants and the number of female applicants is 91. Find the number of male applicants in the recruitment exercise. (4 marks)

2B Discount, profit and loss

2B.1 HKCEE MA 1990 – I – 1

A person bought 10 gold coins at \$3000 each and later sold them all at \$2700 each.

(a) Find the total loss.

(b) Find the percentage loss.

2B.2 HKCEE MA 1994 – I – 6

A merchant bought an article for \$$x$. He put it in his shop for sale at a marked price 70% higher than its cost. The article was then sold to a customer at a discount of 5%.

(a) What was the percentage gain for the merchant by selling the article?

(b) If the customer paid \$2907 for the article, find the value of x.

2B.3 HKCEE MA 1995 – I – 4

Mr. Cheung bought a flat in 1993 for \$2400000. He made a profit of 30% when he sold the flat to Mr. Lee in 1994.

(a) Find the price of the flat that Mr. Lee paid.

(b) Mr. Lee then sold the flat in 1995 for \$3 000000. Find his percentage gain or loss.

2B.4 HKCEE MA 1998 – I – 7

The marked price of a toy car is \$29. It is sold at a discount of 20%.

(a) Find the selling price of the toy car.

(b) If the cost of the toy car is \$18, find the percentage profit.

2B.5 HKCEE MA 2001 – I – 8

The price of a textbook was \$80 last year. The price is increased by 20% this year.

(a) Find the new price.

(b) Peter is given a 20% discount when buying the textbook from a bookstore this year. How much does he pay for this book?

2B.6 HKCEE MA 2003 – I – 5

A handbag costs \$400. The marked price of the handbag is 20% above the cost. It is sold at a 25% discount on the marked price.

(a) Find the selling price of the handbag.

(b) Find the percentage profit or percentage loss.

2B.7 HKCEE MA 2005 – I – 6

The cost of a calculator is \$160. If the calculator is sold at its marked price, then the percentage profit is 25%.

(a) Find the marked price of the calculator.

(b) If the calculator is sold at a 10% discount on the marked price, find the percentage profit or percentage loss.

2B.8 HKCEE MA 2007 – I – 6

The marked price of a vase is \$400. The vase is sold at a discount of 20% on its marked price.

(a) Find the selling price of the vase.

(b) A profit of \$70 is made by selling the vase. Find the percentage profit.

2B.9 HKCEE MA 2011 – I – 7

The marked price of a birthday cake is \$360. The birthday cake is sold at a discount of 45% on its marked price.

(a) Find the selling price of the birthday cake.

(b) If the marked price of the birthday cake is 80% above its cost, determine whether there will be a gain or a loss after selling the birthday cake. Explain your answer.

2B.10 HKDSE MA SP – I – 4

The marked price of a handbag is \$560. It is given that the marked price of the handbag is 40% higher than the cost.

(a) Find the cost of the handbag.

(b) If the handbag is sold at \$460, find the percentage profit.

2B.11 HKDSE MA PP – I – 4

The cost of a chair is \$360. If the chair is sold at a discount of 20% on its marked price, then the percentage profit is 30%. Find the marked price of the chair.

2B.12 HKDSE MA 2014 – I – 6

The marked price of a toy is \$255. The toy is now sold at a discount of 40% on its marked price.

(a) Find the selling price of the toy.

(b) If the percentage profit is 2%, find the cost of the toy.

2B.13 HKDSE MA 2015 – I – 6

The cost of a book is \$250. The book is now sold and the percentage profit is 20%.

(a) Find the selling price of the book.

(b) If the book is sold at a discount of 25% on its marked price, find the marked price of the book.

2B.14 HKDSE MA 2018 – I – 7

The marked price of a vase is 30% above its cost. A loss of \$88 is made by selling the vase at a discount of 40% on its marked price. Find the marked price of the vase.

2B.15 HKDSE MA 2019 – I – 5

A wallet is sold at a discount of 25% on its marked price. The selling price of the wallet is \$690.

(a) Find the marked price of the wallet.

(b) After selling the wallet, the percentage profit is 15%. Find the cost of the wallet.

2C Interest

2C.1 HKCEE MA 1983(A/B) – I – 6

The compound interest on $1000 at 10% per annum for 3 years, compounded yearly, equals the simple interest on another $1000 at r% per annum for the same period of time. Calculate r to 2 decimal places.

2C.2 HKCEE MA 1991 – I – 3

(Also as **8A.6**)

A man buys some British pounds (£) with 150000 Hong Kong dollars (HK$) at the rate £1 = HK$15.00 and puts it on fixed deposit for 30 days. The rate of interest is 14.60% per annum.

(a) How much does he buy in British pounds?

(b) Find the amount in British pounds at the end of 30 days.
(Suppose 1 year = 365 days and the interest is calculated at simple interest.)

(c) If he sells the amount in (b) at the rate of £1 = HK$14.50, how much does he get in Hong Kong dollars?

2C.3 HKCEE MA 1993 – I – 1(a)

What is the simple interest on $100 for 6 months at 3% p.a.?

2C.4 HKCEE MA 1996 I – 12

Bank A offers personal loans at an interest rate of 18% per annum. For each successive month after the day when the loan is taken, loan interest is calculated and an instalment is paid.
(Answers to this question should be corrected to 2 decimal places.)

(a) Mr. Chan took a personal loan of $50000 from Bank A and agreed to repay the bank in monthly instalments of $9000 until the loan is fully repaid (the last instalment may be less than $9000). The outstanding balance of his loan for each of the first three months is shown in Table 1.
   (i) Complete Table 1 until the loan is fully repaid.
   (ii) Find the amount of his last instalment.
   (iii) Calculate the total interest earned by the bank.

(b) Mrs. Lee also took a personal loan of $50 000 from Bank A. She agreed to pay $9000 as the first monthly instalment and increase the amount of each instalment by 20% for every successive month until the loan is fully repaid. The outstanding balance of her loan for the first month is shown in Table 2.
Complete Table 2 until the loan is fully repaid.

(c) Mr. Cheung wants to buy a $50 000 piano for her daughter but he has no savings at hand. He intends to buy the piano by taking a personal loan of $50 000 from Bank A. If he can only save $12000 from his income every month and uses his savings to repay the loan, can he afford to use the repayment scheme as described in (b)? Explain your answer.

Table 1 The outstanding balance of Mr. Chan's loan for each month

Month	Loan Interest ($)	Loan Repaid ($)	Outstanding Balance ($)
1	750.00	8 250.00	41 750.00
2	626.25	8 373.75	33 376.25
3	500.64	8 499.36	24 876.89
4			
5			
6			

Table 2 The outstanding balance of Mrs. Lee's loan for each month

Month	Instalment ($)	Loan Interest ($)	Loan Repaid ($)	Outstanding Balance ($)
1	9 000.00	750.00	8 250.00	41 750.00
2				
3				
4				
5				

2C.5 HKCEE MA 2000 – I – 10

(a) Solve $10x^2+9x-22=0$.

(b) Mr. Tung deposited $10000 in a bank on his 25th birthday and $9000 on his 26th birthday. The interest was compounded yearly at r% p.a., and the total amount he received on his 27th birthday was $22000. Find r.

2C.6 HKCEE MA 2004 – I – 3

A sum of $5000 is deposited at 2% p.a. for 3 years, compounded yearly. Find the interest correct to the nearest dollar.

2 Percentages

2A Basic percentages

2A.1 HKCEE MA 1989 – I – 1

(a) % increase $= \frac{9000-8000}{8000} \times 100\% = 12.5\%$

(b) Amount saved $= \$9000 \times \frac{3}{3+7} = \$2700$

2A.2 HKCEE MA 2002 – I – 6

(a) New radius $= 8 \times (1+10\%) = 8.8$ (cm)

$\Rightarrow$ New area $= \pi(8.8)^2 = 77.44\pi$ (cm^2)

(b) % increase $\frac{77.44\pi - \pi(8)^2}{\pi(8)^2} \times 100\% = 21\%$

2A.3 HKCEE MA 2006 – I – 6

(a) Weight of John $= 60 \div (1+20\%) = 50$ (kg)

(b) Weight of Susan $= 60 \times (1-20\%) = 48 \neq 50$ (kg)

No.

2A.4 HKCEE MA 2008 – I – 8

(a) Number of girls $= 625 \times (1-25\%) = 450$

(b) (i) Required % $= \frac{860}{625+450} \times 100\% = 80\%$

(ii) 80

2A.5 HKCEE MA 2009 – I – 7

(a) Number of female interviewees $= 172 \times (1-75\%) = 43$

(b) Required % $= \frac{43}{172+43} \times 100\% = 20\%$

2A.6 HKCEE MA 2010 – I – 7

(a) Number of badges Tom has $= 50 \times (1-30\%) = 35$

(b) *Method 1*

Total number of badges $= 50 + 35 = 85$, which is odd!

$\therefore$ No.

Method 2

Let Mary give x badges.

$50 - x = 35 + x$

$x = 7.5$, which is not an integer!

No.

2A.7 HKDSE MA 2012 – I – 4

(a) Daily wage of Ada $= \$480 \times (1+20\%) = \$576$

(b) Daily wage of Christine $= \$480 \div (1 - 20\%) = \$600$

$\because$ $600 > 576 > 480$

$\therefore$ Christine

2A.8 HKDSE MA 2016 – I – 5

Let there be x female members.

Number of male members $= 1.4x$

$\Rightarrow$ $1.4x + x = 180$

$x = 75$

$\therefore$ There are 75 female and $1.4(75) = 105$ male members.

$\Rightarrow$ Difference $= 30$

2A.9 on the next page

2B Discount, profit and loss

2B.1 HKCEE MA 1990 – I – 1

(a) Total loss $= \$(3000 - 2700) \times 10 = \$3000$

(b) % loss $= \frac{3000}{3000 \times 10} \times 100\% = 10\%$

2B.2 HKCEE MA 1994 – I – 6

(a) Marked price $= \$1.7x$

Selling price $= \$1.7x(1-5\%) = \$1.615x$

$\therefore$ % gain $= \frac{1.615x - x}{x} \times 100\% = 61.5\%$

(b) $1.615x = 2907 \Rightarrow x = 1800$

2B.3 HKCEE MA 1995 – I – 4

(a) Price $= \$2400000 \times (1+30\%) = \$3120000$

(b) % loss $= \frac{3120000 - 3000000}{3120000} \times 100\% = 3.85\%$

2B.4 HKCEE MA 1998 – I – 7

(a) Selling price $= \$29 \times (1-20\%) = \$23.2$

(b) % profit $= \frac{23.2-18}{18} \times 100\% = 28.9\%$

2B.5 HKCEE MA 2001 – I – 8

(a) New price $= \$80 \times (1+20\%) = \$96$

(b) Amount he pays $= \$96 \times (1-20\%) = \$76.8$

2B.6 HKCEE MA 2003 – I – 5

(a) Marked price $= \$400 \times (1+20\%) = \$480$

$\Rightarrow$ Selling price $= \$480 \times (1-25\%) = \$360$

(b) % loss $= \frac{400-360}{400} \times 100\% = 10\%$

2B.7 HKCEE MA 2005 – I – 6

(a) Marked price $= \$160 \times (1+25\%) = \$200$

(b) Selling price $= \$200 \times (1-10\%) = \$180$

$\therefore$ % profit $= \frac{180-160}{160} \times 100\% = 12.5\%$

2B.8 HKCEE MA 2007 – I – 6

(a) Selling price $= \$400 \times (1-20\%) = \$320$

(b) % profit $= \frac{70}{320-70} \times 100\% = 28\%$

2B.9 HKCEE MA 2011 – I – 7

(a) Selling price $= \$360 \times (1-45\%) = \$198$

(b) Cost $= \$360 \div (1+80\%) = \$200 > \$198$

$\therefore$ Loss

2B.10 HKDSE MA SP – I – 4

(a) Cost $= \$560 \div (1+40\%) = \$400$

(b) % profit $= \frac{460-400}{400} \times 100\% = 15\%$

2B.11 HKDSE MA PP – I – 4

Selling price $= \$360 \times (1+30\%) = \$468$

$\Rightarrow$ Marked price $= \$468 \div (1-20\%) = \$585$

2B.12 HKDSE MA 2014 – I – 6

(a) Selling price $= \$255 \times (1 - 40\%) = \$153$

(b) Cost $= \$153 \div (1 + 2\%) = \$150$

2B.13 HKDSE MA 2015 – I – 6

(a) Selling price $= \$250 \times (1 + 20\%) = \$300$

(b) Marked price $= \$300 \div (1 - 25\%) = \$400$

2B.14 HKDSE MA 2018 – I – 7

Let the marked price be $\$x$. Then

Cost $= \$x \div (1 + 30\%) = \$\frac{10}{13}x$

Selling price $= \$x \times (1 - 40\%) = \$0.6x$

$0.6x + 88 = \frac{10}{13}x \Rightarrow x = 520$

$\therefore$ The marked price is \$520.

2B.15 HKDSE MA 2019 I – 5

(a) Marked price $= \$690 \div (1 \quad 25\%) = \$920$

(b) Cost $\quad \$690 \div (1 + 15\%) = \$600$

**2A.9 HKDSE MA 2020 – I – 5

Let x be the number of female applicants.

Then, the number of male applicants is $x(1 + 28\%) = 1.28x$.

$$\begin{aligned} 1.28x - x &= 91 \\ x &= 325 \end{aligned}$$

The number of male applicants $= 1.28 \times 325$

$= 416$

2C Interest

2C.1 HKCEE MA 1983(A/B) – I – 6

$$\begin{aligned} 1000(1 + 10\%)^3 - 1000 &= 1000 \times r\% \times 3 \\ 331 &= 30r \\ r &= 11.03 \quad (2 \text{ d.p.}) \end{aligned}$$

2C.2 HKCEE MA 1991 – I – 3

(a) $£150000 \div 15 = £10000$

(b) Amount $= 10000 + 10000 \times 14.60\% \times \frac{30}{365}$

$= (£)10120$

(c) $\$10120 \times 14.50 = \$146740$

2C.3 HKCEE MA 1993 – I – 1(a)

Interest $= \$100 \times 3\% \times \frac{6}{12} = \$1.5$

2C.4 HKCEE MA 1996 – I – 12

(a) (i) **Table 1**

4	373.15	8626.85	16250.04
5	243.75	8756.25	7493.79
6	112.41	7493.79	0

(ii) Amount $= 112.41 + 7493.79 = (\$)7606.20$

(iii) Total interest

$1750.00 + 626.25 + 500.64 + 373.15 + 243.75 + 112.41$

$= (\$)2606.20$

(b) **Table 2**

2	10800.00	626.25	10173.75	31576.25
3	12960.00	473.64	12486.36	19089.89
4	15552.00	286.35	15265.65	3824.24
5	3881.60	57.36	3824.24	0

(c)

Month	Savings (\$)	Instalment (\$)	Remaining (\$)
1	12000	9000	3000
2	15000	10800	4200
3	16200	12960	3240
4	15240	15552	

Since the savings (\$15240) would not be enough for another instalment (\$15552), he cannot.

2C.5 HKCEE MA 2000 – I – 10

(a) $x = 1.1$ or -2

(b)
$$\begin{aligned} 10000(1 + r\%)^2 + 9000(1 + r\%) &= 22000 \\ 10(1 + r\%)^2 + 9(1 + r\%) - 22 &= 0 \\ 1 + r\% &= 1.1 \text{ or } -2 \text{ (rejected)} \\ r &= 10 \end{aligned}$$

2C.6 HKCEE MA 2004 – I – 3

Interest $= \$5000(1 + 2\%)^3 - \$5000$

$= \$30.6$ (nearest dollar)

3 Indices and Logarithms

3A Laws of indices

3A.1 HKCEE MA 1987(A) – I – 3(a)

Simplify $\sqrt{\dfrac{3^{5k+2}}{27^k}}$.

3A.2 HKCEE MA 1990 – I – 2(a)

Simplify $\dfrac{a}{\sqrt{a}}$, expressing your answer in index form.

3A.3 HKCEE MA 1993 – I – 5(b)

Simplify and express with positive indices $x\left(\dfrac{x^{-1}}{y^2}\right)^{-3}$.

3A.4 HKCEE MA 1994 – I – 7(a)

Simplify $\dfrac{(a^4b^{-2})^2}{ab}$ and express your answer with positive indices.

3A.5 HKCEE MA 1996 – I – 2

Simplify $\dfrac{a^{\frac{5}{4}}\sqrt[4]{a^3}}{a^{-2}}$.

3A.6 HKCEE MA 1997 – I – 2(a)

Simplify $\dfrac{x^3y^2}{x^{-3}y}$ and express your answer with positive indices.

3A.7 HKCEE MA 1998 – I – 4

Simplify $\dfrac{a^3a^4}{b^{-2}}$ and express your answer with positive indices.

3A.8 HKCEE MA 1999 – I – 1

Simplify $\dfrac{(a^{-3})^2}{a}$ and express your answer with positive indices

3A.9 HKCEE MA 2000 – I – 2

Simplify $\dfrac{x^{-3}y}{x^2}$ and express your answer with positive indices.

3A.10 HKCEE MA 2001 – I – 1

Simplify $\dfrac{m^3}{(mn)^2}$ and express your answer with positive indices.

3A.11 HKCEE MA 2002 – I – 1

Simplify $\dfrac{(ab^2)^2}{a^5}$ and express your answer with positive indices.

3A.12 HKCEE MA 2003 – I – 4

Solve the equation $4^{x+1} = 8$.

3A.13 HKCEE MA 2004 – I – 1

Simplify $\dfrac{(a^{-1}b)^3}{b^2}$ and express your answer with positive indices.

3A.14 HKCEE MA 2005 – I – 2

Simplify $\dfrac{(x^3y)^2}{y^5}$ and express your answer with positive indices.

3A.15 HKCEE MA 2006 – I – 1

Simplify $\dfrac{(a^3)^5}{a^{-6}}$ and express your answer with positive indices.

3A.16 HKCEE MA 2007 – I – 2

Simplify $\dfrac{m^6}{m^9n^{-5}}$ and express your answer with positive indices

3A.17 HKCEE MA 2008 – I – 1

Simplify $\dfrac{(ab)^3}{a^2}$ and express your answer with positive indices.

3A.18 HKCEE MA 2009 – I – 2

Simplify $\dfrac{x^2}{(x^{-7}y)^3}$ and express your answer with positive indices.

3A.19 HKCEE MA 2010 – I – 1

Simplify $a^{14}\left(\dfrac{b^3}{a^2}\right)^5$ and express your answer with positive indices.

3A.20 HKCEE MA 2011 – I – 2

Simplify $\dfrac{x^{65}}{(x^4y^3)^2}$ and express your answer with positive indices.

3A.21 HKDSE MA SP – I – 1

Simplify $\dfrac{(xy)^2}{x^{-5}y^6}$ and express your answer with positive indices.

3A.22 HKDSE MA PP – I – 1

Simplify $\dfrac{(m^5n^{-2})^6}{m^4n^{-3}}$ and express your answer with positive indices.

3A.23 HKDSE MA 2012 – I – 1

Simplify $\dfrac{m^{-12}n^8}{n^3}$ and express your answer with positive indices.

3A.24 HKDSE MA 2013 – I – 1

Simplify $\dfrac{x^{20}y^{13}}{(x^5y)^6}$ and express your answer with positive indices.

3A.25 HKDSE MA 2014 – I – 1

Simplify $\dfrac{(xy^{-2})^3}{y^4}$ and express your answer with positive indices.

3A.26 HKDSE MA 2015 – I – 1

Simplify $\dfrac{m^9}{(m^3n^{-7})^5}$ and express your answer with positive indices.

3A.27 HKDSE MA 2016 – I – 1

Simplify $\dfrac{(x^8y^7)^2}{x^5y^{-6}}$ and express your answer with positive indices.

3A.28 HKDSE MA 2017 – I – 2

Simplify $\dfrac{(m^4n^{-1})^3}{(m^{-2})^5}$ and express your answer with positive indices.

3A.29 HKDSE MA 2018 – I – 2

Simplify $\dfrac{xy^7}{(x^{-2}y^3)^4}$ and express your answer with positive indices.

3A.30 HKDSE MA 2020 – I – 1

Simplify $\dfrac{(mn^{-2})^5}{m^{-4}}$ and express your answer with positive indices.

3B Logarithms

3B.1 HKCEE MA 1986(A) – I – 5(a)

Evaluate $\log_2 8 + \log_2 \dfrac{1}{16}$.

3B.2 HKCEE MA 1987(A) – I – 3(b)

Simplify $\dfrac{\log a^3b^2 - \log ab^2}{\log\sqrt{a}}$.

3B.3 HKCEE MA 1988 – I – 6

Give that $\log 2 = r$ and $\log 3 = s$, express the following in terms of r and s:

(a) $\log 18$,

(b) $\log 15$.

3B.4 HKCEE MA 1990 – I – 2(b)

Simplify $\dfrac{\log(a^2)+\log(b^4)}{\log(ab^2)}$, where $a, b > 0$.

3B.5 HKCEE MA 1991 – I – 7 (Also as **6C.8**.)

Let α and β be the roots of the equation $10x^2+20x+1=0$. Without solving the equation, find the values of

(a) $4^\alpha \times 4^\beta$,

(b) $\log_{10}\alpha + \log_{10}\beta$.

3B.6 HKCEE MA 1992 – I – 2(a)

If $\log x = p$ and $\log y = q$, express $\log xy$ in terms of p and q.

3B.7 HKCEE MA 1994 – I – 7(b)

If $\log 2 = x$ and $\log 3 = y$, express $\log\sqrt{12}$ in terms of x and y.

3B.8 HKCEE MA 1997 – I – 2(b)

Simplify $\dfrac{\log 8 + \log 4}{\log 16}$.

3B.9 HKDSE MA SP – I – 17

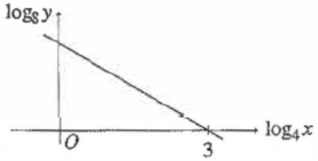
A researcher defined Scale A and Scale B to represent the magnitude of an explosion as shown in the table:

Scale	Formula
A	$M = \log_4 E$
B	$N = \log_8 E$

It is given that M and N are the magnitudes of an explosion on Scale A and Scale B respectively, while E is the relative energy released by the explosion. If the magnitude of an explosion is 6.4 on Scale B, find the magnitude of the explosion on Scale A.

3B.10 HKDSE MA 2014 – I – 15

The graph in the figure shows the linear relation between $\log_4 x$ and $\log_8 y$. The slope and the intercept on the horizontal axis of the graph are $\frac{-1}{3}$ and 3 respectively. Express the relation between x and y in the form $y = Ax^k$, where A and k are constants.



3B.11 HKDSE MA 2017 I 15

Let a and b be constants. Denote the graph of $y = a + \log_b x$ by G. The x intercept of G is 9 and G passes through the point $(243, 3)$. Express x in terms of y.

3C Exponential and logarithmic equations

3C.1 HKCEE MA 1980(3) – I 7

Find x if $\log_3(x-3) + \log_3(x+3) = 3$.

3C.2 HKCEE MA 1981(1) I 5 & HKCEE MA 1981(2) – I – 6

Solve $4^x = 10 \quad 4^{x+1}$.

3C.3 HKCEE MA 1982(1/2) I 2

If $\begin{cases} 4^{x-y} = 4 \\ 4^{x+y} = 16 \end{cases}$, solve for x and y.

3C.4 HKCEE MA 1985(B) I – 3

Solve $2^{2x} - 3(2^x) \quad 4 = 0$.

3C.5 HKCEE MA 1986(A) I 5(b)

If $2\log_{10} x - \log_{10} y = 0$, show that $y = x^2$.

3C.6 HKCEE MA 1987(B) I – 3

Solve the equation $3^{2x} + 3^x - 2 = 0$.

3C.7 HKCEE MA 1993 I 5(a)

If $9^x = \sqrt{3}$, find x.

3C.8 HKCEE MA 1995 I – 7

Solve the following equations without using a calculator:

(a) $3^x = \frac{1}{\sqrt{27}}$;

(b) $\log x + 2\log 4 = \log 48$.

3 Indices and Logarithms

3A Laws of indices

3A.1 HKCEE MA 1987(A) – I – 3(a)

$$\sqrt{\frac{3^{5k+2}}{27^k}}=\left(\frac{3^{5k+2}}{3^{3k}}\right)^{\frac{1}{2}}=(3^{2k+2})^{\frac{1}{2}}=3^{k+1}$$

3A.2 HKCEE MA 1990 – I – 2(a)

$$\frac{a}{\sqrt{a}}=a^{1-\frac{1}{2}}=a^{\frac{1}{2}}$$

3A.3 HKCEE MA 1993 – I – 5(b)

$$x\left(\frac{x^{-1}}{y^2}\right)^{-3}=x\left(\frac{x^{+3}}{y^{-6}}\right)=x^4y^6$$

3A.4 HKCEE MA 1994 I – 7(a)

$$\frac{(a^4b^{-2})^2}{ab}=\frac{a^8b^{-4}}{ab}=\frac{a^{8-1}}{b^{1+4}}=\frac{a^7}{b^5}$$

3A.5 HKCEE MA 1996 – I – 2

$$\frac{a^{\frac{5}{4}}\sqrt[4]{a^3}}{a^{-2}}=\frac{a^{\frac{5}{4}}a^{\frac{3}{4}}}{a^{-2}}=a^{\frac{5}{4}+\frac{3}{4}-(-2)}=a^4$$

3A.6 HKCEE MA 1997 – I – 2(a)

$$\frac{x^3y^2}{x^{-3}y}=x^{3-(-3)}y^{2-1}=x^6y$$

3A.7 HKCEE MA 1998 – I – 4

$$\frac{a^3a^4}{b^{-2}}=a^{3+4}b^2=a^7b^2$$

3A.8 HKCEE MA 1999 – I – 1

$$\frac{(a^{\ 3})^2}{a}\frac{a^{-6}}{a}=\frac{1}{a^{1+6}}=\frac{1}{a^7}$$

3A.9 HKCEE MA 2000 – I – 2

$$\frac{x^{-3}y}{x^2}=\frac{y}{x^{2+3}}=\frac{y}{x^5}$$

3A.10 HKCEE MA 2001 – I – 1

$$\frac{m^3}{(mn)^2}=\frac{m^3}{m^2n^2}=\frac{m}{n^2}$$

3A.11 HKCEE MA 2002 – I – 1

$$\frac{(ab^2)^2}{a^5}=\frac{a^2b^4}{a^5}=\frac{b^4}{a^{5\ 2}}=\frac{b^4}{a^3}$$

3A.12 HKCEE MA 2003 – I – 4

$$2^{2(x+1)}=2^3\ \Rightarrow\ 2x+2=3\ \Rightarrow\ x=\frac{1}{2}$$

3A.13 HKCEE MA 2004 – I – 1

$$\frac{(a^{-1}b)^3}{b^2}=\frac{a^{-3}b^3}{b^2}=\frac{b^{3-2}}{a^3}=\frac{b}{a^3}$$

3A.14 HKCEE MA 2005 – I – 2

$$\frac{(x^3y)^2}{y^5}=\frac{x^6y^2}{y^5}=\frac{x^6}{y^3}$$

3A.15 HKCEE MA 2006 – I – 1

$$\frac{(a^3)^5}{a^{-6}}=\frac{a^{15}}{a^{-6}}=a^{15\ (\ 6)}=a^{21}$$

3A.16 HKCEE MA 2007 – I – 2

$$\frac{m^6}{m^9n^{-5}}=\frac{n^5}{m^{9-6}}=\frac{n^5}{m^3}$$

3A.17 HKCEE MA 2008 – I – 1

$$\frac{(ab)^3}{a^2}=\frac{a^3b^3}{a^2}=ab^3$$

3A.18 HKCEE MA 2009 – I – 2

$$\frac{x^2}{(x^{-7}y)^3}=\frac{x^2}{x^{-21}y^3}=\frac{x^{2+21}}{y^3}=\frac{x^{23}}{y^3}$$

3A.19 HKCEE MA 2010 – I – 1

$$a^{14}\left(\frac{b^3}{a^2}\right)^5=a^{14}\cdot\frac{b^{15}}{a^{10}}=a^4b^{15}$$

3A.20 HKCEE MA 2011 – I – 2

$$\frac{x^{65}}{(x^4y^3)^2}=\frac{x^{65}}{x^8y^6}=\frac{x^{57}}{y^6}$$

3A.21 HKDSE MA SP – I – 1

$$\frac{(xy)^2}{x^{\ 5}y^6}=\frac{x^2y^2}{x^{-5}y^6}=\frac{x^{2+5}}{y^{6-2}}=\frac{x^7}{y^4}$$

3A.22 HKDSE MA PP – I – 1

$$\frac{(m^5n^{-2})^6}{m^4n^{-3}}\frac{m^{30}n^{-12}}{m^4n^{\ 3}}=\frac{m^{30-4}}{n^{-3+12}}=\frac{m^{26}}{n^9}$$

3A.23 HKDSE MA 2012 – I – 1

$$\frac{m^{-12}n^8}{n^3}=\frac{n^{8-3}}{m^{12}}=\frac{n^5}{m^{12}}$$

3A.24 HKDSE MA 2013 – I – 1

$$\frac{x^{20}y^{13}}{(x^5y)^6}=\frac{x^{20}y^{13}}{x^{30}y^6}=\frac{y^7}{x^{10}}$$

3A.25 HKDSE MA 2014 – I – 1

$$\frac{(xy^{-2})^3}{y^4}=\frac{x^3y^{-6}}{y^4}=\frac{x^3}{y^{4+6}}=\frac{x^3}{y^{10}}$$

3A.26 HKDSE MA 2015 – I – 1

$$\frac{m^9}{(m^3n^{-7})^5}=\frac{m^9}{m^{15}n^{-35}}=\frac{n^{35}}{m^6}$$

3A.27 HKDSE MA 2016 – I – 1

$$\frac{(x^8y^7)^2}{x^5y^{-6}}=\frac{x^{16}y^{14}}{x^5y^{\ 6}}=x^{16-5}y^{14\ (-6)}=x^{11}y^{20}$$

3A.28 HKDSE MA 2017 – I – 2

$$\frac{(m^4n^{-1})^3}{(m^{-2})^5}=\frac{m^{12}n^{-3}}{m^{-10}}=\frac{m^{12-(\ 10)}}{n^{3}}=\frac{m^{22}}{n^3}$$

3A.29 HKDSE MA 2018 – I – 2

$$\frac{xy^7}{(x^{-2}y^3)^4}=\frac{xy^7}{x^{-8}y^{12}}=\frac{x^{1+8}}{y^{12-7}}=\frac{x^9}{y^5}$$

3A.30 HKDSE MA 2020 – I – 1

$$\begin{aligned}\frac{(mn^{-2})^5}{m^{-4}}&=m^{5-(-4)}n^{-2\times5}\\&=m^9n^{-10}\\&=\frac{m^9}{n^{10}}\end{aligned}$$

3B Logarithms

3B.1 HKCEE MA 1986(A) – I – 5(a)

$$\log_2 8+\log_2\frac{1}{16}=\log_2 2^3+\log_2 2^{-4}=3+(-4)=-1$$

3B.2 HKCEE MA 1987(A) I – 3(b)

$$\frac{\log a^3b^2-\log ab^2}{\log\sqrt{a}}=\frac{\log\frac{a^3b^2}{ab^2}}{\frac{1}{2}\log a}=\frac{\log a^2}{\frac{1}{2}\log a}=\frac{2\cancel{\log a}}{\frac{1}{2}\cancel{\log a}}=4$$

3B.3 HKCEE MA 1988 – I – 6

(a) $\log 18=\log 2\cdot 3^2=\log 2+2\log 3=r+2s$

(b) $\log 15=\log\frac{3\times 10}{2}=\log 3+1-\log 2=s+1-r$

3B.4 HKCEE MA 1990 – I – 2(b)

$$\frac{\log(a^2)+\log(b^4)}{\log(ab^2)}=\frac{\log a^2b^4}{\log ab^2}=\frac{\log(ab^2)^2}{\log ab^2}=\frac{2\cancel{\log ab^2}}{\cancel{\log ab^2}}=2$$

3B.5 HKCEE MA 1991 – I – 7

$$\begin{cases}\alpha+\beta=\ \ 2\\ \alpha\beta=\frac{1}{10}\end{cases}$$

(a) $4^\alpha\times 4^\beta=4^{\alpha+\beta}=4^{-2}=\frac{1}{16}$

(b) $\log_{10}\alpha+\log_{10}\beta=\log_{10}\alpha\beta=\log_{10}\frac{1}{10}=-1$

3B.6 HKCEE MA 1992 – I – 2(a)

$\log xy=\log x+\log y=p+q$

3B.7 HKCEE MA 1994 I – 7(b)

$$\log\sqrt{12}=\frac{1}{2}\log 2^2\cdot 3=\frac{1}{2}(2\log 2+\log 3)=\frac{2x+y}{2}$$

3B.8 HKCEE MA 1997 – I – 2(b)

$$\frac{\log 8+\log 4\qquad 3\log 2+2\log 2}{\log 16\qquad\quad 4\log 2}=\frac{5\cancel{\log 2}}{4\cancel{\log 2}}=\frac{5}{4}$$

3B.9 HKDSE MA SP – I – 17

Method 1

$6.4=\log_8 E \Rightarrow E=8^{6.4}$

$$\begin{aligned}\therefore\ M=\log_4 E=\log_4(8^{6.4})&=\frac{\log_2 8^{6.4}}{\log_2 4}\\&=\frac{\log_2 2^{3(6.4)}}{\log_2 2^2}=\frac{19.2}{2}=9.6\end{aligned}$$

Method 2

$$\begin{cases}M=\log_4 E\\N=\log_8 E\end{cases}\Rightarrow\begin{cases}E=4^M\\E=8^N\end{cases}\Rightarrow\begin{aligned}4^M&=8^N\\2^{2M}&=2^{3N}\\M&=\frac{3}{2}N=\frac{3}{2}(6.4)=9.6\end{aligned}$$

3B.10 HKDSE MA 2014 – I – 15

Method 1

From the graph, $(\log_4 x,\log_8 y)=(3,0)$ and Slope $=\frac{-1}{3}$.

Using point-slope form, the equation is:

$$\begin{aligned}\log_8 y-0&=\frac{-1}{3}(\log_4 x-3)\\\log_8 y&=\frac{-1}{3}\log_4 x+1\\&=\log_4\left(x^{\frac{-1}{3}}\cdot 4\right)\\\frac{\log_2 y}{\log_2 8}&=\frac{\log_2 4x^{\frac{-1}{3}}}{\log_2 4}\\\frac{\log_2 y}{3}&=\frac{\log_2 4x^{\frac{-1}{3}}}{2}\\\log_2 y&=\frac{3}{2}\log_2 4x^{\frac{-1}{3}}\\&=\log_2\left(4x^{\frac{-1}{3}}\right)^{\frac{3}{2}}=\log_2 8x^{\frac{-1}{2}}\\\Rightarrow\ y&=8x^{\frac{-1}{2}}\end{aligned}$$

Method 2

$(\log_4 x,\log_8 y)=(3,0)\Rightarrow (x,y)=(64,1)$

Let the point of the line cutting the vertical axis be $(0,b)$.

$\frac{b-0}{0-3}=\frac{-1}{3}\Rightarrow b=1$

$\therefore (\log_4 x,\log_8 y)=(0,1)\Rightarrow (x,y)=(1,8)$

Putting into $y=Ax^k$, $\begin{cases}8=A\\1=A(64)^k\Rightarrow 1=8^{1+2k}\Rightarrow k=\frac{-1}{2}\end{cases}$

Hence, $y=8x^{\frac{-1}{2}}$.

Method 3

$$\begin{aligned}y=Ax^k\Rightarrow\ \log_2 y&=\log_2 Ax^k=\log_2 A+k\log_2 x\\\frac{\log_8 y}{\log_8 2}&=\log_2 A+k\frac{\log_4 x}{\log_4 2}\\3\log_8 y&=\log_2 A+2k\log_4 x\\\log_8 y&=\frac{2k}{3}\log_4 x+\frac{1}{3}\log_2 A\end{aligned}$$

From theory of straight lines,

$$\begin{cases}\frac{-1}{3}=\text{Slope}=\frac{2k}{3}\Rightarrow k=\frac{-1}{2}\\3=x\text{-intercept}=-\frac{\frac{1}{3}\log_2 A}{\frac{2k}{3}}=\frac{-1}{2k}\log_2 A\Rightarrow A=2^3=8\end{cases}$$

Hence, $y=8x^{\frac{-1}{2}}$.

3B.11 HKDSE MA 2017 – I – 15

G passes through $(9,0)$ and $(243,3)$

$\Rightarrow\begin{cases}0=a+\log_b 9\\3=a+\log_b 243\end{cases}\Rightarrow 3=\log_b 243-\log_b 9=\log_b\frac{243}{9}$

$\Rightarrow b^3=27\Rightarrow b=9\Rightarrow a=-\log_b 9=-2$

$\therefore y=-2+\log_3 x\Rightarrow \log_3 x=y+2\Rightarrow x=3^{y+2}$

3C Exponential and logarithmic equations

3C.1 HKCEE MA 1980(3) – I – 7

$$\begin{aligned}\log_3(x-3)+\log_3(x+3)&=3\\\log_3(x-3)(x+3)&=3\\x^2-9&=27\\x&=6\text{ or }\ \ 6\text{ (rejected)}\end{aligned}$$

3C.2 HKCEE MA 1981(1) – I – 5 & 1981(2) – I – 6

$$\begin{aligned}4^x&=10-4^{x+1}\\4^x&=10-4^x\cdot 4\\(1+4)4^x&=10\\4^x&=2\Rightarrow x=\frac{1}{2}\end{aligned}$$

3C.3 HKCEE MA 1982(1/2) – I – 2

$$\begin{cases}4^{x-y}=4\Rightarrow x\ \ \ y=1\\4^{x+y}=16\Rightarrow x+y=2\end{cases}\Rightarrow\begin{cases}x=\frac{3}{2}\\y=\frac{1}{2}\end{cases}$$

3C.4 HKCEE MA 1985(B) – I – 3

$$\begin{aligned}2^{2x}\ \ \ 3(2^x)\ \ \ 4&=0\\(2^x)^2\ \ \ 3(2^x)\ \ \ 4&=0\\(2^x-4)(2^x+1)&=0\\2^x&=4\text{ or }-1\text{ (rejected)}\Rightarrow x=2\end{aligned}$$

3C.5 HKCEE MA 1986(A) – I – 5(b)

$$\begin{aligned}2\log_{10}x-\log_{10}y&=0\\\log_{10}x^2&=\log_{10}y\\x^2&=y\end{aligned}$$

3C.6 HKCEE MA 1987(B) – I – 3

$$\begin{aligned}3^{2x}+3^x-2&=0\\(3^x)^2+(3^x)-2&=0\\(3^x+2)(3^x-1)&=0\\3^x&=-2\text{ (rejected) or }1\Rightarrow x=0\end{aligned}$$

3C.7 HKCEE MA 1993 – I – 5(a)

$9^x=\sqrt{3}$

$3^{2x}=3^{\frac{1}{2}}\Rightarrow 2x=\frac{1}{2}\Rightarrow x=\frac{1}{4}$

3C.8 HKCEE MA 1995 – I – 7

(a) $3^x=\frac{1}{\sqrt{27}}=27^{\frac{-1}{2}}=(3^3)^{\frac{-1}{2}}$

$x=\frac{-3}{2}$

(b) $\log x+2\log 4=\log 48$

$\log x+\log 4^2=\log 48$

$\log 16x=\log 48\Rightarrow 16x=48\Rightarrow x=3$

4 Polynomials

4A Factorization, H.C.F. and L.C.M. of polynomials

4A.1 HKCEE MA 1980(1/1*/3) I 2

Factorize

(a) $a(3b-c)+c-3b$,
(b) x^4-1.

4A.2 HKCEE MA 1981(2/3) I 5

Factorize $(1+x)^4-(1-x^2)^2$.

4A.3 HKCEE MA 1983(A/B) – I – 1

Factorise $(x^2+4x+4)-(y-1)^2$.

4A.4 HKCEE MA 1984(A/B) I – 4

Factorize

(a) $x^2y+2xy+y$,
(b) $x^2y+2xy+y-y^3$.

4A.5 HKCEE MA 1985(A/B) I – 1

(a) Factorize a^4-16 and a^3-8.
(b) Find the L.C.M. of a^4-16 and a^3-8.

4A.6 HKCEE MA 1986(A/B) I 1

Factorize

(a) x^2-2x-3,
(b) $(a^2+2a)^2-2(a^2+2a)-3$.

4A.7 HKCEE MA 1987(A/B) I 1

Factorize

(a) x^2-2x+1,
(b) $x^2-2x+1-4y^2$.

4A.8 HKCEE MA 1993 – I 2(e)

Find the H.C.F. and L.C.M. of $6x^2y^3$ and $4xy^2z$.

4A.9 HKCEE MA 1995 I 1(b)

Find the H.C.F. of $(x-1)^3(x+5)$ and $(x-1)^2(x+5)^3$.

4A.10 HKCEE MA 1997 – I 1

Factorize

(a) x^2-9,
(b) $ac+bc-ad-bd$.

4A.11 HKCEE MA 2003 – I 3

Factorize

(a) $x^2-(y-x)^2$,
(b) $ab-ad-bc+cd$.

4A.12 HKCEE MA 2004 – I 6

Factorize

(a) $a^2-ab+2a-2b$,
(b) $169y^2-25$.

4A.13 HKCEE MA 2005 – I 3

Factorize

(a) $4x^2-4xy+y^2$,
(b) $4x^2-4xy+y^2-2x+y$.

4A.14 HKCEE MA 2007 I – 3

Factorize

(a) $r^2+10r+25$,
(b) $r^2+10r+25-s^2$.

4A.15 HKCEE MA 2009 – I 3

Factorize

(a) a^2b+ab^2,
(b) $a^2b+ab^2+7a+7b$.

4A.16 HKCEE MA 2010 – I – 3

Factorize

(a) $m^2+12mn+36n^2$,
(b) $m^2+12mn+36n^2-25k^2$.

4A.17 HKCEE MA 2011 – I – 3

Factorize

(a) $81m^2-n^2$,
(b) $81m^2 \quad n^2+18m-2n$.

4A.18 HKDSE MA SP – I – 3

Factorize

(a) $3m^2 - mn - 2n^2$,

(b) $3m^2 \quad mn - 2n^2 - m + n$.

4A.19 HKDSE MA PP – I – 3

Factorize

(a) $9x^2 - 42xy + 49y^2$,

(b) $9x^2 - 42xy + 49y^2 - 6x + 14y$.

4A.20 HKDSE MA 2012 – I – 3

Factorize

(a) $x^2 - 6xy + 9y^2$,

(b) $x^2 - 6xy + 9y^2 + 7x - 21y$.

4A.21 HKDSE MA 2013 – I – 3

Factorize

(a) $4m^2 - 25n^2$,

(b) $4m^2 - 25n^2 + 6m - 15n$.

4A.22 HKDSE MA 2014 – I – 2

Factorize

(a) $a^2 - 2a - 3$,

(b) $ab^2 + b^2 + a^2 - 2a - 3$.

4A.23 HKDSE MA 2015 – I – 4

Factorize

(a) $x^3 + x^2y - 7x^2$,

(b) $x^3 + x^2y - 7x^2 - x - y + 7$.

4A.24 HKDSE MA 2016 I 4

Factorize

(a) $5m - 10n$,

(b) $m^2 + mn \quad 6n^2$,

(c) $m^2 + mn - 6n^2 - 5m + 10n$.

4A.25 HKDSE MA 2017 – I – 3

Factorize

(a) $x^2 - 4xy + 3y^2$,

(b) $x^2 - 4xy + 3y^2 + 11x - 33y$.

4A.26 HKDSE MA 2018 I – 5

Factorize

(a) $9r^3 - 18r^2s$,

(b) $9r^3 - 18r^2s - rs^2 + 2s^3$.

4A.27 HKDSE MA 2019 – I – 4

Factorize

(a) $4m^2 - 9$,

(b) $2m^2n + 7mn - 15n$,

(c) $4m^2 - 9 - 2m^2n - 7mn + 15n$.

4A.28 HKDSE MA 2020 – I – 2

Factorize

(a) $\alpha^2 + \alpha - 6$,

(b) $\alpha^4 + \alpha^3 - 6\alpha^2$.

4B Division algorithm, remainder theorem and factor theorem

4B.1 HKCEE MA 1980(1*/3) – I – 13(a)

It is given that $f(x) = 2x^2 + ax + b$.

(i) If $f(x)$ is divided by $(x-1)$, the remainder is -5. If $f(x)$ is divided by $(x+2)$, the remainder is 4. Find the values of a and b.

(ii) If $f(x) = 0$, find the value of x.

4B.2 HKCEE MA 1981(2) – I – 3 and HKCEE MA 1981(3) – I – 2

Let $f(x) = (x+2)(x-3) + 3$. When $f(x)$ is divided by $(x-k)$, the remainder is k. Find k.

4B.3 HKCEE MA 1984(A/B) – I – 1

If $3x^2 - kx - 2$ is divisible by $x - k$, where k is a constant, find the two values of k.

4B.4 HKCEE MA 1985(A/B) – I – 4

Given $f(x) = ax^2 + bx - 1$, where a and b are constants. $f(x)$ is divisible by $x-1$. When divided by $x+1$, $f(x)$ leaves a remainder of 4. Find the values of a and b.

4B.5 HKCEE MA 1987(A/B) – I – 2

Find the values of a and b if $2x^3 + ax^2 + bx - 2$ is divisible by $x-2$ and $x+1$.

4B.6 HKCEE MA 1989 – I – 3

Given that $(x+1)$ is a factor of $x^4 + x^3 - 8x + k$, where k is a constant,

(a) find the value of k,

(b) factorize $x^4 + x^3 - 8x + k$.

4B.7 HKCEE MA 1990 – I – 7

(a) Find the remainder when $x^{1000} + 6$ is divided by $x+1$.

(b) (i) Using (a), or otherwise, find the remainder when $8^{1000} + 6$ is divided by 9.

(ii) What is the remainder when 8^{1000} is divided by 9?

4B.8 HKCEE MA 1990 – I – 11 (Continued from **15B.6.**)

A solid right circular cylinder has radius r and height h. The volume of the cylinder is V and the total surface area is S.

(a) (i) Express S in terms of r and h.

(ii) Show that $S = 2\pi r^2 + \dfrac{2V}{r}$.

(b) Given that $V = 2\pi$ and $S = 6\pi$, show that $r^3 - 3r + 2 = 0$. Hence find the radius r by factorization.

(c) *[Out of syllabus]*

4B.9 HKCEE MA 1992 – I – 2(b)

Find the remainder when $x^3 - 2x^2 + 3x - 4$ is divided by $x - 1$.

4B.10 HKCEE MA 1993 – I – 2(d)

Find the remainder when $x^3 + x^2$ is divided by $x - 1$.

4B.11 HKCEE MA 1994 – I – 3

When $(x+3)(x-2)+2$ is divided by $x-k$, the remainder is k^2. Find the value(s) of k.

4B.12 HKCEE MA 1995 – I – 2

(a) Simplify $(a+b)^2 - (a-b)^2$.

(b) Find the remainder when $x^3 + 1$ is divided by $x+2$.

4B.13 HKCEE MA 1996 – I – 4

Show that $x+1$ is a factor of $x^3 - x^2 - 3x - 1$.
Hence solve $x^3 - x^2 - 3x - 1 = 0$. (Leave your answers in surd form.)

4B.14 HKCEE MA 1998 – I – 9

Let $f(x) = x^3 + 2x^2 - 5x - 6$.

(a) Show that $x-2$ is a factor of $f(x)$.

(b) Factorize $f(x)$.

4B.15 HKCEE MA 2000 – I – 6

Let $f(x) = 2x^3 + 6x^2 - 2x - 7$. Find the remainder when $f(x)$ is divided by $x+3$.

4B.16 HKCEE MA 2001 – I – 2

Let $f(x) = x^3 - x^2 + x - 1$. Find the remainder when $f(x)$ is divided by $x-2$.

4B.17 HKCEE MA 2002 – I – 4

Let $f(x) = x^3 - 2x^2 - 9x + 18$.

(a) Find $f(2)$.

(b) Factorize $f(x)$.

4B.18 HKCEE MA 2005 – I – 10 (Continued from **8C.16.**)

It is known that $f(x)$ is the sum of two parts, one part varies as x^3 and the other part varies as x.
Suppose $f(2) = -6$ and $f(3) = 6$.

(a) Find $f(x)$.

(b) Let $g(x) = f(x) - 6$.

(i) Prove that $x-3$ is a factor of $g(x)$.

(ii) Factorize $g(x)$.

4B.19 HKCEE MA 2007 – I – 14 (To continue as **8C.18.**)

(a) Let $f(x) = 4x^3 + kx^2 - 243$, where k is a constant. It is given that $x+3$ is a factor of $f(x)$.

(i) Find the value of k.

(ii) Factorize $f(x)$.

4B.20 HKDSE MA SP – I – 10

(a) Find the quotient when $5x^3 + 12x^2 - 9x - 7$ is divided by $x^2 + 2x - 3$.

(b) Let $g(x) = (5x^3 + 12x^2 - 9x - 7) - (ax + b)$, where a and b are constants. It is given that $g(x)$ is divisible by $x^2 + 2x - 3$.

(i) Write down the values of a and b.

(ii) Solve the equation $g(x) = 0$.

4B.21 HKDSE MA PP – I – 10

Let $f(x)$ be a polynomial. When $f(x)$ is divided by $x-1$, the quotient is $6x^2+17x-2$. It is given that $f(1)=4$.

(a) Find $f(-3)$.

(b) Factorize $f(x)$.

4B.22 HKDSE MA 2012 – I – 13 (To continue as **7B.17**.)

(a) Find the value of k such that $x-2$ is a factor of $kx^3-21x^2+24x-4$.

4B.23 HKDSE MA 2013 – I – 12

Let $f(x)=3x^3-7x^2+kx-8$, where k is a constant. It is given that $f(x)\equiv(x-2)(ax^2+bx+c)$, where a, b and c are constants.

(a) Find a, b and c.

(b) Someone claims that all the roots of the equation $f(x)=0$ are real numbers. Do you agree? Explain your answer.

4B.24 HKDSE MA 2014 – I – 7

Let $f(x)=4x^3 \quad 5x^2-18x+c$, where c is a constant. When $f(x)$ is divided by $x-2$, the remainder is 33.

(a) Is $x+1$ a factor of $f(x)$? Explain your answer.

(b) Someone claims that all the roots of the equation $f(x)=0$ are rational numbers. Do you agree? Explain your answer.

4B.25 HKDSE MA 2015 – I 11

Let $f(x)=(x-2)^2(x+h)+k$, where h and k are constants. When $f(x)$ is divided by $x-2$, the remainder is -5. It is given that $f(x)$ is divisible by $x-3$.

(a) Find h and k.

(b) Someone claims that all the roots of the equation $f(x)=0$ are integers. Do you agree? Explain your answer.

4B.26 HKDSE MA 2016 – I – 14

Let $p(x)=6x^4+7x^3+ax^2+bx+c$, where a, b and c are constants. When $p(x)$ is divided by $x+2$ and when $p(x)$ is divided by $x-2$, the two remainders are equal. It is given that $p(x)\equiv(lx^2+5x+8)(2x^2+mx+n)$, where l, m and n are constants.

(a) Find l, m and n.

(b) How many real roots does the equation $p(x)=0$ have? Explain your answer.

4B.27 HKDSE MA 2017 I – 14

Let $f(x)=6x^3-13x^2-46x+34$. When $f(x)$ is divided by $2x^2+ax+4$, the quotient and the remainder are $3x+7$ and $bx+c$ respectively, where a, b and c are constants.

(a) Find a.

(b) Let $g(x)$ be a quadratic polynomial such that when $g(x)$ is divided by $2x^2+ax+4$, the remainder is $bx+c$.

(i) Prove that $f(x)-g(x)$ is divisible by $2x^2+ax+4$.

(ii) Someone claims that all the roots of the equation $f(x)-g(x)=0$ are integers. Do you agree? Explain your answer.

4B.28 HKDSE MA 2018 – I – 12

Let $f(x)=4x(x+1)^2+ax+b$, where a and b are constants. It is given that $x-3$ is a factor of $f(x)$. When $f(x)$ is divided by $x+2$, the remainder is $2b+165$.

(a) Find a and b.

(b) Someone claims that the equation $f(x)=0$ has at least one irrational root. Do you agree? Explain your answer.

4B.29 HKDSE MA 2019 – I – 11

Let $p(x)$ be a cubic polynomial. When $p(x)$ is divided by $x-1$, the remainder is 50. When $p(x)$ is divided by $x+2$, the remainder is 52. It is given that $p(x)$ is divisible by $2x^2+9x+14$.

(a) Find the quotient when $p(x)$ is divided by $2x^2+9x+14$.

(b) How many rational roots does the equation $p(x)=0$ have? Explain your answer.

4 Polynomials

4A Factorization, H.C.F. and L.C.M. of polynomials

4A.1 HKCEE MA 1980(1/1*/3) – I – 2

(a) $a(3b-c)+c-3b=(3b-c)(a-1)$

(b) $x^4-1=(x-1)(x+1)(x^2+1)$

4A.2 HKCEE MA 1981(2/3) – I – 5

$$\begin{aligned}(1+x)^4-(1-x^2)^2&=[(1+x)^2]^2-(1-x^2)^2\\&=[(1+x)^2-(1-x^2)][(1+x)^2+(1-x^2)]\\&=(2x+2x^2)(2+2x)=4x(1+x)^2\end{aligned}$$

4A.3 HKCEE MA 1983(A/B) – I – 1

$$\begin{aligned}(x^2+4x+4)-(y-1)^2&=(x+2)^2-(y-1)^2\\&=[(x+2)-(y-1)][(x+2)+(y-1)]\\&=(x-y+3)(x+y+1)\end{aligned}$$

4A.4 HKCEE MA 1984(A/B) – I – 4

(a) $x^2y+2xy+y=y(x^2+2x+1)=y(x+1)^2$

(b) $$\begin{aligned}x^2y+2xy+y-y^3&=y(x+1)^2-y^3\\&=y[(x+1)^2-y^2]\\&=y(x+1-y)(x+1+y)\end{aligned}$$

4A.5 HKCEE MA 1985(A/B) – I – 1

(a) $a^4-16=(a-2)(a+2)(a^2+4)$

$a^3-8=(a-2)(a^2+2a+4)$

(b) L.C.M. $=(a-2)(a+2)(a^2+4)(a^2+2a+4)$

4A.6 HKCEE MA 1986(A/B) – I – 1

(a) $x^2-2x-3=(x-3)(x+1)$

(b) $(a^2+2a)^2-2(a^2+2a)-3$
$=[(a^2+2a)-3][(a^2+2a)+1]=(a+3)(a-1)(a+1)^2$

4A.7 HKCEE MA 1987(A/B) – I – 1

(a) $x^2-2x+1=(x-1)^2$

(b) $$\begin{aligned}x^2-2x+1-4y^2&=(x-1)^2-(2y)^2\\&=(x-1-2y)(x-1+2y)\end{aligned}$$

4A.8 HKCEE MA 1993 – I – 2(c)

H.C.F. $=2xy^2$, L.C.M. $=12x^2y^3z$

4A.9 HKCEE MA 1995 – I – 1(b)

H.C.F. $=(x-1)^2(x+5)$

4A.10 HKCEE MA 1997 – I – 1

(a) $x^2-9=(x-3)(x+3)$

(b) $ac+bc-ad-bd=c(a+b)-d(a+b)=(a+b)(c-d)$

4A.11 HKCEE MA 2003 – I – 3

(a) $x^2-(y-x)^2=[x-(y-x)][x+(y-x)]=y(2x-y)$

(b) $ab-ad-bc+cd=a(b-d)-c(b-d)=(b-d)(a-c)$

4A.12 HKCEE MA 2004 – I – 6

(a) $a^2-ab+2a-2b=a(a-b)+2(a-b)=(a-b)(a+2)$

(b) $169y^2-25=(13y)^2-5^2=(13y-5)(13y+5)$

4A.13 HKCEE MA 2005 – I – 3

(a) $4x^2-4xy+y^2=(2x-y)^2$

(b) $$\begin{aligned}4x^2-4xy+y^2-2x+y&=(2x-y)^2-(2x-y)\\&=(2x-y)(2x-y-1)\end{aligned}$$

4A.14 HKCEE MA 2007 – I – 3

(a) $r^2+10r+25=(r+5)^2$

(b) $r^2+10r+25-s^2=(r+5)^2-s^2=(r+5-s)(r+5+s)$

4A.15 HKCEE MA 2009 – I – 3

(a) $a^2b+ab^2=ab(a+b)$

(b) $$\begin{aligned}a^2b+ab^2+7a+7b&=ab(a+b)+7(a+b)\\&=(a+b)(ab+7)\end{aligned}$$

4A.16 HKCEE MA 2010 – I – 3

(a) $m^2+12mn+36n^2=(m+6n)^2$

(b) $$\begin{aligned}m^2+12mn+36n^2-25k^2&=(m+6n)^2-(5k)^2\\&=(m+6n-5k)(m+6n+5k)\end{aligned}$$

4A.17 HKCEE MA 2011 – I – 3

(a) $81m^2-n^2=(9m-n)(9m+n)$

(b) $$\begin{aligned}81m^2-n^2+18m-2n&=(9m-n)(9m+n)+2(9m-n)\\&=(9m-n)(9m+n+2)\end{aligned}$$

4A.18 HKDSE MA SP – I – 3

(a) $3m^2-mn-2n^2=(3m+2n)(m-n)$

(b) $$\begin{aligned}3m^2-mn-2n^2-m+n&=(3m+2n)(m-n)-(m-n)\\&=(m-n)(3m+2n-1)\end{aligned}$$

4A.19 HKDSE MA PP – I – 3

(a) $9x^2-42xy+49y^2=(3x-7y)^2$

(b) $$\begin{aligned}9x^2-42xy+49y^2-6x+14y&=(3x-7y)^2-2(3x-7y)\\&=(3x-7y)(3x-7y-2)\end{aligned}$$

4A.20 HKDSE MA 2012 – I – 3

(a) $x^2-6xy+9y^2=(x-3y)^2$

(b) $$\begin{aligned}x^2-6xy+9y^2+7x-21y&=(x-3y)^2+7(x-3y)\\&=(x-3y)(x-3y+7)\end{aligned}$$

4A.21 HKDSE MA 2013 – I – 3

(a) $4m^2-25n^2=(2m-5n)(2m+5n)$

(b) $4m^2-25n^2+6m-15n$
$=(2m-5n)(2m+5n)+3(2m-5n)$
$=(2m-5n)(2m+5n+3)$

4A.22 HKDSE MA 2014 – I – 2

(a) $a^2-2a-3=(a-3)(a+1)$

(b) $$\begin{aligned}ab^2+b^2+a^2-2a-3&=b^2(a+1)+(a-3)(a+1)\\&=(a+1)(b^2+a-3)\end{aligned}$$

4A.23 HKDSE MA 2015 – I – 4

(a) $x^3+x^2y-7x^2=x^2(x+y-7)$

(b) $$\begin{aligned}x^3+x^2y-7x^2-x-y+7&=x^2(x+y-7)-(x+y-7)\\&=(x+y-7)(x^2-1)\\&=(x+y-7)(x-1)(x+1)\end{aligned}$$

4A.24 HKDSE MA 2016 – I – 4

(a) $5m-10n=5(m-2n)$

(b) $m^2+mn-6n^2=(m+3n)(m-2n)$

(c) $m^2+mn-6n^2-5m+10n$
$=(m+3n)(m-2n)-5(m-2n)=(m-2n)(m+3n-5)$

4A.25 HKDSE MA 2017 – I – 3

(a) $x^2-4xy+3y^2=(x-3y)(x-y)$

(b) $$\begin{aligned}x^2-4xy+3y^2+11x-33y&=(x-3y)(x-y)+11(x-3y)\\&=(x-3y)(x-y+11)\end{aligned}$$

4A.26 HKDSE MA 2018 – I – 5

(a) $9r^3-18r^2s=9r^2(r-2s)$

(b) $$\begin{aligned}9r^3-18r^2s-rs^2+2s^3&=9r^2(r-2s)-s^2(r-2s)\\&=(r-2s)(9r^2-s^2)\\&=(r-2s)(3r-s)(3r+s)\end{aligned}$$

4A.27 HKDSE MA 2019 – I – 4

(a) $4m^2-9=(2m-3)(2m+3)$

(b) $2m^2n+7mn-15n=n(2m^2+7m-5)=n(2m-3)(m+5)$

(c) $4m^2-9-2m^2n-7mn+15n$
$=(2m-3)(2m+3)-n(2m-3)(m+5)$
$=(2m-3)[(2m+3)-n(m+5)]$
$=(2m-3)(2m-mn-5n+3)$

4A.28 HKDSE MA 2020 – I – 2

2a $\alpha^2+\alpha-6=(\alpha+3)(\alpha-2)$

b $\alpha^4+\alpha^3-6\alpha^2=\alpha^2(\alpha^2+\alpha-6)$
$=\alpha^2(\alpha+3)(\alpha-2)$

4B Division algorithm, remainder theorem and factor theorem

4B.1 HKCEE MA 1980(1*/3) – I – 13(a)

(a) (i) $\begin{cases}5=f(1)=24a+b & \Rightarrow\ a+b=-7\\4=f(-2)=8-2a+b & \Rightarrow\ 2a-b=4\end{cases}$

$\Rightarrow \begin{cases}a=-1\\b=-6\end{cases}$

(ii) $f(x)=0$

$2x^2-x-6=0$

$(2x+3)(x-2)=0\ \Rightarrow\ x=-\frac{3}{2}$ or 2

4B.2 HKCEE MA 1981(2) – I – 3 and 1981(3) – I – 2

$$\begin{aligned}k=f(k)&=(k+2)(k-3)+3\\k&=k^2-k-3\\k^2-2k-3&=0\\(k-3)(k+1)&=0\ \Rightarrow\ k=3\text{ or }-1\end{aligned}$$

4B.3 HKCEE MA 1984(A/B) – I – 1

$\because$ $x-k$ is a factor

$\therefore$ $3(k)^2-k(k)-2=0\ \Rightarrow\ k^2=1\ \Rightarrow\ k=\pm1$

4B.4 HKCEE MA 1985(A/B) – I – 4

$\begin{cases}0=f(1)=a+b-1 & \Rightarrow\ a+b=1\\4=f(-1)=a-b-1 & \Rightarrow\ a-b=5\end{cases}\ \Rightarrow\ \begin{cases}a=3\\b=-2\end{cases}$

4B.5 HKCEE MA 1987(A/B) – I – 2

$\begin{cases}2(2)^3+a(2)^2+b(2)-2=0\\2(-1)^3+a(-1)^2+b(-1)-2=0\end{cases}$

$\Rightarrow \begin{cases}4a+2b=-14\\a-b=4\end{cases}\ \Rightarrow\ \begin{cases}a=-1\\b=-5\end{cases}$

4B.6 HKCEE MA 1989 – I – 3

(a) $(-1)^4+(-1)^3-8(-1)+k=0\ \Rightarrow\ k=-8$

(b) $$\begin{aligned}x^4+x^3-8x+k&=x^4+x^3-8x-8\\&=x^3(x+1)-8(x+1)\\&=(x+1)(x^3-8)\\&=(x+1)(x-2)(x^2+2x+4)\end{aligned}$$

4B.7 HKCEE MA 1990 – I – 7

(a) Remainder $=(-1)^{1000}+6=7$

(b) (i) By (a), the remainder when $(8)^{1000}+6$ is divided by $(8)+1=9$ is 7.

(ii) Remainder $=7-6=1$

4B.8 HKCEE MA 1990 – I – 11

(a) (i) $S=2\pi r^2+2\pi rh$

(ii) $V=\pi r^2h\ \Rightarrow\ h=\dfrac{V}{\pi r^2}$

$\therefore\ S=2\pi r^2+2\pi r\left(\dfrac{V}{\pi r^2}\right)=2\pi r^2+\dfrac{2V}{r}$

(b) $6\pi=2\pi r^2+\dfrac{2(2\pi)}{r}$

$3r=r^3+2\ \Rightarrow\ r^3-3r+2=0$

Since $(1)^3-3(1)+2=0$, $r-1$ is a factor.

$\therefore\ r^3-3r+2=(r-1)(r^2+r-2)=0$

$(r-1)(r+2)(r-1)=0$

$r=-2$ (rej.) or 1

4B.9 <u>HKCEE MA 1992 – I – 2(b)</u>

Remainder $= (1)^3 - 2(1)^2 + 3(1) - 4 = -2$

4B.10 <u>HKCEE MA 1993 – I – 2(d)</u>

Remainder $= (1)^3 + (1)^2 = 2$

4B.11 <u>HKCEE MA 1994 – I – 3</u>

Remainder $= k^2 = (k+3)(k-2)+2$

$k^2 + k - 4 = k^2 \Rightarrow k = 4$

4B.12 <u>HKCEE MA 1995 – I – 2</u>

(a) $(a+b)^2 - (a-b)^2 = [(a+b) - (a-b)][(a+b)+(a-b)]$
$= (2b)(2a) = 4ab$

(b) Remainder $= (-2)^3 + 1 = -7$

4B.13 <u>HKCEE MA 1996 – I – 4</u>

$\because (-1)^3 - (-1)^2 - 3(-1) - 1 = 0$
$\therefore x+1$ is a factor.

$x^3 - x^2 - 3x - 1 = 0$
$(x+1)(x^2 - 2x - 1) = 0$
$x = -1 \text{ or } \dfrac{2 \pm \sqrt{4+4}}{2} = -1 \text{ or } 1 \pm \sqrt{2}$

4B.14 <u>HKCEE MA 1998 – I – 9</u>

(a) $\because f(2) = (2)^3 + 2(2)^2 - 5(2) - 6 = 0$
$\therefore x-2$ is a factor.

(b) $f(x) = (x-2)(x^2+4x+3) = (x-2)(x+1)(x+3)$

4B.15 <u>HKCEE MA 2000 – I – 6</u>

Remainder $= f(-3) = 2(-3)^3 + 6(-3)^2 - 2(-3) - 7 = -1$

4B.16 <u>HKCEE MA 2001 – I – 2</u>

Remainder $= f(2) = (2)^3 - (2)^2 + (2) - 1 = 5$

4B.17 <u>HKCEE MA 2002 – I – 4</u>

(a) $f(2) = (2)^3 - 2(2)^2 - 9(2) + 18 = 0$

(b) $\because f(2) = 0$
$\therefore x-2$ is a factor of $f(x)$.
$f(x) = (x-2)(x^2-9) = (x-2)(x-3)(x+3)$

4B.18 <u>HKCEE MA 2005 – I – 10</u>

(a) Let $f(x) = hx^3 + kx$.

$\begin{cases} -6 = f(2) = 8h + 2k \Rightarrow 4h + k = -3 \\ 6 = f(3) = 27h + 3k \Rightarrow 9h + k = 2 \end{cases} \Rightarrow \begin{cases} h = 1 \\ k = -7 \end{cases}$

$\therefore f(x) = x^3 - 7x$

(b) $g(x) = x^3 - 7x - 6$

(i) $\because g(3) = (3)^3 - 7(3) - 6 = 0$
$\therefore x - 3$ is a factor of $g(x)$.

(ii) $g(x) = (x-3)(x^2+3x+2) = (x-3)(x+1)(x+2)$

4B.19 <u>HKCEE MA 2007 – I – 14</u>

(a) (i) $0 = f(-3) = 4(-3)^3 + k(-3)^2 - 243 \Rightarrow k = 39$

(ii) $f(x) = (x+3)(4x^2 + 27x - 81)$
$= (x+3)(4x-9)(x+9)$

4B.20 <u>HKDSE MA SP – I – 10</u>

(a)

$$\begin{array}{r} 5x + 2 \\ x^2+2x-3 \overline{\big) \, 5x^3 + 12x^2 - 9x^2 - 7} \\ \underline{5x^3 + 10x^2 - 15x} \\ 2x^2 + 6x - 7 \\ \underline{2x^2 + 4x - 6} \\ 2x - 1 \end{array}$$

$\therefore$ Quotient $= 5x + 2$

(b) (i) From (a),
$5x^3 + 12x^2 - 9x - 7 = (5x+2)(x^2+2x-3) + (2x-1)$
Hence, $(5x^3 + 12x^2 - 9x - 7) - (2x - 1)$ is a multiple of $x^2 + 2x - 3$.
$\therefore a = 2,\ b = -1$

(ii) $(5x+2)(x^2+2x-3) = 0$
$x = -\dfrac{2}{5}$ or $(x+3)(x-1) = 0 \Rightarrow x = -\dfrac{2}{5}$ or -3 or 1

4B.21 <u>HKDSE MA PP – I – 10</u>

(a) Since it is given that the remainder when $f(x)$ is divided by $x-1$ is 4,
$f(x) \equiv (x-1)(6x^2 + 17x - 2) + 4$
$\therefore f(-3) = (-3-1)[6(-3)^2 + 17(-3) - 2] + 4 = 0$

(b) From (a), $x+3$ is a factor of $f(x)$.
$\therefore f(x) = 6x^3 + 11x^2 - 19x + 6$
$= (x+3)(6x^2 - 7x + 2) = (x+3)(3x-1)(x-2)$

4B.22 <u>HKDSE MA 2012 – I – 13</u>

(a) $0 = k(2)^3 - 21(2)^2 + 24(2) - 4 \Rightarrow k = 5$

4B.23 <u>HKDSE MA 2013 – I – 12</u>

(a) Given: $x-2$ is a factor.
$\therefore 0 = 3(2)^3 - 7(2)^2 + k(2) - 8 \Rightarrow k = 6$
Hence, $f(x) = 3x^3 - 7x^2 + 6x - 8 = (x-2)(3x^2 - x + 4)$
$\Rightarrow a = 3,\ b = -1,\ c = 4$

(b) Δ of $3x^2 - x + 4 = -47 < 0$
$\therefore$ Roots for $3x^2 - x + 4 = 0$ are not real.
Hence, $f(x) = 0$ only has 1 real root. Disagreed.

4B.24 <u>HKDSE MA 2014 – I – 7</u>

(a) $-33 = f(2) = 32 - 20 - 36 + c \Rightarrow c = -9$
$\Rightarrow f(x) = 4x^3 - 5x^2 - 18x - 9$
$\because f(-1) = -4 - 5 + 18 - 9 = 0$,
$\therefore x+1$ is a factor of $f(x)$.

(b) $f(x) = (x+1)(4x^2 - 9x - 9) = (x+1)(4x+3)(x-3)$
$\therefore$ The roots are -1, $\dfrac{-3}{4}$ and 3, which are all rational. Yes.

4B.25 <u>HKDSE MA 2015 – I – 11</u>

(a) $\begin{cases} -5 = f(2) = k \\ 0 = f(3) = (3-2)^2(3+h) + k \end{cases} \Rightarrow \begin{cases} h = 2 \\ k = -5 \end{cases}$

(b) $f(x) = (x-2)^2(x+2) - 5 = x^3 - 2x^2 - 3x + 3$
$= (x-3)(x^2 + x - 1)$
$\therefore$ The roots of $f(x) = 0$ are 3 and $\dfrac{-1 \pm \sqrt{1+4}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$, which are not integers. Disagreed.

4B.26 <u>HKDSE MA 2016 – I – 14</u>

(a) $p(-2) = p(2)$
$96 - 56 + 4a - 2b + c = 96 + 56 + 4a + 2b + c$
$b = -28$

Thus, we have
$6x^4 + 7x^3 + ax^2 - 28x + c \equiv (lx^2 + 5x + 8)(2x^2 + mx + n)$
$\Rightarrow \begin{cases} 6 = 2l \Rightarrow l = 3 \\ 7 = (3)m + 10 \Rightarrow m = -1 \\ -28 = 8(-1) + 5n \Rightarrow n = -4 \end{cases}$

(b) $p(x) = (3x^2 + 5x + 8)(2x^2 - x - 4)$
Δ of $3x^2 + 5x + 8 = -71 < 0 \Rightarrow$ No real root
Δ of $2x^2 - x - 4 = 33 < 0 \Rightarrow$ 2 distinct real roots
$\therefore p(x) = 0$ has 2 real roots.

4B.27 <u>HKDSE MA 2017 – I – 14</u>

(a) Using the division algorithm,
$f(x) \equiv (3x+7)(2x^2 + ax + 4) + (bx + c) \Rightarrow$
$6x^3 - 13x^2 - 46x + 34 \equiv (3x+7)(2x^2 + ax + 4) + (bx + c)$

<u>Method 1</u>
Expand and compare coefficients of like terms.

<u>Method 2</u>
$\begin{cases} f(0) = 34 = 28 + c \Rightarrow c = 6 \\ f(1) = -19 = 10(6+a) + (b+6) \Rightarrow 10a + b = -85 \\ f(2) = -62 = 13(12+2a) + (2b+6) \Rightarrow 13a + b = -112 \end{cases}$
$\Rightarrow b = 5,\ a = -9$

(b) (i) $\begin{cases} f(x) = (3x+7)(2x^2 - 9x + 4) + (bx + c) \\ g(x) = k(2x^2 - 9x + 4) + (bx + c) \end{cases}$
$f(x) - g(x) = (3x+7)(2x^2 - 9x + 4) - k(2x^2 - 9x + 4)$
$= (2x^2 - 9x + 4)(3x + 7 - k)$,
which has a factor of $2x^2 - 9x + 4$ indeed.

(ii) Roots of $2x^2 - 9x + 4 = (2x-1)(x-4)$ are 4 and $\dfrac{1}{2}$, which is not an integer. Disagreed.

4B.28 <u>HKDSE MA 2018 – I – 12</u>

(a) $\begin{cases} 0 = f(3) = 192 + 3a + b \Rightarrow 3a + b = -192 \\ 2b + 165 = f(-2) = -8 - 2a + b \Rightarrow 2a + b = -173 \end{cases}$
$\Rightarrow \begin{cases} a = -19 \\ b = -135 \end{cases}$

(b) $f(x) = 4x(x+1)^2 - 19x - 135 = 4x^3 + 8x - 15x - 135$
$= (x-3)(4x^2 + 20x + 45)$
Roots of $f(x) = 0$ are 3 and $\dfrac{-20 \pm \sqrt{400 - 720}}{8}$ which are unreal. Disagreed.

4B.29 <u>HKDSE MA 2019 – I – 11</u>

(a) Let $p(x) = (ax + b)(2x^2 + 9x + 14)$.
$\begin{cases} 50 = p(1) = 25(a+b) \Rightarrow a + b = 2 \\ -52 = p(-2) = 4(-2a + b) \Rightarrow 2a - b = -13 \end{cases}$
$\Rightarrow \begin{cases} a = 5 \\ b = -3 \end{cases} \Rightarrow$ Required quotient $= ax + b = 5x - 3$

(b) $p(x) = 0 \Rightarrow 5x - 3 = 0$ or $2x^2 + 9x + 14 = 0$
$\because \Delta$ of $2x^2 + 9x + 14 = -31 < 0$
$\therefore 2x^2 + 9x + 14 = 0$ has no real rt, and thus no rational rt.
$\therefore$ The only real root of $p(x) = 0$ is $\dfrac{3}{5}$ which is rational.
i.e. There is 1 rational root.

5 Formulas

5.1 HKCEE MA 1980(1/1*) – I – 7

Given that $a\left(1+\frac{x}{100}\right)=b\left(1-\frac{x}{100}\right)$, express x in terms of a and b.

5.2 HKCEE MA 1981(2) – I – 2

If $x=(a+by^2)^{\frac{1}{3}}$, express y in terms of a, b and x.

5.3 HKCEE MA 1993 I – 2(b)

If $2xy+3=6x$, express y in terms of x.

5.4 HKCEE MA 1996 – I – 1

Make r the subject of the formula $h=a+r(1+p^2)$.
If $h=8$, $a=6$ and $p=\ 4$, find the value of r.

5.5 HKCEE MA 1998 – I – 5

Make x the subject of the formula $b=2x+(1-x)a$.

5.6 HKCEE MA 1999 I 2

Make x the subject of the formula $a=b+\frac{c}{x}$.

5.7 HKCEE MA 2000 – I – 1

Let $C=\frac{5}{9}(F-32)$. If $C=30$, find F.

5.8 HKCEE MA 2001 – I – 6

Make x the subject of the formula $y=\frac{1}{2}(x+3)$.
If the value of y is increased by 1, find the corresponding increase in the value of x.

5.9 HKCEE MA 2003 I – 1

Make m the subject of the formula $mx=2(m+c)$.

5.10 HKCEE MA 2004 – I – 2

Make x the subject of the formula $y=\frac{2}{a-x}$.

5.11 HKCEE MA 2005 I – 1

Make a the subject of the formula $P=ab+2bc+3ac$.

5.12 HKCEE MA 2007 I – 1

Make p the subject of the formula $5p-7=3(p+q)$.

5.13 HKCEE MA 2008 I – 6

It is given that $\frac{2s+t}{s+2t}=\frac{3}{4}$.

(a) Express t in terms of s.

(b) If $s+t=959$, find s and t.

5.14 HKCEE MA 2009 – I – 1

Make n the subject of the formula $3n \quad \frac{5m}{2}=4$.

5.15 HKCEE MA 2010 – I – 5

Consider the formula $3(2c+5d+4)=39d$.

(a) Make c the subject of the above formula.

(b) If the value of d is decreased by 1, how will the value of c be changed?

5.16 HKCEE MA 2011 I 1

Make k the subject of the formula $\frac{mk-t}{k+t}=4$.

5.17 HKDSE MA SP – I 2

Make b the subject of the formula $a(b+7)=a+b$.

5.18 HKDSE MA PP – I – 2

Make a the subject of the formula $\frac{5+b}{1-a}=3b$.

5.19 HKDSE MA 2012 I – 2

Make a the subject of the formula $\frac{3a+b}{8}=b-1$.

5.20 HKDSE MA 2013 I 2

Make k the subject of the formula $\frac{3}{h}-\frac{1}{k}=2$.

5.21 HKDSE MA 2014 – I – 5

Consider the formula $2(3m+n)=m+7$.

(a) Make n the subject of the above formula.

(b) If the value of m is increased by 2, write down the change in the value of n.

5.22 <u>HKDSE MA 2015 – I – 2</u>

Make b the subject of the formula $\dfrac{4a+5b-7}{b}=8$.

5.23 <u>HKDSE MA 2016 – I – 2</u>

Make x the subject of the formula $Ax=(4x+B)C$.

5.24 <u>HKDSE MA 2017 – I – 1</u>

Make y the subject of the formula $k=\dfrac{3x-y}{y}$.

5.25 <u>HKDSE MA 2018 – I – 1</u>

Make b the subject of the formula $\dfrac{a+4}{3}=\dfrac{b+1}{2}$.

5.26 <u>HKDSE MA 2019 – I – 1</u>

Make h the subject of the formula $9(h+6k)=7h+8$.

5 Formulas

5.1 HKCEE MA 1980(1/1*) – I – 7

$\frac{a(100+x)}{\cancel{100}} = \frac{b(100-x)}{\cancel{100}}$

$100a+ax=100b-bx \quad \Rightarrow \quad x=\frac{100(b-a)}{a+b}$

5.2 HKCEE MA 1981(2) – I – 2

$x^3=a+by^2$

$y^2=\frac{x^3-a}{b} \quad \Rightarrow \quad y=\pm\sqrt{\frac{x^3-a}{b}}$

5.3 HKCEE MA 1993 – I – 2(b)

$y=\frac{6x-3}{2x}$

5.4 HKCEE MA 1996 – I – 1

$r=\frac{h-a}{1+p^2}$

Hence, $r=\frac{(8)-(6)}{1+(-4)^2}=\frac{2}{17}$

5.5 HKCEE MA 1998 – I – 5

$x=\frac{b-a}{2-a}$

5.6 HKCEE MA 1999 – I – 2

$x=\frac{c}{a-b}$

5.7 HKCEE MA 2000 – I – 1

$(30)=\frac{5}{9}(F-32) \quad \Rightarrow \quad F=96$

5.8 HKCEE MA 2001 – I – 6

$x=2y-3$

If $y'=y+1$, $x'=2y'-3$

$\qquad =2(y+1)-3=2y-1$

$\therefore$ Increase in $x=x'-x=(2y-1)-(2y-3)=2$

5.9 HKCEE MA 2003 – I – 1

$m=\frac{2c}{x-2}$

5.10 HKCEE MA 2004 – I – 2

Method 1 $\quad ay-xy=2$

$\qquad ay-2=xy \quad \Rightarrow \quad x=\frac{ay-2}{y}$

Method 2 $\quad a-x=\frac{2}{y}$

$\qquad a=\frac{2}{y}+x \quad \Rightarrow \quad x=a-\frac{2}{y}$

5.11 HKCEE MA 2005 – I – 1

$a=\frac{P-2bc}{b+3c}$

5.12 HKCEE MA 2007 – I – 1

$p=\frac{3q+7}{2}$

5.13 HKCEE MA 2008 – I – 6

(a) $4(2s+t)=3(s+2t) \quad \Rightarrow \quad t=\frac{5}{2}s$

(b) $s+\left(\frac{5}{2}s\right)=959 \quad \Rightarrow \quad s=254 \quad \Rightarrow \quad t=\frac{5}{2}(254)=635$

5.14 HKCEE MA 2009 – I – 1

$n=\frac{8+5m}{3}$

5.15 HKCEE MA 2010 – I – 5

(a) $c=4d-2$

(b) $d'=d-1 \quad \Rightarrow \quad c'=4d'-2$

$\qquad =4(d-1)-2=4d-6$

$\therefore$ Change in $c=c'-c=(4d-6)-(4d-2)=-4$

i.e. a decrease of 4.

5.16 HKCEE MA 2011 – I – 1

$k=\frac{5t}{m-4}$

5.17 HKDSE MA SP – I – 2

$b=\frac{6a}{1-a}$

5.18 HKDSE MA PP – I – 2

$a=\frac{2b-5}{3b}$

5.19 HKDSE MA 2012 – I – 2

$a=\frac{7b-8}{3}$

5.20 HKDSE MA 2013 – I – 2

$k=\frac{h}{3-2h}$

5.21 HKDSE MA 2014 – I – 5

(a) $n=\frac{7-5m}{2}$

(b) $m'=m+2 \quad \Rightarrow \quad n'=\frac{7-5m'}{2}$

$\qquad =\frac{7-5(m+2)}{2}=\frac{-3-5m}{2}$

$\therefore$ Change in $n=n'-n=\frac{-3-5m}{2}-\frac{7-5m}{2}=-5$

5.22 HKDSE MA 2015 – I – 2

$b=\frac{4a-7}{3}$

5.23 HKDSE MA 2016 – I – 2

$x=\frac{BC}{A-4C}$

5.24 HKDSE MA 2017 – I – 1

$y=\frac{3x}{k+1}$

5.25 HKDSE MA 2018 – I – 1

$b=\frac{2a+5}{3}$

5.26 HKDSE MA 2019 – I – 1

$h=\frac{8-54k}{2}=4-27k$

6 Identities, Equations and the Number System

6A Simple equations

6A.1 HKCEE MA 1980(1*/3) – I 13(b)

Solve the equation $1-2x=\sqrt{2-x}$.

6A.2 HKCEE MA 1982(2/3) I – 7

Solve $x-\sqrt{x+1}=5$.

6A.3 HKCEE MA 1984(A) – I – 3

Expand $(1+\sqrt{2})^4$ and express your answer in the form $a+b\sqrt{2}$ where a and b are integers.

6A.4 HKCEE MA 1984(A/B) I – 6

Solve $x-5\sqrt{x}-6=0$.

6A.5 HKCEE MA 2003 – I – 6

There are only two kinds of tickets for a cruise: first-class tickets and economy class tickets. A total of 600 tickets are sold. The number of economy-class tickets sold is three times that of first class tickets sold. If the price of a first class ticket is \$850 and that of an economy class ticket is \$500, find the sum of money for the tickets sold.

6A.6 HKCEE MA 2004 I 7

The prices of an orange and an apple are \$2 and \$3 respectively. A sum of \$46 is spent buying some oranges and apples. If the total number of oranges and apples bought is 20, find the number of oranges bought.

6A.7 HKCEE MA 2007 – I – 7

The consultation fees charged to an elderly patient and a non elderly patient by a doctor are \$120 and \$160 respectively. On a certain day, there were 67 patients consulted the doctor and the total consultation fee charged was \$9000. How many elderly patients consulted the doctor on that day?

6A.8 HKCEE MA 2008 – I – 3

(a) Write down all positive integers m such that $m+2n=5$, where n is an integer.

(b) Write down all values of k such that $2x^2+5x+k\equiv(2x+m)(x+n)$, where m and n are positive integers.

6A.9 HKCEE MA 2009 – I – 6

The total number of stamps owned by John and Mary is 300. If Mary buys 20 stamps from a post office, the number of stamps owned by her will be 4 times that owned by John. Find the number of stamps owned by John.

6A.10 HKCEE MA 2010 I 6

The cost of a bottle of orange juice is the same as the cost of 2 bottles of milk. The total cost of 3 bottles of orange juice and 5 bottles of milk is \$66. Find the cost of a bottle of milk.

6A.11 HKDSE MA SP – I – 5

In a football league, each team gains 3 points for a win, 1 point for a draw and 0 point for a loss. The champion of the league plays 36 games and gains a total of 84 points. Given that the champion does not lose any games, find the number of games that the champion wins.

6A.12 HKDSE MA 2012 – I – 5

There are 132 guards in an exhibition centre consisting of 6 zones. Each zone has the same number of guards. In each zone, there are 4 more female guards than male guards. Find the number of male guards in the exhibition centre.

6A.13 HKDSE MA 2013 – I – 4

The price of 7 pears and 3 oranges is \$47 while the price of 5 pears and 6 oranges is \$49. Find the price of a pear.

6A.14 HKDSE MA 2015 – I – 7

The number of apples owned by Ada is 4 times that owned by Billy. If Ada gives 12 of her apples to Billy, they will have the same number of apples. Find the total number of apples owned by Ada and Billy.

6A.15 HKDSE MA 2017 – I – 4

There are only two kinds of admission tickets for a theatre: regular tickets and concessionary tickets. The prices of a regular ticket and a concessionary ticket are \$126 and \$78 respectively. On a certain day, the number of regular tickets sold is 5 times the number of concessionary tickets sold and the sum of money for the admission tickets sold is \$50 976. Find the total number of admission tickets sold that day.

6A.16 HKDSE MA 2019 – I – 3

The length and the breadth of a rectangle are 24 cm and $(13+r)$ cm respectively. If the length of a diagonal of the rectangle is $(17-3r)$ cm, find r.

6B Nature of roots of quadratic equations

6B.1 HKCEE MA 1988 – I 4

The quadratic equation $9x^2-(k+1)x+1=0$ ....(*) has equal roots.

(a) Find the two possible values of the constant k.

(b) If k takes the negative value obtained, solve equation (*).

6B.2 HKCEE MA 2007 – I – 5

Let k be a constant. If the quadratic equation $x^2+14x+k=0$ has no real roots, find the range of values of k.

6B.3 HKCEE AM 1980 – I 1

Find the range of values of k for which the equation $2x^2+x+5=k(x+1)^2$ has no real roots.

6B.4 HKCEE AM 1998 – I – 3

The quadratic equations $x^2-6x+2k=0$ and $x^2-5x+k=0$ have a common root α. (i.e. α is a root of both equations.)
Show that $\alpha=k$ and hence find the value(s) of k.

6C Roots and coefficients of quadratic equations

6C.1 HKCEE MA 1980(1/1*/3) – I – 3

What is the product of the roots of the quadratic equation $2x^2+kx-5=0$?
If one of the roots is 5, find the other root and the value of k.

6C.2 HKCEE MA 1982(2/3) – I – 1

If $a-b=10$ and $ab=k$, express a^2+b^2 in terms of k.

6C.3 HKCEE MA 1983(B) I 14

(To continue as **10C.1**.)

α and β are the roots of the quadratic equation $x^2-2mx+n=0$, where m and n are real numbers.

(a) Find, in terms of m and n,
 (i) $(m-\alpha)+(m-\beta)$,
 (ii) $(m-\alpha)(m-\beta)$.

(b) Find, in terms of m and n, the quadratic equation having roots $m-\alpha$ and $m-\beta$.

6C.4 HKCEE MA 1985(A/B) I – 5

Let α and β be the roots of $x^2+kx+1=0$, where k is a constant.

(a) Find, in terms of k,
 (i) $(\alpha+2)+(\beta+2)$,
 (ii) $(\alpha+2)(\beta+2)$.

(b) Suppose $\alpha+2$ and $\beta+2$ are the roots of $x^2+px+q=0$, where p and q are constants. Find p and q in terms of k.

6C.5 HKCEE MA 1986(A/B) I – 7

If $\dfrac{1}{m}+\dfrac{1}{n}=\dfrac{1}{a}$ and $m+n=b$, express the following in terms of a and b

(a) mn,

(b) m^2+n^2.

6C.6 HKCEE MA 1987(A/B) I – 5

α and β are the roots of the quadratic equation $kx^2-4x+2k=0$, where k ($k\neq 0$) is a constant. Express the following in terms of k:

(a) $\alpha^2+\beta^2$,

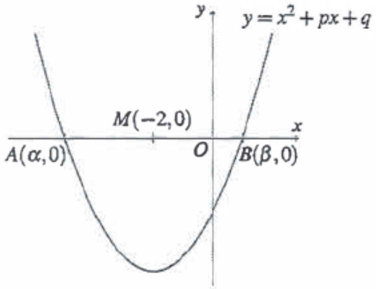
(b) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$.

6C.7 HKCEE MA 1990 – I 6

In the figure, the curve $y=x^2+px+q$ cuts the x axis at the two points $A(\alpha,0)$ and $B(\beta,0)$. $M(\ 2,0)$ is the mid point of AB.

(a) Express $\alpha+\beta$ in terms of p. Hence find the value of p.

(b) If $\alpha^2+\beta^2=26$, find the value of q.



6C.8 HKCEE MA 1991 – I – 7 (Also as **3B.5**.)

Let α and β be the roots of the equation $10x^2+20x+1=0$. Without solving the equation, find the values of

(a) $4^{\alpha}\times 4^{\beta}$,

(b) $\log_{10}\alpha+\log_{10}\beta$.

6C.9 HKCEE MA 1993 – I – 2(f)

If $(x-1)(x+2)=x^2+rx+s$, find r and s.

6C.10 HKCEE MA 1993 – I – 6

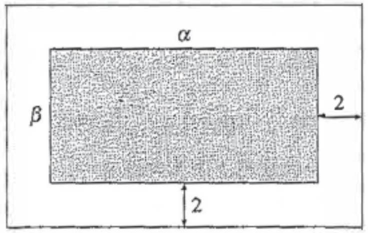
The length α and the breadth β of a rectangular photograph are the roots of the equation $2x^2-mx+500=0$. The photo graph is mounted on a piece of rectangular cardboard, leaving a uniform border of width 2 as shown in the figure.

(a) Find the area of the photograph.

(b) Find, in terms of m,

  (i) the perimeter of the photograph,

  (ii) the area of the border.



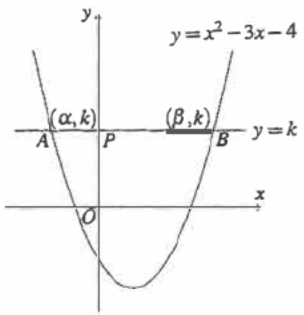
6C.11 HKCEE MA 1995 – I – 8

In the figure, the line $y=k\ (k>0)$ cuts the curve $y=x^2-3x-4$ at the points $A(\alpha,k)$ and $B(\beta,k)$.

(a) (i) Find the value of $\alpha+\beta$.

  (ii) Express $\alpha\beta$ in terms of k.

(b) If the line AB cuts the y-axis at P and $BP=2PA$, find the value of k.



6C.12 HKCEE MA 1997 – I – 8

The roots of the equation $2x^2-7x+4=0$ are α and β.

(a) Write down the values of $\alpha+\beta$ and $\alpha\beta$.

(b) Find the quadratic equation whose roots are $\alpha+2$ and $\beta+2$.

6C.13 (HKCEE AM 1984 – I – 5)

Let α and β be the roots of the equation $x^2-2x-(m^2-m+1)=0$, where m is a real number.

(a) Show that $(\alpha-\beta)^2>0$ for any value of m.

(b) Find the minimum value of $\sqrt{(\alpha-\beta)^2}$.

6C.14 HKCEE AM 1987 – I – 5

The equation $x^2+4x+p=0$, where p is a real constant, has distinct real roots α and β.

(a) Find the range of values of p.

(b) If $\alpha^2+\beta^2+\alpha^2\beta^2+3(\alpha+\beta)-19=0$, find the value of p.

6C.15 HKCEE AM 1989 – I – 11 ***[Difficult]***

(a) Let α,β be the roots of the equation $x^2+px+q=0$ ......(*), where p and q are real constants. Find, in terms of p and q,

  (i) $\alpha^2+\beta^2$,

  (ii) $\alpha^3+\beta^3$,

  (iii) $(\alpha^2-\beta-1)(\beta^2-\alpha-1)$.

(b) If the square of one root of (*) minus the other root equals 1, use (a), or otherwise, to show that $q^2-3(p-1)q+(p-1)^2(p+1)=0$ ....(**).

(c) Find the range of values of p such that the quadratic equation (**) in q has real roots.

(d) Suppose k is a real constant. If the square of one root of $4x^2+5x+k=0$ minus the other root equals 1, use the result in (b), or otherwise, to find the value of k.

6C.16 HKCEE AM 1990 – I – 4

α,β are the roots of the quadratic equation $x^2-(k+2)x+k=0$.

(a) Find $\alpha+\beta$ and $\alpha\beta$ in terms of k.

(b) If $(\alpha+1)(\beta+2)=4$, show that $\alpha=-2k$. Hence find the two values of k.

6C.17 HKCEE AM 1991 – I – 7 (To continue as **10C.10**.)

p, q and k are real numbers satisfying the following conditions: $\begin{cases} p+q+k=2, \\ pq+qk+kp=1. \end{cases}$

(a) Express pq in terms of k.

(b) Find a quadratic equation, with coefficients in terms of k, whose roots are p and q.

6C.18 HKCEE AM 1992 – I – 9

α,β are the roots of the quadratic equation $x^2+(p+1)x+(p-1)=0$, where p is a real number.

(a) Show that α,β are real and distinct.

(b) Express $(\alpha-2)(\beta-2)$ in terms of p.

(c) Given $\beta<2<\alpha$.

  (i) Using the result of (b), show that $p<-\frac{5}{3}$.

  (ii) If $(\alpha-\beta)^2<24$, find the range of possible values of p. Hence write down the possible integral value(s) of p.

6C.19 HKCEE AM 1993 – I – 3

α,β are the roots of the equation $x^2+px+q=0$ and $\alpha+3,\ \beta+3$ are the roots of the equations $x^2+qx+p=0$. Find the values of p and q.

6C.20 (HKCEE AM 1995 – I – 10) ***[Difficult]*** (To continue as **10C.13**.)

Let $f(x)=12x^2+2px-q$ and $g(x)=12x^2+2qx-p$, where p, q are distinct real numbers. α,β are the roots of the equation $f(x)=0$ and α,γ are the roots of the equation $g(x)=0$.

(a) Using the fact that $f(\alpha)=g(\alpha)$, find the value of α. Hence show that $p+q=3$.

(b) Express β and γ in terms of p.

6C.21 HKCEE AM 1998 – I – 2

α,β are the roots of the quadratic equation $x^2-2x+7=0$. Find the quadratic equation whose roots are $\alpha+2$ and $\beta+2$.

6C.22 HKCEE AM 2000 – I – 7

α and β are the roots of the quadratic equation $x^2+(p-2)x+p=0$, where p is real.

(a) Express $\alpha+\beta$ and $\alpha\beta$ in terms of p.

(b) If α and β are real such that $\alpha^2+\beta^2=11$, find the value(s) of p.

6C.23 (HKCEE AM 2011 – I – 7)

Let α and β be the roots of the quadratic equation $x^2+(k+2)x+k=0$, where k is real.

(a) Prove that α and β are real and distinct.

(b) If $\alpha=\sqrt{\beta^2}$, find the value of k.

6C.24 HKDSE MA PP – I – 17 (Continued from **6D.1**.)

(a) Express $\frac{1}{1+2i}$ in the form of $a+bi$, where a and b are real numbers.

(b) The roots of the quadratic equation $x^2+px+q=0$ are $\frac{10}{1+2i}$ and $\frac{10}{1-2i}$. Find

(i) p and q,

(ii) the range of values of r such that the quadratic equation $x^2+px+q=r$ has real roots.

6D Complex numbers

6D.1 HKDSE MA PP – I – 17 (To continue as **6C.24**.)

(a) Express $\frac{1}{1+2i}$ in the form of $a+bi$, where a and b are real numbers.

6A Simple equations

6A.1 HKCEE MA 1980(1*/3) – I – 13(b)

(b) $(1-2x)^2 = 2-x$

$4x^2-3x-1=0$

$(4x+1)(x-1)=0 \Rightarrow x = \quad \frac{1}{4}$ or 1 (rejected)

6A.2 HKCEE MA 1982(2/3) – I – 7

$x-5 \quad \sqrt{x+1}$

$(x-5)^2 = x+1$

$x^2-11x+24=0 \Rightarrow x=8$ or 3 (rejected)

6A.3 HKCEE MA 1984(A) – I – 3

$$(1+\sqrt{2})^4 = \left[(1+\sqrt{2})^2\right]^2 = (1+2\sqrt{2}+2)^2$$
$$= (3+2\sqrt{2})^2$$
$$= 9+12\sqrt{2}+8 = 17+12\sqrt{2}$$

6A.4 HKCEE MA 1984(A/B) – I – 6

Let $\sqrt{x}=u \Rightarrow \quad u^2-5u \quad 6=0$

$u=6$ or -1

$\sqrt{x}=6$ or -1 (rejected) $\Rightarrow x=36$

6A.5 HKCEE MA 2003 – I – 6

Let x and y first- and economy-class tickets be sold respectively.

$\begin{cases} x+y=600 \\ y=3x \end{cases} \Rightarrow \begin{cases} x=150 \\ y=450 \end{cases}$

$\therefore$ Sum of money $= 150 \times \$850 + 450 \times \$500 =$ \$352500

6A.6 HKCEE MA 2004 – I – 7

Let x oranges and y apples be bought.

$\begin{cases} 2x+3y=46 \\ x+y=20 \end{cases} \Rightarrow \begin{cases} x=14 \\ y=6 \end{cases}$

$\therefore$ 14 oranges were bought.

6A.7 HKCEE MA 2007 – I – 7

Let there be x elderly patients.

Then there were $67-x$ non-elderly patients.

$120x+160(67-x)=9000$

$10720-40x=9000$

$x=(10720-9000)\div 6=43$

.. There were 43 elderly patients.

6A.8 HKCEE MA 2008 – I – 3

(a) $m=1$ or 3 (corresponding $n=2$ or 1)

(b) $2x^2+5x+k \equiv 2x^2+(m+2n)x+mn$

Comparing coefficients of like terms, $\begin{cases} 5=m+2n \\ k=mn \end{cases}$

$\therefore$ Possible values of k are $(1)(2)=2$ and $(3)(1)=3$ only

6A.9 HKCEE MA 2009 – I – 6

Let John own x stamps.

Then Mary owns $300-x$ stamps.

$(300-x)+20=4x$

$320=5x \Rightarrow x=64$

$\therefore$ John owns 64 stamps.

6A.10 HKCEE MA 2010 I 6

Let \$$2x$ and \$$x$ be the costs of 1 orange juice and 1 bottle of milk respectively.

$3(2x)+5(x)=66$

$11x=66 \Rightarrow x=6$

$\therefore$ The cost of a bottle of milk is \$6.

6A.11 HKDSE MA SP – I – 5

Let the champion win x games.

Then it has $36-x$ draws.

$3(x)+1(36-x)=84$

$2x=48 \Rightarrow x=24$

$\therefore$ The champion wins 24 games.

6A.12 HKDSE MA 2012 – I – 5

Let there be x male guards.

Then there are $132 \quad x$ female guards.

$\frac{132 \quad x}{6} = \frac{x}{6}+4$

$132-x=x+24 \Rightarrow x=54$

.. There are 54 male guards.

6A.13 HKDSE MA 2013 – I – 4

Let the prices of a pear and an orange be \$$x$ and \$$y$ respectively.

$\begin{cases} 7x+3y=47 & (1) \\ 5x+6y=49 & (2) \end{cases}$

$2(1)-(2): \quad 9x=45 \Rightarrow x=5$

$\therefore$ The price of a pear is \$5.

6A.14 HKDSE MA 2015 – I – 7

Let Ada and Billy own $4x$ and x apples.

$4x \quad 12=x+12$

$3x=24 \Rightarrow x=8$

$\therefore$ Billy owns 8 apples and Ada $4(8)=32$ apples.

6A.15 HKDSE MA 2017 – I – 4

Let x regular and y concessionary tickets be sold that day.

$\begin{cases} x=5y \\ 126x+78y=50976 \end{cases} \Rightarrow \begin{cases} y=72 \\ x=5(72)=360 \end{cases}$

$\therefore$ $360+72=432$ tickets were sold that day

6A.16 HKDSE MA 2019 – I – 3

$(17-3r)^2 = 24^2+(13+r)^2$

$289-102r+9r^2=576+169+26r+r^2$

$8r^2-128r-456=0 \Rightarrow r=-3$ or 19 (rejected)

6B Nature of roots of quadratic equations

6B. 1 HKCEE MA 1988 – I – 4

(a) $\Delta = 0$

$(k+1)^2 \quad 36 = 0$

$k+1 = \pm 6 \quad \Rightarrow \quad k = 5$ or -7

(b) When $k=-7$, (*) becomes

$9x^2+6x+1=0$

$(3x+1)^2=0 \quad \Rightarrow \quad x=-\frac{1}{3}$ (repeated)

6B.2 HKCEE MA 2007 – I – 5

$\Delta < 0$

$14^2-4k<0$

$4k>196 \quad \Rightarrow \quad k>49$

6B. 3 HKCEE AM 1980 – I – 1

$2x^2+x+5=k(x+1)^2 \Rightarrow (2 \quad k)x^2+(1 \quad 2k)x+(5 \quad k)=0$

No real roots $\Rightarrow$ $\Delta<0$

$(1-2k)^2-4(2-k)(5 \quad k)<0$

24k $\;$ 39 < 0 ⇒ k < 39/24

6B.4 HKCEE AM 1998 – I – 3

$\begin{cases}\alpha^2-6\alpha+2k=0 & (1)\\ \alpha^2-5\alpha+k=0 & (2)\end{cases}$

$(1)-(2) \quad \Rightarrow \quad -\alpha+k=0 \quad \Rightarrow \quad \alpha=k$

Hence the equation becomes

$k^2-6k+2k=0$

$k^2 \quad 4k=0 \quad \Rightarrow \quad k=0$ or 4

6C Roots and coefficients of quadratic equations

6C. 1 HKCEE MA 1980(1/1*/3) – I – 3

product of rt=-5/2, k=-9

6C. 2 HKCEE MA 1982(2/3) – I – 1

$a^2+b^2=(a \quad b)^2 \quad 2ab=(10)^2-2(k)=100-2k$

6C. 3 HKCEE MA 1983(B) – I – 14

(a) $\begin{cases}\alpha+\beta=2m\\ \alpha\beta=n\end{cases}$

(i) $(m \quad \alpha)+(m \quad \beta)=2m \quad (\alpha+\beta)=2m \quad (2m)=0$

(ii) $(m \quad \alpha)(m \quad \beta)=m^2 \quad (\alpha+\beta)m+\alpha\beta$
$=m^2-(2m)m+(n)=n-m^2$

(b) By (a), the equation is

$x^2 \quad (\text{sum})x+(\text{product})=0$

$x^2-(0)x+(n-m^2)=0 \quad \Rightarrow \quad x^2+n-m^2=0$

6C. 4 HKCEE MA 1985(A/B) – I – 5

$\begin{cases}\alpha+\beta= \quad k\\ \alpha\beta=1\end{cases}$

(a) (i) $(\alpha+2)+(\beta+2)=(\alpha+\beta)+4=4 \quad k$

(ii) $(\alpha+2)(\beta+2)=\alpha\beta+2(\alpha+\beta)+4=5 \quad 2k$

(b) $p=-(\text{sum of roots})=-(4-k)=k-4$

$q=\text{product of roots}=5 \quad 2k$

6C. 5 HKCEE MA 1986(A/B) – I – 7

(a) $\frac{1}{a}=\frac{1}{m}+\frac{1}{n}=\frac{m+n}{mn}=\frac{b}{mn} \quad \Rightarrow \quad mn=\frac{b}{a}$

(b) $m^2+n^2=(m+n)^2-2mn=(b)^2-2\left(\frac{b}{a}\right)=b^2-\frac{2b}{a}$

6C. 6 HKCEE MA 1987(A/B) I – 5

$\begin{cases}\alpha+\beta=\frac{4}{k}\\ \alpha\beta=2\end{cases}$

(a) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(\frac{4}{k}\right)^2-2(2)=\frac{16}{k^2}-4$

(b) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{\frac{16}{k^2} \quad 4}{2}=\frac{8}{k^2} \quad 2$

6C. 7 HKCEE MA 1990 – I – 6

(a) $\alpha+\beta=-p \quad \Rightarrow \quad -2=\frac{\alpha+\beta}{2}=\frac{-p}{2} \quad \Rightarrow \quad p=4$

(b) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=26$

$4^2 \quad 2(q)=26 \quad \Rightarrow \quad q=-5$

6C. 8 HKCEE MA 1991 – I – 7

$\begin{cases}\alpha+\beta= \quad \frac{20}{10}= \quad 2\\ \alpha\beta=\frac{1}{10}\end{cases}$

(a) $4^{\alpha}\times 4^{\beta}=4^{\alpha+\beta}=4^{-2}=\frac{1}{16}$

(b) $\log_{10}\alpha+\log_{10}\beta=\log_{10}\alpha\beta=\log_{10}\frac{1}{10}= \quad 1$

6C.9 HKCEE MA 1993 – I – 2(f)

$r=$ (sum of roots) $=-[1+(-2)]=1$ $\qquad$ $s=$ product of roots $=(1)(-2)=-2$

6C. 10 HKCEE MA 1993 – I – 6

(a) From the equation, $\alpha\beta=\frac{500}{2}=250$

$\therefore$ Area of photograph $=250$

(b) (i) Perimeter $=2(\alpha+\beta)=2\left(\frac{m}{2}\right)=m$

(ii) Area of border $=(\alpha+4)(\beta+4) \quad \alpha\beta$
$=4(\alpha+\beta)+16=4m+16$

6C. 11 HKCEE MA 1995 – I – 8

(a) α and β are the roots of the equation $(k)=x^2-3x-4$

$\Rightarrow x^2 \quad 3x \quad 4 \quad k=0$

(i) $\alpha+\beta=3$

(ii) $\alpha\beta= \quad 4 \quad k$

(b) $BP=2PA \quad \Rightarrow \quad \beta=2(-\alpha)=-2\alpha$

Hence, $\alpha+\beta=4 \quad \Rightarrow \quad \alpha+(\quad 2\alpha)=3$

$\alpha= \quad 3 \quad \Rightarrow \quad \beta=6$

$\therefore \quad (-3)(6)=\alpha\beta= \quad 4 \quad k \quad \Rightarrow \quad k=14$

6C. 12 HKCEE MA 1997 – I – 8

(a) $\alpha+\beta=\frac{7}{2}, \quad \alpha\beta=\frac{4}{2}=2$

(b) Sum of roots $=(\alpha+2)+(\beta+2)$
$=(\alpha+\beta)+4=\left(\frac{7}{2}\right)+4=\frac{15}{2}$

Product of roots $=(\alpha+2)(\beta+2)$
$=\alpha\beta+2(\alpha+\beta)+4$
$=(2)+2\left(\frac{7}{2}\right)+4=13$

Hence, required equation is $x^2-\frac{15}{2}x+13=0$

$\Rightarrow 2x^2 \quad 15x+26=0$

6C. 13 (HKCEE AM 1984 – I – 5)

(a) $\begin{cases}\alpha+\beta=2\\ \alpha\beta=-(m^2-m+1)\end{cases}$

$(\alpha \quad \beta)^2=(\alpha+\beta)^2 \quad 4\alpha\beta=(2)^2+4(m^2 \quad m+1)$

= 4m²-4m+8

=(x-1/2)²+7/4

=>7/4>0

for any value of m.

(b) From (a), minimum of $(\alpha-\beta)^2=$ 7/4

$\therefore$ minimum of $\sqrt{(\alpha \quad \beta)^2}=$ 7/4

6C. 14 HKCEE AM 1987 – I – 5

(a) $\Delta>0 \quad \Rightarrow \quad 16-4p>0 \quad \Rightarrow \quad p<4$

(b) $\begin{cases}\alpha+\beta= \quad 4\\ \alpha\beta=p\end{cases}$

$0=\alpha^2+\beta^2+\alpha^2\beta^2+3(\alpha+\beta) \quad 19$
$=(\alpha+\beta)^2-2\alpha\beta+(\alpha\beta)^2+3(\alpha+\beta)-19$
$=(\quad 4)^2-2(p)+(p)^2+3(-4)-19$
$=p^2-2p-15=(p-5)(p+3)$

$\Rightarrow \quad p=5$ (rejected) or -3

6C. 15 HKCEE AM 1989 – I – 11

(a) $\begin{cases}\alpha+\beta=-p\\ \alpha\beta=q\end{cases}$

(i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(\quad p)^2 \quad 2q$
$=p^2-2q$

(ii) $\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$
$=(\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]$
$=(-p)[(-p)^2 \quad 3(q)]=3pq \quad p^3$

(iii) $(\alpha^2-\beta-1)(\beta^2-\alpha-1)$
$=\alpha^2\beta^2-(\alpha^3+\beta^3) \quad (\alpha^2+\beta^2)+\alpha\beta+(\alpha+\beta)+1$
$=(q)^2-(3pq-p^3)-(p^2-2q)+(q)+(-p)+1$
$p^3 \quad p^2+q^2-3pq+3q-p+1$

(b) The given information means either

$\alpha^2-\beta=1$ or $\beta^2 \quad \alpha=1$

$\Rightarrow \quad (\alpha^2-\beta-1)(\beta^2-\alpha-1)=0$

$p^3 \quad p^2+q^2 \quad 3pq+3q \quad p+1=0$

$q^2 \quad 3(p \quad 1)q+p^3 \quad p^2 \quad p+1=0$

$q^2-3(p \quad 1)q+p^2(p \quad 1)-(p-1)=0$

$q^2-3(p \quad 1)q+(p-1)(p^2 \quad 1)=0$

$q^2-3(p-1)q+(p-1)^2(p+1)=0$

(c) $\Delta\geq 0$

$9(p \quad 1)^2 \quad 4(p \quad 1)^2(p+1)\geq 0$

$(p-1)^2[9-4(p+1)]\geq 0$

$(p-1)^2(5-4p)\geq 0$

Since $(p-1)^2\geq 0$, $5-4p\geq 0 \quad \Rightarrow \quad p<\frac{5}{4}$

(d) $4x^2+5x+k=0 \quad \Leftrightarrow \quad x^2+\frac{5}{4}x+\frac{k}{4}=0$

Put $p=\frac{5}{4}$ and $q=\frac{k}{4}$ into (b):

$\left(\frac{k}{4}\right)^2-3\left(\frac{1}{4}\right)\left(\frac{k}{4}\right)+\left(\frac{1}{4}\right)^2\left(\frac{9}{4}\right)=0$

$4k^2-12k+9=0 \quad \Rightarrow \quad k=\frac{3}{2}$

6C. 16 HKCEE AM 1990 – I – 4

(a) $\alpha+\beta=k+2, \quad \alpha\beta=k$

(b) $(\alpha+1)(\beta+2)=4$

$\alpha\beta+2\alpha+\beta+2=4$

$\alpha\beta+(\alpha+\beta)+\alpha+2=4$

$(k+2)+(k)+\alpha+2=4 \quad \Rightarrow \quad \alpha= \quad 2k$

Hence, putting $\alpha= \quad 2k$ into the equation:

$(-2k)^2-(k+2)(-2k)+k=0$

$6k^2-3k=0 \quad \Rightarrow \quad k=0$ or $\frac{1}{2}$

6C. 17 HKCEE AM 1991 – I – 7

(a) From the first equation, $p+q=2 \quad k$

From the second equation, $pq+k(p+q)=1$

$pq=1 \quad k(2 \quad k)$
$=(k+1)^2$

(b) Sum of roots $=p+q=2 \quad k$

Product of roots $=(k+1)^2$

$\therefore$ Required equation: $x^2-(2-k)x+(k+1)^2=0$

6C.18 HKCEE AM 1992 – I – 9

(a) $\Delta = (p+1)^2 - 4(p-1) = p^2 - 2p + 5$
$= (p+1)^2 + 4 \geq 4 > 0$

Hence, the two roots are real and distinct.

(b) $\begin{cases} \alpha+\beta = -(p+1) \\ \alpha\beta = p-1 \end{cases}$

$\therefore (\alpha-2)(\beta-2) = \alpha\beta - 2(\alpha+\beta) + 4$
$= (p-1) + 2(p+1) + 4 = 3p+5$

(c) (i) $\beta < 2 < \alpha \Rightarrow \alpha - 2 > 0$ and $\beta - 2 < 0$
$\therefore (\alpha-2)(\beta-2) < 0$
$3p+5 < 0 \Rightarrow p < -\frac{5}{3}$

(ii) $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta < 24$
$(p+1)^2 - 4(p-1) < 24$
$(p-1)^2 < 20$
$1 - \sqrt{20} < p < 1 + \sqrt{20}$

Together with (c)(i), $1 - \sqrt{20} < p < -\frac{5}{3}$

$\therefore$ Possible integral values $= -3$ and -2

6C.19 HKCEE AM 1993 – I – 3

$\begin{cases} \alpha+\beta = -p \\ \alpha\beta = q \end{cases}$ and $\begin{cases} (\alpha+3)+(\beta+3) = -q \\ (\alpha+3)(\beta+3) = p \end{cases}$

$\Rightarrow \begin{cases} \alpha+\beta = -q-6 \\ \alpha\beta = p - 3(\alpha+\beta) - 9 = 4p - 9 \end{cases}$

$\therefore \begin{cases} -p = -q-6 \\ q = 4p-9 \end{cases} \Rightarrow \begin{cases} p = 1 \\ q = -5 \end{cases}$

6C.20 (HKCEE AM 1995 – I – 10)

(a) $f(\alpha) = g(\alpha)$
$12\alpha^2 + 2p\alpha - q = 12\alpha^2 + 2q\alpha - p$
$2\alpha\cancel{(p-q)} = -\cancel{(p-q)}$ ($\because$ p, q are distinct)
$2\alpha = -1 \Rightarrow \alpha = -\frac{1}{2}$

(b) $\alpha+\beta = -\frac{2p}{12} \Rightarrow \beta = \frac{-p}{6} + \frac{1}{2}$
$\alpha\gamma = \frac{-p}{12} \Rightarrow \gamma = \frac{-p}{12} \div \frac{-1}{2} = \frac{p}{6}$

6C.21 HKCEE AM 1998 – I – 2

$\begin{cases} \alpha+\beta = 2 \\ \alpha\beta = 7 \end{cases}$

Sum of roots $= (\alpha+2) + (\beta+2) = (\alpha+\beta) + 4$
$= (2) + 4 = 6$

Product of roots $= (\alpha+2)(\beta+2) = \alpha\beta + 2(\alpha+\beta) + 4$
$= (7) + 2(2) + 4 = 15$

Required equation: $x^2 - 6x + 15 = 0$

6C.22 HKCEE AM 2000 – I – 7

(a) $\alpha+\beta = 2 - p$, $\alpha\beta = p$

(b) $\alpha^2 + \beta^2 = 11$
$(\alpha+\beta)^2 - 2\alpha\beta = 11$
$(2-p)^2 - 2(p) = 11$
$p^2 - 6p - 7 = 0 \Rightarrow p = 7$ or -1

6C.23 (HKCEE AM 2011 – I – 7)

(a) $\Delta = (k+2)^2 - 4k = k^2 + 4$
$\geq 0 + 4 > 0$

$\therefore$ The roots are real and distinct.

(b) If $\alpha = \sqrt{\beta^2}$ and $\alpha \neq \beta$ (from (a)),
then $\alpha = -\beta$.
$\therefore \alpha + \beta = 0 \Rightarrow k = -2$

6C.24 HKDSE MA PP – I – 17

(a) $\frac{1}{1+2i} = \frac{1(1-2i)}{(1+2i)(1-2i)} = \frac{1-2i}{1^2+2^2} = \frac{1}{5} - \frac{2}{5}i$

(b) (i) By (a), the roots are $10\left(\frac{1}{5} - \frac{2}{5}i\right) = 2 - 4i$ and $2 + 4i$.

$\therefore \begin{cases} p = -(\text{sum of roots}) = -4 \\ q = \text{product of roots} = 2^2 + 4^2 = 20 \end{cases}$

(ii) The equation becomes $x^2 - 4x + (20 - r) = 0$.
$\Delta \geq 0$
$16 - 4(20 - r) \geq 0 \Rightarrow r \geq 16$

6D Complex numbers

6D.1 HKDSE MA PP – I – 17

(a) $\frac{1}{1+2i} = \frac{1(1-2i)}{(1+2i)(1-2i)} = \frac{1-2i}{1^2+2^2} = \frac{1}{5} - \frac{2}{5}i$

7 Functions and Graphs

7A General functions

7A.1 HKCEE MA 1992 – I – 4

(a) Factorize

(i) x^2-2x,

(ii) x^2-6x+8.

(b) Simplify $\dfrac{1}{x^2-2x}+\dfrac{1}{x^2-6x+8}$.

7A.2 HKCEE MA 1993 – I – 2(a)

Let $f(x)=\dfrac{x^2+1}{x-1}$. Find $f(3)$.

7A.3 HKCEE MA 2006 – I – 10

Let $f(x)=(x-a)(x-b)(x+1)-3$, where a and b are positive integers with $a<b$. It is given that $f(1)=1$.

(a) (i) Prove that $(a-1)(b-1)=2$.

(ii) Write down the values of a and b.

(b) Let $g(x)=x^3-6x^2-2x+7$. Using the results of (a)(ii), find $f(x)-g(x)$.
Hence find the exact values of all the roots of the equation $f(x)=g(x)$.

7A.4 HKDSE MA 2016 – I – 3

Simplify $\dfrac{2}{4x-5}+\dfrac{3}{1-6x}$.

7A.5 HKDSE MA 2019 – I – 2

Simplify $\dfrac{3}{7x-6}-\dfrac{2}{5x-4}$.

7B Quadratic functions and their graphs

7B.1 HKCEE MA 1982(1/2/3) – I – 11

In the figure, O is the origin. The curve $C_1: y=x^2-10x+k$ (where k is a fixed constant) intersects the x-axis at the points A and B.

(a) By considering the sum and the product of the roots of $x^2-10x+k=0$, or otherwise,

(i) find $OA+OB$,

(ii) find $OA\times OB$ in terms of k.

(b) M and N are the mid-points of OA and OB respectively (see the figure).

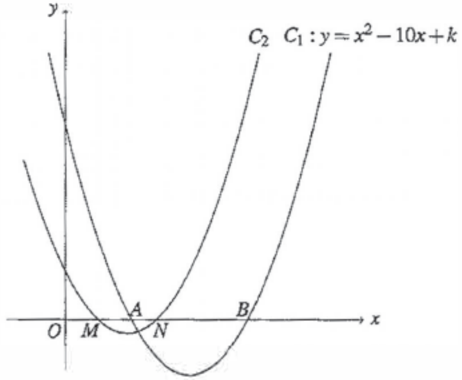
(i) Find $OM+ON$.

(ii) Find $OM\times ON$ in terms of k.

(c) Another curve $C_2: y=x^2+px+r$ (where p and r are fixed constants) passes through the points M and N.

(i) Using the results in (b) or otherwise, find the value of p and express r in terms of k.

(ii) If $OM=2$, find k.



7B.2 HKCEE MA 1992 – I – 9

The figure shows the graph of $y=2x^2-4x+3$, where $x\geq 0$. $P(a,b)$ is a variable point on the graph. A rectangle $OAPB$ is drawn with A and B lying on the x and y axes respectively.

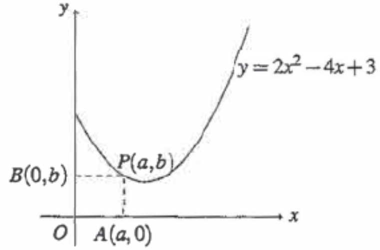
(a) (i) Find the area of rectangle $OAPB$ in terms of a.

(ii) Find the two values of a for which $OAPB$ is a square.

(b) Suppose the area of $OAPB=\dfrac{3}{2}$.

(i) Show that $4a^3-8a^2+6a-3=0$.

(ii) *[Out of syllabus]*



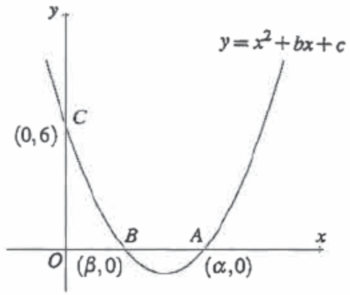
7B.3 HKCEE MA 1994 – I – 8

In the figure, the curve $y=x^2+bx+c$ meets the y-axis at $C(0,6)$ and the x axis at $A(\alpha,0)$ and $B(\beta,0)$, where $\alpha>\beta$.

(a) Find c and hence find the value of $\alpha\beta$.

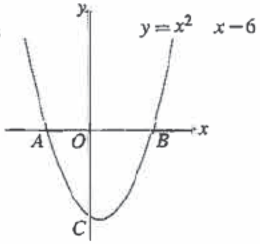
(b) Express $\alpha+\beta$ in terms of b.

(c) Using the results in (a) and (b), express $(\alpha-\beta)^2$ in terms of b. Hence find the area of $\triangle ABC$ in terms of b.



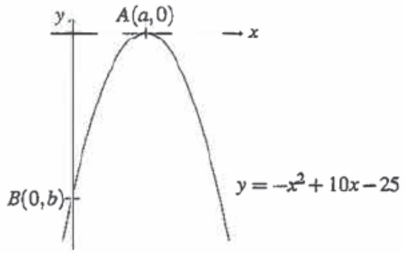
7B.4 HKCEE MA 1999 – I – 7

The graph of $y=x^2-x-6$ cuts the x-axis at $A(a,0)$, $B(b,0)$ and the y-axis at $C(0,c)$ as shown in the figure. Find a, b and c.



7B.5 HKCEE MA 2004 – I – 4

In the figure, the graph of $y=-x^2+10x-25$ touches the x-axis at $A(a,0)$ and cuts the y-axis at $B(0,b)$. Find a and b.



7B.6 HKCEE MA 2008 – I – 11

Consider the function $f(x)=x^2+bx-15$, where b is a constant. It is given that the graph of $y=f(x)$ passes through the point $(4,9)$.

(a) Find b. Hence, or otherwise, find the two x-intercepts of the graph of $y=f(x)$.

(b) Let k be a constant. If the equation $f(x)=k$ has two distinct real roots, find the range of values of k.

(c) Write down the equation of a straight line which intersects the graph of $y=f(x)$ at only one point.

7B.7 HKCEE MA 2009 – I – 12

In the figure, R is the vertex of the graph of $y=-2(x-11)^2+23$.

(a) Write down

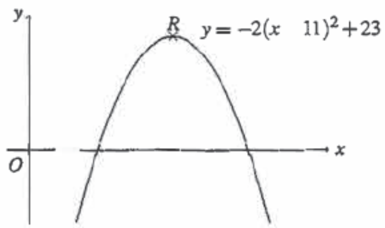
  (i) the equation of the axis of symmetry of the graph,

  (ii) the coordinates of R.

(b) It is given that $P(p,5)$ and $Q(q,5)$ are two distinct points lying on the graph. Find

  (i) the distance between P and Q;

  (ii) the area of the quadrilateral $PQRS$, where S is a point lying on the x axis.



7B.8 HKCEE MA 2010 – I – 16 (To continue as 7E.1.)

Let $f(x)=\frac{1}{2}x-\frac{1}{144}x^2-6$.

(a) (i) Using the method of completing the square, find the coordinates of the vertex of the graph of $y=f(x)$.

7B.9 HKCEE MA 2011 – I – 11 (Continued from 8C.20.)

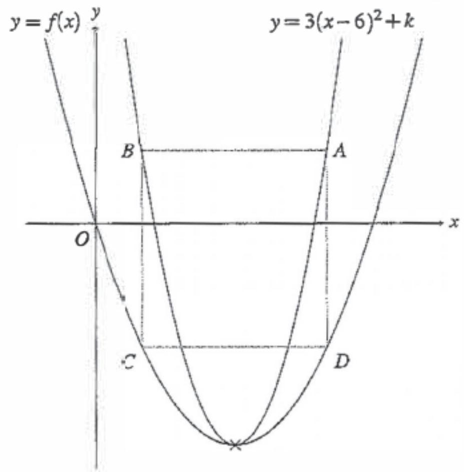
It is given that $f(x)$ is the sum of two parts, one part varies as x^2 and the other part varies as x. Suppose that $f(-2)=28$ and $f(6)=-36$.

(a) Find $f(x)$.

(b) The figure shows the graph of $y=3(x-6)^2+k$ and the graph of $y=f(x)$, where k is a constant. The two graphs have the same vertex.

  (i) Find the value of k.

  (ii) It is given that A and B are points lying on the graph of $y=3(x-6)^2+k$ while C and D are points lying on the graph of $y=f(x)$. Also, $ABCD$ is a rectangle and AB is parallel to the x axis. The x coordinate of A is 10. Find the area of the rectangle $ABCD$.



7B.10 HKCEE AM 1988 – I – 10 (To continue as 10C.9.)

Let $f(x)=x^2+2x-1$ and $g(x)=-x^2+2kx-k^2+6$ (where k is a constant.)

(a) Suppose the graph of $y=f(x)$ cuts the x axis at the points P and Q, and the graph of $y=g(x)$ cuts the x-axis at the points R and S.

  (i) Find the lengths of PQ and RS.

  (ii) Find, in terms of k, the x-coordinate of the mid-point of RS.
  If the mid points of PQ and RS coincide with each other, find the value of k.

(b) If the graphs of $y=f(x)$ and $y=g(x)$ intersect at only one point, find the possible values of k; and for each value of k, find the point of intersection.

7B.11 HKCEE AM 1991 – I – 9 (To continue as 10C.11.)

Let $f(x)=x^2+2x-2$ and $g(x)=-2x^2-12x-23$.

(a) Express $g(x)$ in the form $a(x+b)^2+c$, where a, b and c are real constants.
Hence show that $g(x)<0$ for all real values of x.

(b) Let k_1 and k_2 $(k_1>k_2)$ be the two values of k such that the equation $f(x)+kg(x)=0$ has equal roots.

  (i) Find k_1 and k_2.

7B.12 (HKCEE AM 1993 – I – 10)

$C(k)$ is the curve $y=\frac{1}{k+1}[2x^2+(k+7)x+4]$, where k is a real number not equal to -1.

(a) If $C(k)$ cuts the x axis at two points P and Q and $PQ=1$, find the value(s) of k.

(b) Find the range of values of k such that $C(k)$ does not cut the x-axis.

(c) (i) Find the points of intersection of the curves $C(1)$ and $C(-2)$.

  (ii) Show that $C(k)$ passes through the two points in (c)(i) for all values of k.

7B.13 HKCEE AM 1998 – I – 11

Let $f(x) = x^2 - kx$, where k is a real constant, and $g(x) = \quad x$.

(a) Show that the least value of $f(x)$ is $\frac{k^2}{4}$ and find the corresponding value of x.

(b) Find the coordinates of the two intersecting points of curves $y = f(x)$ and $y = g(x)$.

(c) Suppose $k = 3$.
  (i) In the same diagram, sketch the graphs of $y = f(x)$ and $y = g(x)$ and label their intersecting points.
  (ii) Find the range of values of x such that $f(x) \le g(x)$.
  Hence find the least value of $f(x)$ within this range of values of x.

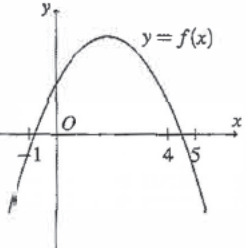
(d) Suppose $k = \frac{3}{2}$. Find the least value of $f(x)$ within the range of values of x such that $f(x) \le g(x)$.

7B.14 HKCEE AM 2000 – I – 12

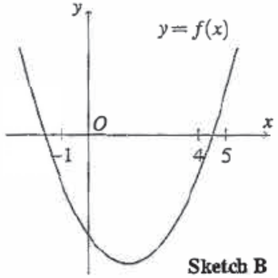
Consider the function $f(x) = x^2 - 4mx - (5m^2 - 6m + 1)$, where $m > \frac{1}{3}$.

(a) Show that the equation $f(x) = 0$ has distinct real roots.

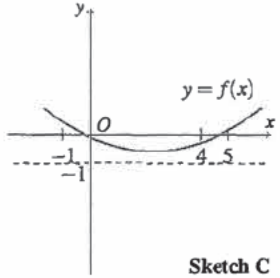
(b) Let α and β be the roots of the equation $f(x) = 0$, where $\alpha < \beta$.
  (i) Express α and β in terms of m.
  (ii) Furthermore, it is known that $4 < \beta < 5$.
    (1) Show that $1 < m < \frac{6}{5}$.
    (2) The following figure shows three sketches of the graph of $y = f(x)$ drawn by three students. Their teacher points out that the three sketches are all incorrect. Explain why each of the sketches is incorrect.



Sketch A



Sketch B



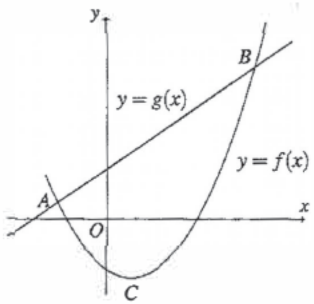
Sketch C

7B.15 HKCEE AM 2002 – 11

Let $f(x) = x^2 - 2x - 6$ and $g(x) = 2x + 6$. The graphs of $y = f(x)$ and $y = g(x)$ intersect at points A and B (see the figure). C is the vertex of the graph of $y = f(x)$.

(a) Find the coordinates of points A, B and C.

(b) Write down the range of values of x such that $f(x) \le g(x)$.
  Hence write down the value(s) of k such that the equation $f(x) = k$ has only one real root in this range.



7B.16 HKCEE AM 2003 17

Let $f(x) = \quad (x - a)^2 + b$, where a and b are real. Point P is the vertex of the graph of $y = f(x)$.

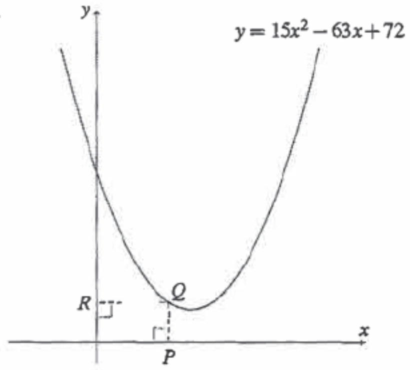
(a) Write down the coordinates of point P.

(b) Let $g(x)$ be a quadratic function such that the coefficient of x^2 is 1 and the vertex of the graph of $y = g(x)$ is the point $Q(b, a)$. It is given that the graph of $y = f(x)$ passes through point Q.
  (i) Write down $g(x)$ and show that the graph of $y = g(x)$ passes through point P.
  (ii) Furthermore, the graph of $y = f(x)$ touches the x-axis. For each of the possible cases, sketch the graphs of $y = f(x)$ and $y = g(x)$ in the same diagram.

7B.17 HKDSE MA 2012 – I 13 (Continued from **4B.22.**)

(a) Find the value of k such that $x - 2$ is a factor of $kx^3 - 21x^2 + 24x - 4$.

(b) The figure shows the graph of $y = 15x^2 - 63x + 72$. Q is a variable point on the graph in the first quadrant. P and R are the feet of the perpendiculars from Q to the x axis and the y axis respectively.
  (i) Let $(m, 0)$ be the coordinates of P. Express the area of the rectangle $OPQR$ in terms of m.
  (ii) Are there three different positions of Q such that the area of the rectangle $OPQR$ is 12? Explain your answer.



7B.18 HKDSE MA 2015 I 18 (To continue as 7E.2.)

Let $f(x) = 2x^2 - 4kx + 3k^2 + 5$, where k is a real constant.

(a) Does the graph of $y = f(x)$ cut the x axis? Explain your answer.

(b) Using the method of completing the square, express, in terms of k, the coordinates of the vertex of the graph of $y = f(x)$.

7B.19 HKDSE MA 2016 – I – 18 (To continue as 7E.3.)

Let $f(x) = \frac{-1}{3}x^2 + 12x \quad 121$.

(a) Using the method of completing the square, find the coordinates of the vertex of the graph of $y = f(x)$.

7B.20 HKDSE MA 2017 – I 18

The equation of the parabola Γ is $y = 2x^2 - 2kx + 2x - 3k + 8$, where k is a real constant. Denote the straight line $y = 19$ by L.

(a) Prove that L and Γ intersect at two distinct points.

(b) The points of intersection of L and Γ are A and B.
  (i) Let a and b be the x coordinates of A and B respectively. Prove that $(a \quad b)^2 = k^2 + 4k + 23$.
  (ii) Is it possible that the distance between A and B is less than 4? Explain your answer.

7B.21 HKDSE MA 2018 – I – 18 (Continued from 8C.29 and to continue as 7E.4.)

It is given that $f(x)$ partly varies as x^2 and partly varies as x. Suppose that $f(2)=60$ and $f(3)=99$.

(a) Find $f(x)$.

(b) Let Q be the vertex of the graph of $y=f(x)$ and R be the vertex of the graph of $y=27-f(x)$.

(i) Using the method of completing the square, find the coordinates of Q.

7B.22 HKDSE MA 2020 – I –

Let $p(x)=4x^2+12x+c$, where c is a constant. The equation $p(x)=0$ has equal roots. Find

(a) c,

(b) the x-intercept(s) of the graph of $y=p(x)-169$.

(5 marks)

7B.23 HKDSE MA 2020 – I – 17

Let $g(x)=x^2-2kx+2k^2+4$, where k is a real constant.

(a) Using the method of completing the square, express, in terms of k, the coordinates of the vertex of the graph of $y=g(x)$. (2 marks)

(b) On the same rectangular coordinate system, let D and E be the vertex of the graph of $y=g(x+2)$ and the vertex of the graph of $y=-g(x-2)$ respectively. Is there a point F on this rectangular coordinate system such that the coordinates of the circumcentre of ΔDEF are $(0,3)$? Explain your answer. (4 marks)

7C Extreme values of quadratic functions

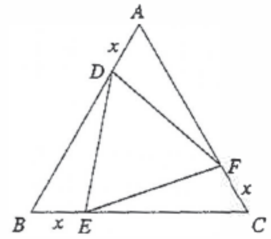
7C.1 HKCEE MA 1985(A/B) – I – 13 (Continued from 14A.3 and to continue as 10C.2.)

In the figure, ABC is an equilateral triangle. $AB=2$. D, E, F are points on AB, BC, CA respectively such that $AD=BE=CF=x$.

(a) By using the cosine formula or otherwise, express DE^2 in terms of x.

(b) Show that the area of $\triangle DEF=\frac{\sqrt{3}}{4}(3x^2-6x+4)$.

Hence, by using the method of completing the square, find the value of x such that the area of $\triangle DEF$ is smallest.



7C.2 HKCEE MA 1982(1/2) – I – 12 (Continued from 8C.1.)

The price of a certain monthly magazine is x dollars per copy. The total profit on the sale of the magazine is P dollars. It is given that $P=Y+Z$, where Y varies directly as x and Z varies directly as the square of x. When x is 20, P is 80 000; when x is 35, P is 87 500.

(a) Find P when $x=15$.

(b) Using the method of completing the square, express P in the form $P=a \quad b(x \quad c)^2$ where a, b and c are constants. Find the values of a, b and c.

(c) Hence, or otherwise, find the value of x when P is a maximum.

7C.3 HKCEE MA 1988 – I – 10 (Continued from 8C.5.)

A variable quantity y is the sum of two parts. The first part varies directly as another variable x, while the second part varies directly as x^2. When $x=1$, $y=-5$; when $x=2$, $y=-8$.

(a) Express y in terms of x. Hence find the value of y when $x=6$.

(b) Express y in the form $(x \quad p)^2-q$, where p and q are constants. Hence find the least possible value of y when x varies.

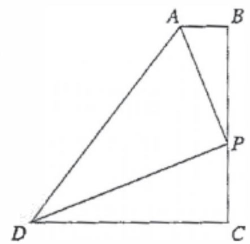
7C.4 HKCEE MA 2011 – I – 12

In the figure, $ABCD$ is a trapezium, where AB is parallel to CD. P is a point lying on BC such that $BP=x$ cm. It is given that $AB=3$ cm, $BC=11$ cm, $CD=k$ cm and $\angle ABP=\angle APD=90°$.

(a) Prove that $\triangle ABP \sim \triangle PCD$.

(b) Prove that $x^2 \quad 11x+3k=0$.

(c) If k is an integer, find the greatest value of k.



7C.5 HKCEE AM 1986 I – 3

The maximum value of the function $f(x)=4k+18x \quad kx^2$ (k is a positive constant) is 45. Find k.

7C.6 HKCEE AM 1996 – I – 4

Given $x^2 \quad 6x+11=(x+a)^2+b$, where x is real.

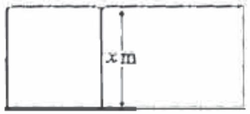
(a) Find the values of a and b. Hence write down the least value of $x^2-6x+11$.

(b) Using (a), or otherwise, write down the range of possible values of $\frac{1}{x^2-6x+11}$.

7C.7 HKDSE MA 2013 – I – 17

(a) Let $f(x) = 36x - x^2$. Using the method of completing the square, find the coordinates of the vertex of the graph of $y = f(x)$.

(b) The length of a piece of string is 108 m. A guard cuts the string into two pieces. One piece is used to enclose a rectangular restricted zone of area A m². The other piece of length x m is used to divide this restricted zone into two rectangular regions as shown in the figure.

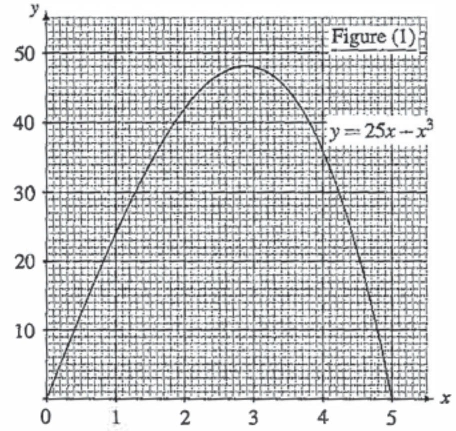


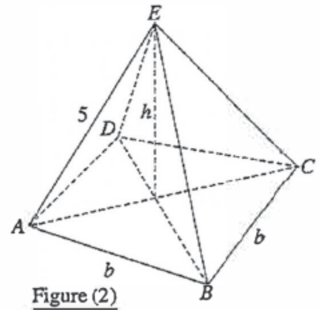
(i) Express A in terms of x.

(ii) The guard claims that the area of this restricted zone can be greater than 500 m². Do you agree? Explain your answer.

7D Solving equations using graphs of functions

7D.1 HKCEE MA 1980(3) I 16





(a) Figure (1) shows the graph of $y = 25x - x^3$ for $0 \le x \le 5$. By adding a suitable straight line to the graph, solve the equation $30 = 25x - x^3$, where $0 \le x \le 5$. Give your answers correct to 2 significant figures.

(b) Figure (2) shows a right pyramid with a square base $ABCD$. $AB = b$ units and $AE = 5$ units. The height of the pyramid is h units and its volume is V cubic units.

(i) Express b in terms of h. Hence show that $V = \frac{2}{3}(25h - h^3)$.

(ii) Using (a), find the two values of h such that $V = 20$.
(Your answers should be correct to 2 significant figures.)

(iii) ***[Out of syllabus]***

7D.2 HKCEE MA 1981(1) – I – 11

A piece of wire 20 cm long is bent into a rectangle. Let one side of the rectangle be x cm long and the area be y cm².

(a) Show that $y = 10x - x^2$.

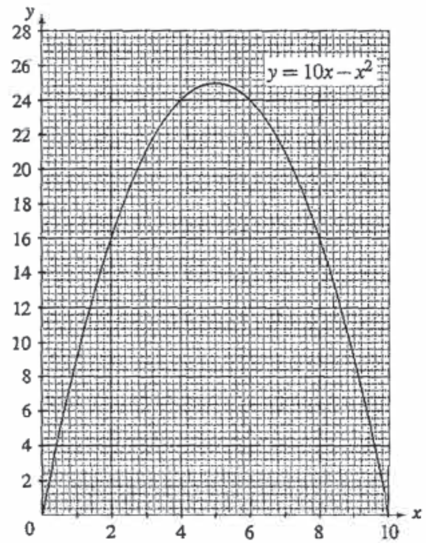
(b) The figure shows the graph of $y = 10x - x^2$ for $0 \le x \le 10$. Using the graph, find

(i) the value of y, correct to 1 decimal place, when $x = 3.4$,

(ii) the values of x, correct to 1 decimal place, when the area of the rectangle is 12 cm²,

(iii) the greatest area of the rectangle,

(iv) ***[Out of syllabus]***



7D.3 HKCEE MA 1983(A) – I – 14

Equal squares each of side k cm are cut from the four corners of a square sheet of paper of side 7 cm (see Figure (1)). The remaining part is folded along the dotted lines to form a rectangular box as shown in Figure (2).

(a) Show that the volume V of the rectangular box, in cm^3, is $V = 4k^3 - 28k^2 + 49k$.

(b) Figure (3) shows the graph of $y = 4x^3 - 28x^2 + 49x$ for $0 \le x \le 5$. Draw a suitable straight line in Figure (3) and use it to find all the possible values of x such that $4x^3 - 28x^2 + 49x - 20 = 0$.
(Give the answers to 1 decimal place.)

(c) Using the results of (a) and (b), deduce the values of k such that the volume of the box is $20 cm^3$.
(Give the answers to 1 decimal place.)

(d) ***[Out of syllabus]***

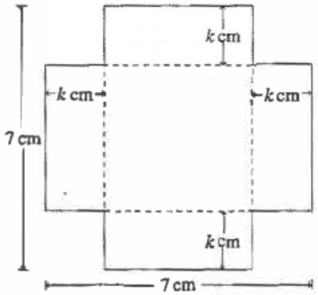


Figure (1)

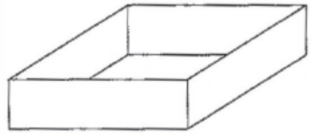


Figure (2)

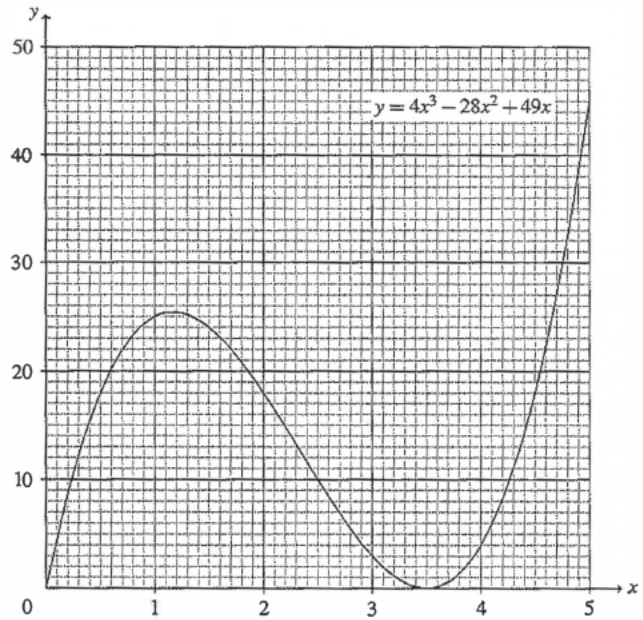


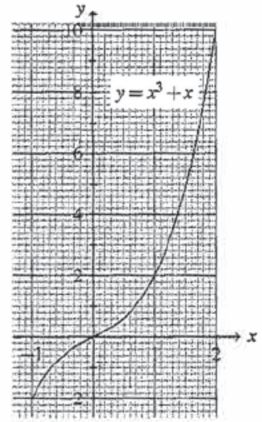
Figure (3)

7D.4 HKCEE MA 1985(A) – I – 12

The figure shows the graph of $y = x^3 + x$ for $-1 \le x \le 2$.

(a) (i) Draw a suitable straight line in the figure and hence find, correct to 1 decimal place, the real root of the equation $x^3 + x - 1 = 0$.
(ii) *[**Out of syllabus**. The result $x = 0.68$ (correct to 2 d.p.) is obtained for the equation in (i).]*

(b) (i) Expand and simplify the expression $(x+1)^4 - (x-1)^4$.
(ii) Using the result in (a)(ii), find, correct to 2 decimal places, the real root of the equation $(x+1)^4 - (x-1)^4 = 8$.



7D.5 HKCEE MA 1985(B) – I – 12

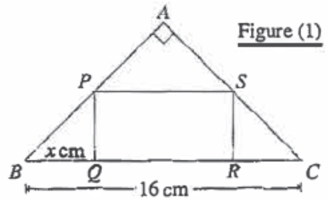
In Figure (1), ABC is an isosceles triangle with $\angle A = 90°$. $PQRS$ is a rectangle inscribed in $\triangle ABC$. $BC = 16$ cm, $BQ = x$ cm.

(a) Show that the area of $PQRS = 2(8x - x^2)$ cm^2.

(b) Figure (2) shows the graph of $y = 8x - x^2$ for $0 \le x \le 8$.
Using the graph,
(i) find the value of x such that the area of $PQRS$ is greatest;
(ii) find the two values of x, correct to 1 decimal place, such that the area of $PQRS$ is 28 cm^2.

(c) ***[Out of syllabus]***

Figure (1)



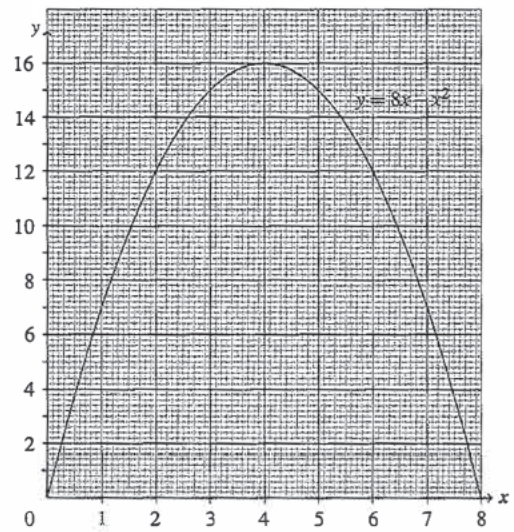


Figure (2)

7D.6 HKCEE MA 1986(B) – I 14

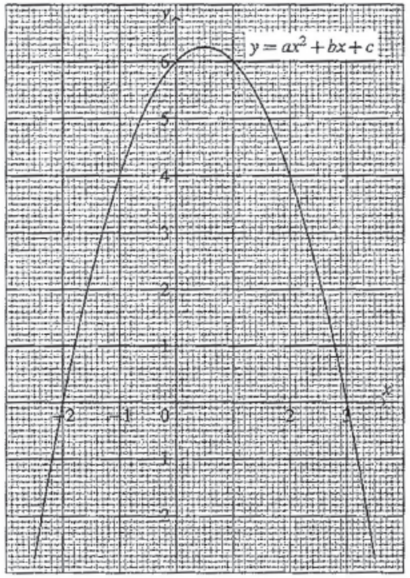
The figure shows the graph of $y=ax^2+bx+c$.

(a) Find the value of c and hence the values of a and b.

(b) Solve the following equations by adding a suitable straight line to the figure for each case. Give your answers correct to 1 decimal place.

(i) $(x+2)(x-3)=-1$,

(ii) *[Out of syllabus]*



7D.7 HKCEE MA 1987(A) I – 14

The figure shows the graph of $y=x^3-6x^2+9x$.

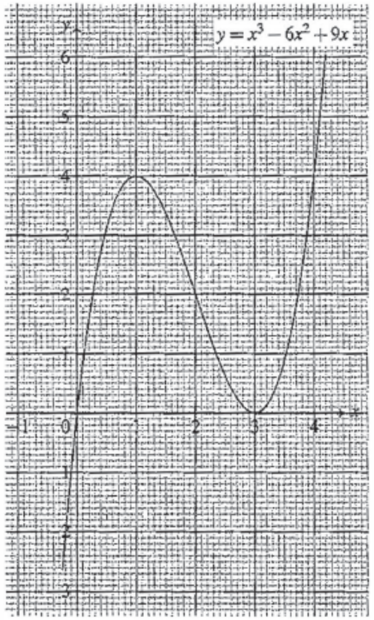
(a) By adding suitable straight lines to the figure, find, correct to 1 decimal place, the real roots of the following equations:

(i) $x^3-6x^2+9x-1=0$,

(ii) *[Out of syllabus]*

(b) *[Out of syllabus]*

(c) From the figure, find the range of values of k such that the equation $x^3-6x^2+9x-k=0$ has three distinct real roots.



7D.8 HKCEE MA 1997 I – 13

Miss Lee makes and sells handmade leather belts and handbags. She finds that if a batch of x belts is made, where $1\le x\le 11$, the cost per belt $\$B$ is given by $B=x^2-20x+120$. The figure shows the graph of the function $y=x^2-20x+120$.

(a) Use the given graph to write down the number(s) of belts in a batch that will make the cost per belt

(i) a minimum,

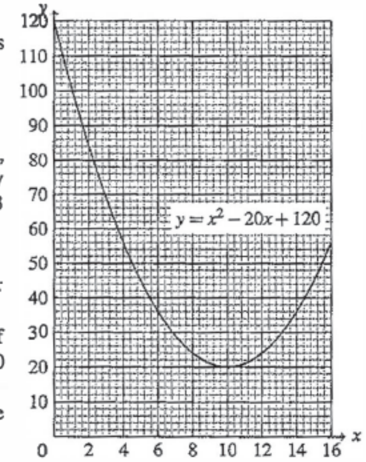
(ii) less than \$90.

(b) Miss Lee also finds that if a batch of x handbags is made, where $1\le x\le 8$, the cost per handbag $\$H$ is given by $H=x^2-17x+c$ (c is a constant). When a batch of 3 handbags is made, the cost per handbag is \$144.

(i) Find c.

(ii) *[Out of syllabus The following result is obtained: When $H=120$, $x=6$.]*

(iii) Miss Lee made a batch of 10 belts and a batch of 6 handbags. She managed to sell 6 belts at \$100 each and 4 handbags at \$300 each while the remaining belts and handbags sold at half of their respective cost. Find her gain or loss.



7D.9 HKCEE MA 2000 – I – 18

(Continued from 8C.11.)

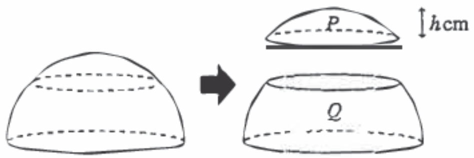


Figure (1)

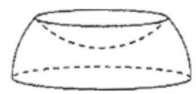


Figure (2)

Figure (1) shows a solid hemisphere of radius 10 cm. It is cut into two portions, P and Q, along a plane parallel to its base. The height and volume of P are h cm and V cm^3 respectively. It is known that V is the sum of two parts. One part varies directly as h^2 and the other part varies directly as h^3. $V=\frac{29}{3}\pi$ when $h=1$ and $V=81\pi$ when $h=3$.

(a) Find V in terms of h and π.

(b) A solid congruent to P is carved away from the top of Q to form a container as shown in Figure (2).

(i) Find the surface area of the container (excluding the base).

(ii) It is known that the volume of the container is $\frac{1400}{3}\pi$ cm^3. Show that $h^3-30h^2+300=0$.

(iii) Using the graph in Figure (3) and a suitable method, find the value of h correct to 2 decimal places.

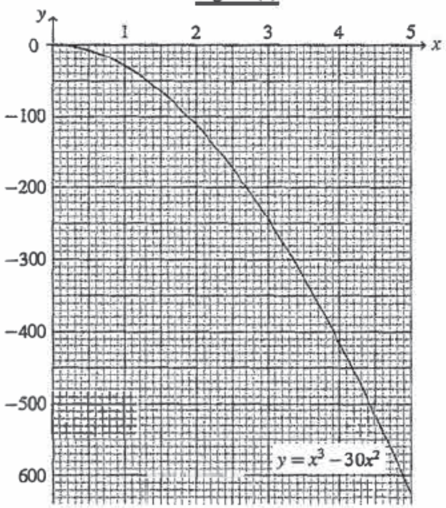


Figure (3)

7E Transformation of graphs of functions

7E.1 HKCEE MA 2010 – I – 16 (Continued from **7B.8**.)

Let $f(x) = \frac{1}{2}x \quad \frac{1}{144}x^2 \quad 6$.

(a) (i) Using the method of completing the square, find the coordinates of the vertex of the graph of $y = f(x)$.

(ii) If the graph of $y = g(x)$ is obtained by translating the graph of $y = f(x)$ leftwards by 4 units and upwards by 5 units, find $g(x)$.

(iii) If the grpah of $y = h(x)$ is obtained by translating the graph of $y = 2^{f(x)}$ leftwards by 4 units and upwards by 5 units, find $h(x)$.

(b) A researcher performs an experiment to study the relationship between the number of bacteria A (u hundred million) and the temperature (s °C) under some controlled conditions. From the data of u and s recorded in Table (1), the researcher suggests using the formula $u = 2^{f(s)}$ to describe the relationship.

s	a_1	a_2	a_3	a_4	a_5	a_6	a_7
u	b_1	b_2	b_3	b_4	b_5	b_6	b_7

Table (1)

(i) According to the formula suggested by the researcher, find the temperature at which the number of the bacteria is 8 hundred million.

(ii) The researcher then performs another experiment to study the relationship between the number of bacteria B (v hundred million) and the temperature (t °C) under the same controlled conditions and the data of v and t are recorded in Table (2).

t	a_1-4	a_2-4	a_3-4	a_4-4	a_5-4	a_6-4	a_7-4
v	b_1+5	b_2+5	b_3+5	b_4+5	b_5+5	b_6+5	b_7+5

Table (2)

Using the formula suggested by the research, propose a formula to express v in terms of t.

7E.2 HKDSE MA 2015 – I – 18 (Continued from **7B.18**.)

Let $f(x) = 2x^2 - 4kx + 3k^2 + 5$, where k is a real constant.

(a) Does the graph of $y = f(x)$ cut the x axis? Explain your answer.

(b) Using the method of completing the square, express, in terms of k, the coordinates of the vertex of the graph of $y = f(x)$.

(c) In the same rectangular system, let S and T be moving points on the graph of $y = f(x)$ and the graph of $y = 2 - f(x)$ respectively. Denote the origin by O. Someone claims that when S and T are nearest to each other, the circumcentre of $\triangle OST$ lies on the x axis. Is the claim correct? Explain your answer.

7E.3 HKDSE MA 2016 – I – 18 (Continued from **7B.19**.)

Let $f(x) = \frac{-1}{3}x^2 + 12x - 121$.

(a) Using the method of completing the square, find the coordinates of the vertex of the graph of $y = f(x)$.

(b) The graph of $y = g(x)$ is obtained by translating the graph of $y = f(x)$ vertically. If the graph of $y = g(x)$ touches the x-axis, find $g(x)$.

(c) Under a transformation, $f(x)$ is changed to $\frac{-1}{3}x^2 - 12x - 121$. Describe the geometric meaning of the transformation.

7E.4 HKDSE MA 2018 – I 18 (Continued from **7B.21**.)

It is given that $f(x)$ partly varies as x^2 and partly varies as x. Suppose that $f(2) = 60$ and $f(3) = 99$.

(a) Find $f(x)$.

(b) Let Q be the vertex of the graph of $y = f(x)$ and R be the vertex of the graph of $y = 27 \quad f(x)$.

(i) Using the method of completing the square, find the coordinates of Q.

(ii) Write down the coordinates of R.

(iii) The coordinates of the point S are $(56, 0)$. Let P be the circumcentre of $\triangle QRS$. Describe the geometric relationship between P, Q and R. Explain your answer.

7E.5 HKDSE MA 2019 I – 19 (To continue as **16C.56**.)

Let $f(x) = \frac{1}{\cancel{1+k}}(x^2 + (6k \quad 2)x + (9k + 25))$, where k is a positive constant. Denote the point $(4, 33)$ by F.

(a) Prove that the graph of $y = f(x)$ passes through F.

(b) The graph of $y = g(x)$ is obtained by reflecting the graph of $y = f(x)$ with respect to the y-axis and then translating the resulting graph upwards by 4 units. Let U be the vertex of the graph of $y = g(x)$. Denote the origin by O.

(i) Using the method of completing the square, express the coordinates of U in terms of k.

7 Functions and Graphs

7A General functions

7A.1 HKCEE MA 1992 – I – 4

(a) (i) $x^2-2x=x(x-2)$

(ii) $x^2-6x+8=(x-2)(x-4)$

(b) $\dfrac{1}{x^2-2x}+\dfrac{1}{x^2-6x+8}\quad\dfrac{1}{x(x-2)}+\dfrac{1}{(x-2)(x-4)}$

$$=\frac{(x-4)+(x)}{x(x-2)(x-4)}$$
$$=\frac{2x-4}{x(x-2)(x-4)}$$
$$=\frac{2\cancel{(x-2)}}{x\cancel{(x-2)}(x-4)}=\frac{2}{x(x-4)}$$

7A.2 HKCEE MA 1993 – I – 2(a)

$$f(3)=\frac{(3)^2+1}{(3)-1}=5$$

7A.3 HKCEE MA 2006 – I – 10

(a) (i) $1=f(1)=(1-a)(1-b)(2)-3$

$\Rightarrow\ (a-1)(b-1)=2$

(ii) Since $a-1$ and $b-1$ are both integers and $b-1>a-1$,

$\begin{cases}a-1=1\\b-1=2\end{cases}\Rightarrow\begin{cases}a=2\\b=3\end{cases}$

(b) $f(x)-g(x)$

$=(x\quad 2)(x-3)(x+1)-3-(x^3-6x^2-2x+7)$

$=2x^2+3x-4$

$\therefore\quad f(x)=g(x)$

$\Rightarrow\ 2x^2+3x-4=0$

$$x==\frac{-3\pm\sqrt{9+32}}{4}=\frac{-3\pm\sqrt{41}}{4}$$

7A.4 HKDSE MA 2016 – I – 3

$$\frac{2}{4x-5}+\frac{3}{1-6x}=\frac{2(1-6x)+3(4x-5)}{(4x-5)(1-6x)}$$
$$=\frac{-13}{(4x-5)(1-6x)}$$

7A.5 HKDSE MA 2019 – I – 2

$$\frac{3}{7x-6}-\frac{2}{5x-4}=\frac{3(5x-4)-2(7x-6)}{(7x-6)(5x-4)}\quad\frac{x}{(7x-6)(5x\quad 4)}$$

7B Quadratic functions

7B.1 HKCEE MA 1982(1/2/3) – I – 11

(a) Since OA and OB are the roots of the equation,

(i) $OA+OB=10$

(ii) $OA\times OB=k$

(b) (i) $OM+ON=\dfrac{OA}{2}+\dfrac{OB}{2}=\dfrac{OA+OB}{2}=5$

(ii) $OM\times ON=\left(\dfrac{OA}{2}\right)\left(\dfrac{OB}{2}\right)=\dfrac{OA\times OB}{4}=\dfrac{k}{4}$

(c) (i) $-p=OM+ON=5\ \Rightarrow\ p=5$

$r=OM\times ON=\dfrac{k}{4}$

(ii) $OM+ON=5\ \Rightarrow\ ON=5-2=3$

$\therefore\ \dfrac{k}{4}=OM\times ON\ \Rightarrow\ k=4\times2\times3=24$

7B.2 HKCEE MA 1992 – I – 9

(a) (i) $b=2a^2-4a+3$

$\therefore$ Area of $OAPB=a(2a^2-3a+3)=2a^3-4a^2+3a$

(ii) When $a=2a^2-4a+3$,

$2a^2-5a+3=0\ \Rightarrow\ a=1$ or $\dfrac{3}{2}$

(b) (i) $2a^3-4a^2+3a=\dfrac{3}{2}$

$4a^3-8a^2+6a=3$

$4a^3-8a^2+6a-3=0$

(ii) *[Out of syllabus]*

7B.3 HKCEE MA 1994 – I – 8

(a) $c=y$-intercept $=6$

$\therefore\ \alpha\beta=$ product of roots $=6$

(b) $\alpha+\beta=$ sum of roots $=-b$

(c) $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=(-b)^2-4(6)$

$=b^2-24$

$\therefore$ Area of $\Delta ABC=\dfrac{1}{2}(\alpha-\beta)(6)$

$=3(\alpha-\beta)=3\sqrt{b^2-24}$

7B.4 HKCEE MA 1999 – I – 7

$c=y$-intercept $=-6$

When $y=0$, $x^2-x-6=0\ \Rightarrow\ x=-2$ or 3

$\therefore\ a=-2,\ b=3$

7B.5 HKCEE MA 2004 – I – 4

$b=y$-intercept $=-25$

Put $(a,0)$: $0=-a^2+10a-25\ \Rightarrow\ a=5$ (repeated)

7B.6 HKCEE MA 2008 – I – 11

(a) Put $(4,9)$: $9=(4)^2+b(4)-15\ \Rightarrow\ b=2$

Hence, $0=x^2+2x\quad15=(x+5)(x-3)$

$\Rightarrow\ x$-intercept $=-5$ and 3

(b) $x^2+2x-15=k\ \Rightarrow\ x^2+2x-(15+k)=0$

$\because$ 2 distinct roots

$\therefore\quad\Delta>0$

$4+4(15+k)>0\ \Rightarrow\ k>-16$

(c) When $\Delta=0$, there is only 1 intersection. i.e. $k=-16$.

$\therefore$ Required line is $y=-16$.

7B.7 HKCEE MA 2009 – I – 12

(a) (i) $x=11$

(ii) $(11,23)$

(b) (i) Put $y=5$: $\quad5=-2(x-11)^2+23$

$(x-11)^2=9\ \Rightarrow\ x=11\pm3=8$ or 14

$\therefore$ Distance between P and $Q=14-8=6$

(ii) Regardless of the position of S, for ΔPQS,

$PQ=6$, Corresponding height $=5$

$\therefore$ Area of $PQRS$

$=$ Area of $\Delta PQR+$ Area of ΔPQS

$=\frac{1}{2}(6)(23-5)+\frac{1}{2}(6)(5)=69$

7B.8 HKCEE MA 2010 – I – 16

(a) (i) $f(x)=\dfrac{-1}{144}(x^2-72x)-6$

$$=\frac{-1}{144}(x^2-72x+36^2-36^2)-6$$
$$=\frac{-1}{144}(x-36)^2+3\ \Rightarrow\ \text{Vertex}=(36,3)$$

7B.9 HKCEE MA 2011 – I – 11

(a) Let $f(x)=hx^2+kx$.

$\begin{cases}28=f(-2)=4h-2k\\-36=f(6)=36h+6k\end{cases}\Rightarrow\begin{cases}h=1\\k=-12\end{cases}$

$\therefore\ f(x)=x^2-12x$

(b) (i) $f(x)=x^2-12x=(x\quad6)^2-36\ \Rightarrow\ k=-36$

(ii) Put $x=10$.

$y=3(10-6)^2-36=2\ \Rightarrow\ A=(10,2)$

$y=(10)^2-12(10)=-20\ \Rightarrow\ D=(10,-20)$

Since the graphs are symmetric about the common axis of symmetry $x=6$,

$B=(6-(10-6),2)=(2,2)$

$C=(10-(10-6),-20)=(2,-20)$

$\therefore$ Area of $ABCD=(2-(-20))(10-2)=176$

7B.10 HKCEE AM 1988 – I – 10

(a) (i) For $f(x)$, $\begin{cases}\text{Sum of rts}=-2\\\text{Prod of rts}=-1\end{cases}$

For $g(x)$, $\begin{cases}\text{Sum of rts}=2k\\\text{Prod of rts}=k^2-6\end{cases}$

$PQ=$ Difference of rts of $f(x)$

$=\sqrt{(-2)^2-4(\quad1)}=\sqrt{8}$

$RS=$ Difference of rts of $g(x)$

$=\sqrt{(2k)^2-4(k^2-6)}=\sqrt{24}$

(ii) Mid-pt of $RS=\left(\dfrac{\text{Sum of rts}}{2},0\right)=(k,0)$

If this is also the mid-point of PQ, $k=\dfrac{-2}{2}=-1$.

(b) $\begin{cases}y=f(x)\\y=g(x)\end{cases}\Rightarrow\ x^2+2x-1=-x^2+2kx-k^2+6$

$2x^2+2(1-k)x+k^2-7=0\ \ldots\ (*)$

$\Delta=4(1-k)^2-8(k^2-7)=0$

$k^2+2k-15=0\ \Rightarrow\ k=\quad5$ or 3

For $k=-5$, $(*)$ becomes $2x^2+12x+18=0$

$2(x+3)^2=0$

$x=-3$

$\Rightarrow$ Intersection $=(-3,(-3)^2+2(-3)-1)=(-3,2)$

For $k=3$, $(*)$ becomes $2x^2\quad4x+2=0$

$2(x-1)^2=0$

$x=1$

$\Rightarrow$ Intersection $=(1,1^2+2(1)-1)=(1,2)$

7B.11 HKCEE AM 1991 – I – 9

(a) $g(x)=-2x^2-12x-23=-2(x^2+6x+9-9)-25$

$=-2(x+3)^2-5$

$\le-5<0$

(b) (i) $f(x)+kg(x)=0$

$(x^2+2x-2)+k(-2x^2-12x-23)=0$

$(1-2k)x^2+2(1-6k)x-(2+23k)=0$

Equal rts $\Rightarrow\ \Delta=0$

$4(1-6k)^2+4(1\quad2k)(2+23k)=0$

$10k^2-7k-3=0$

$k=1$ or $\dfrac{-3}{10}$

$\therefore\ k_1=1,\ k_2=\dfrac{-3}{10}$

7B.12 (HKCEE AM 1993 – I – 10)

(a) Put $y=0$: $\dfrac{1}{k+1}[2x^2+(k+7)x+4]=0$

$2x^2+(k+7)x+4=0$

$\because$ Sum of rts $=-\dfrac{k+7}{2}$, Product of rts $=2$

$\therefore\quad PQ=$ Difference of rts

$1=\sqrt{\left(\quad\dfrac{k+7}{2}\right)^2-4(2)}$

$1=\dfrac{(k+7)^2}{4}-8$

$(k+7)^2=36$

$k=\pm6-7=-13$ or -1 (rejected)

(b) *Method 1*

From (a), PQ does not exist when

$\left(\dfrac{k+7}{2}\right)^2-8<0$

$(k+7)^2<32$

$-7-\sqrt{32}<k<-7+\sqrt{32}$

Method 2

$\Delta<0$

$\left(\dfrac{k+7}{k+1}\right)^2-4\left(\dfrac{2}{k+1}\right)\left(\dfrac{4}{k+1}\right)<0$

$(k+7)^2-32<0$

$(k+7)^2<32$

$-7-\sqrt{32}<k<-7+\sqrt{32}$

(c) (i) $\begin{cases}C(1):y=\frac{1}{2}(2x^2+8x+4)=x^2+4x+2\\C(-2):y=-1(2x^2+5x+4)=-2x^2-5x-4\end{cases}$

$\Rightarrow\ 3x^2+9x+6=0$

$x=-2$ or $-1\ \Rightarrow\ y=-2$ or -1

$\therefore$ Pts of intersection are $(-2,-2)$ and $(-1,-1)$.

(ii) Put $x=-2$ into $C(k)$:

RHS $=\dfrac{1}{k+1}[2(-2)^2+(k+7)(-2)+4]$

$=\dfrac{1}{k+1}(-2k-2)=-2$

$\therefore\ (-2,-2)$ is on $C(k)$ for any k.

Put $x=-1$ into $C(k)$:

RHS $=\dfrac{1}{k+1}[2(-1)^2+(k+7)(-1)+4]$

$=\dfrac{1}{k+1}(-k-1)=-1$

$\therefore\ (-1,-1)$ is on $C(k)$ for any k.

7B.13 HKCEE AM 1998 – I – 11

(a) $f(x)=x^2-kx=x^2-kx+\left(\frac{k}{2}\right)^2-\left(\frac{k}{2}\right)^2$

$=\left(x-\frac{k}{2}\right)^2-\frac{k^2}{4}$

$\therefore$ Least value $=-\frac{k^2}{4}$. Corresponding $x=\frac{k}{2}$

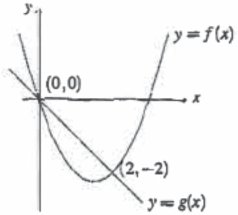
(b) $\begin{cases}y=x^2-kx\\y=-x\end{cases}\Rightarrow x^2-kx=-x$

$x(x-k+1)=0$

$x=0$ or $k-1\Rightarrow y=0$ or $1-k$

$\therefore$ The intersections are $(0,0)$ and $(k-1,1-k)$.

(c) (i)



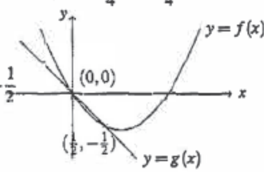
(ii) $f(x)\le g(x)\Rightarrow 0\le x\le 2$

$\therefore$ Least value of $f(x)=-\frac{(3)^2}{4}=-\frac{9}{4}$

(d) $f(x)\le g(x)$

$\Rightarrow 0\le x\le\frac{1}{2}$

$\Rightarrow$ Least $f(x)=-\frac{1}{2}$



7B.14 HKCEE AM 2000 – I – 12

(a) Δ of $f(x)=(-4m)^2+4(5m^2-6m+1)$

$=36m^2-24m+4$

$=4(3m-1)^2\ge 0$

Since $m\ne\frac{1}{3}$, $\Delta\ne 0$.

Thus, $\Delta>0$, and $f(x)$ has 2 distinct real roots.

(b) (i) $x=\frac{4m\pm\sqrt{\Delta}}{2}=\frac{4m\pm 2(3m-1)}{2}$

$\Rightarrow \beta=\frac{4m+2(3m-1)}{2}=5m-1$

$\alpha=\frac{4m-2(3m-1)}{2}=-m+1$

(ii) (1) $4<\beta=5m-1<5\Rightarrow 5<5m<6$

$\Rightarrow 1<m<\frac{6}{5}$

(2) Sketch A:

The parabola should open upwards as the leading coefficient is positive.

Sketch B:

$1<m<\frac{6}{5}\Rightarrow -\frac{1}{5}<\alpha=-m+1<0$

The root should be larger than -1.

Sketch C:

$f(x)=x^2-4mx-(5m^2-6m+1)$

$=x^2-4mx+4m^2-9m^2+6m-1$

$=(x-2m)^2-(3m-1)^2$

$\Rightarrow$ Min value of $f(x)=-(3m-1)^2$

$1<m<\frac{6}{5}\Rightarrow -4.225<-(3m-1)^2<-4$

Thus the min value should be smaller than -1.

7B.15 HKCEE AM 2002 – 11

(a) $f(x)=x^2-2x-6=(x-1)^2-7\Rightarrow C=(1,-7)$

$\begin{cases}y=x^2-2x-6\\y=2x+6\end{cases}$

$\Rightarrow x^2-2x-6=2x+6$

$x^2-4x-12=0\Rightarrow x=6$ or -2

$\therefore A=(-2,2(-2)+6)=(-2,2)$

$B=(6,2(6)+6)=(6,18)$

(b) $f(x)\le g(x)$ when $-2\le x\le 6$

In this range, the horizontal line $y=k$ intersects the parabola $y=f(x)$ at one point, and thus $f(x)=k$ has only one root.

$\therefore 2<k\le 6$ or $k=-7$

7B.16 HKCEE AM 2003 – 17

Let $f(x)=-(x-a)^2+b$, where a and b are real. Point P is the vertex of the graph of $y=f(x)$.

(a) $P=(a,b)$

(b) (i) $g(x)=(x-b)^2+a$

Since $Q(b,a)$ is on the graph of $y=f(x)$,

$a=-(b-a)^2+b\Rightarrow (b-a)^2=b-a$

$\because g(a)=(a-b)^2+a$

$=(b-a)+a=b$

$\therefore (a,b)=P$ lies on $y=g(x)$.

(ii) $y=f(x)$ touches the x-axis $\Rightarrow b=0$

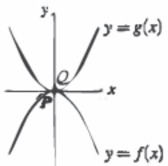
From (b)(i), $(b-a)^2=(b-a)=0$

$(b-a)(b-a-1)=0$

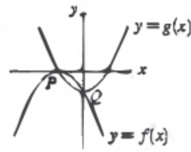
$\Rightarrow a=b$ or $a=b-1$

Thus, there are two cases:

Case 1: $a=b=0$



Case 2: $a=-1, b=0$



7B.17 HKDSE MA 2012 – I – 13

(a) $0=k(2)^3-21(2)^2+24(2)-4\Rightarrow k=5$

(b) $P=(m,0)\Rightarrow Q=(m,15m^2-63m+72)$

$\therefore$ Area of $OPQR=m(15m^2-63m+72)$

$=15m^3-63m^2+72m$

(c) $15m^3-63m^2+72m=12$

$3(5m^3-21m^2+24m-4)=0$

$(m-2)(5m^2-11m+2)=0$ (by (a))

$(m-2)(5m-1)(m-2)=0$

$m=2,\frac{1}{5}$ or -2 (rejected as P is in Quad I)

7B.18 HKDSE MA 2015 – I – 18

(a) $\Delta=(-4k)^2-4(2)(3k^2+5)=-8k^2-40$

$\le -40<0$

$\therefore$ It does not cut the x-axis.

(b) $f(x)=2x^2-4kx+3k^2+5$

$=2(x^2-2kx+k^2-k^2)+3k^2+5$

$=2(x-k)^2+k^2+5$

$\therefore$ Vertex $=(k,k^2+5)$

7B.19 HKDSE MA 2016 – I – 18

(a) $f(x)=-\frac{1}{3}(x^2-36x)-121$

$=-\frac{1}{3}(x^2-36x+18^2-18^2)-121$

$=-\frac{1}{3}(x-18)^2-13$

$\therefore$ Vertex $=(18,-13)$

7B.20 HKDSE MA 2017 – I – 18

(a) $\begin{cases}y=2x^2-2kx+2x-3k+8\\y=19\end{cases}$

$\Rightarrow 2x^2+2(1-k)x-(3k+11)=0$

$\Delta=4(1-k)^2+8(3k+11)$

$=4(k^2-2k+1+6k+22)$

$=4(k^2+4k+23)$

$=4(k+2)^2+76\ge 76>0$

$\therefore$ There are 2 distinct intersections.

(b) (i) $\begin{cases}a+b=-\frac{2(1-k)}{2}=k-1\\ab=\frac{-(3k+11)}{2}\end{cases}$

$(a-b)^2=(a+b)^2-4ab$

$=(k-1)^2+2(3k+11)=k^2+4k+23$

(ii) $(a-b)^2=(k+2)^2+19$

Minimum value of $(a-b)^2=19$

$\Rightarrow$ Minimum distance of $AB=\sqrt{19}>4$

$\therefore$ NO

7B.21 HKDSE MA 2018 – I – 18

(a) Let $f(x)=hx^2+kx$.

$\begin{cases}60=f(2)=4h+2k\\99=f(3)=9h+3k\end{cases}\Rightarrow\begin{cases}h=3\\k=24\end{cases}$

$\therefore f(x)=3x^2+24x$

(b) (i) $f(x)=3(x^2+8x)=3(x^2+8x+16-16)$

$=3(x+4)^2-48$

$\therefore Q=(-4,-48)$

7B.22 HKDSE MA 2020 – I – 7

7a Since the equation $p(x)=0$, i.e. $4x^2+12x+c=0$, has equal roots,

$\Delta=0$

$12^2-4(4)(c)=0$

$c=9$

b Put $y=0$,

$0=p(x)-169$

$4x^2+12x+9-169=0$

$x^2+3x-40=0$

$(x+8)(x-5)=0$

$x=-8$ or 5

Therefore, the x-intercepts of the graph of $y=p(x)-169$ are -8 and 5

7C Extreme values of quadratic functions

7C.1 HKCEE MA 1985(A/B) – I – 13

(a) $DE^2=BD^2+BE^2-2\cdot BD\cdot BE\cos\angle B$

$=(2-x)^2+x^2-2(2-x)(x)\cos 60°$

$=3x^2-6x+4$

(b) Area of $\triangle DEF=\frac{1}{2}DE\cdot DE\sin 60°$

$=\frac{1}{2}(3x^2-6x+4)\cdot\frac{\sqrt{3}}{2}$

$=\frac{\sqrt{3}}{4}(3x^2-6x+4)$

$=\frac{3\sqrt{3}}{4}\left(x^2-2x+\frac{4}{3}\right)$

$=\frac{3\sqrt{3}}{4}\left(x^2-2x+1+\frac{1}{3}\right)$

$=\frac{3\sqrt{3}}{4}(x-1)^2+\frac{\sqrt{3}}{4}$

$\therefore$ Minimum area is attained when $x=1$.

7C.2 HKCEE MA 1982(1/2) – I – 12

(a) Let $P=ax+bx^2$.

$\begin{cases}80000=20a+400b\Rightarrow a+20b=4000\\87500=35a+1225b\Rightarrow a+35b=2500\end{cases}$

$\Rightarrow\begin{cases}a=6000\\b=-100\end{cases}\Rightarrow P=6000x-100x^2$

Hence, when $x=15$, $P=5000(15)-100(15)^2=67500$.

(b) $P=-100(x^2-60x)=-100(x^2-60x+30^2-30^2)$

$=90000-(x-30)^2$

i.e. $a=90000$, $b=1$, $c=30$

(c) When P is maximum, $x=30$.

7C.3 HKCEE MA 1988 – I – 10

(a) Let $y=ax+bx^2$

$\begin{cases}-5=a+b\\-8=2a+4b\end{cases}\Rightarrow\begin{cases}a=-6\\b=1\end{cases}\Rightarrow y=x^2-6x$

Hence, when $x=6$, $y=(6)^2-6(6)=0$

(b) $y=x^2-6x+9-9=(x-3)^2-9$

$\therefore$ Least possible value of $y=-9$

7C.4 HKCEE MA 2011 – I – 12

(a) $\angle C=180°-\angle B=90°$ (int. $\angle$s, $AB//DC$)

$\angle DPC=180°-\angle APD-\angle APB$ (adj. $\angle$s on st. line)

$=90°-\angle APB$

$\angle PAB=180°-\angle B-\angle APB$ ($\angle$ sum of $\triangle$)

$=90°-\angle APB=\angle DPC$

In $\triangle ABP$ and $\triangle PCD$,

$\angle B=\angle C=90°$ (proved)

$\angle DPC=\angle PAB$ (proved)

$\angle PDC=\angle APB$ ($\angle$ sum of $\triangle$)

$\therefore\triangle ABP\sim\triangle PCD$ (AAA)

(b) $\frac{AB}{BP}=\frac{PC}{CD}$ (corr. sides, $\sim\triangle$s)

$\frac{3}{x}=\frac{11-x}{k}$

$3k=11x-x^2\Rightarrow x^2-11x+3k=0$

(c) $\Delta\ge 0\Rightarrow(-11)^2-4(3k)\ge 0\Rightarrow k\le\frac{121}{12}$

Hence, the greatest integral value of k is 10.

7B.23 HKDSE MA 2020 I – 17

17a $g(x)=x^2-2kx+2k^2+4$

$=x^2-2kx+\left(\frac{2k}{2}\right)^2+2k^2+4-\left(\frac{2k}{2}\right)^2$

$=(x-k)^2+k^2+4$

Therefore, the coordinates of the vertex of the graph of $y=f(x)$ are (k,k^2+4).

b Since the graph of $y=g(x+2)$ can be obtained by translating be graph of $y=g(x)$ leftwards by 2 units, we know that $D=(k-2,k^2+4)$.

Since the graph of $y=-g(x-2)$ can be obtained by translating the graph of $y=g(x)$ rightwards by 2 units followed by reflecting the resulting graph along the x-axis, we know that $E=(k+2,-(k^2+4))=(k+2,-k^2-4)$.

Let M be the mid-point of DE and O be the circumcentre of ΔDEF.

$M=\left(\frac{(k-2)+(k+2)}{2},\frac{(k^2+4)+(-k^2-4)}{2}\right)$

$=(k,0)$

Suppose there exists such a point F.

$OM \perp DE$ (circumcentre of ΔDEF)

The slope of OM × The slope of $DE=-1$

$\frac{0-3}{k-0}\times\frac{(k^2+4)-(-k^2-4)}{(k-2)-(k+2)}=-1$

$-6(k^2+4)=4k$

$3k^2+2k+12=0$

$\Delta=2^2-4(3)(12)$

$=-140$

<0

Hence, there is no real solution to k so contradiction arises.
Therefore, there is no such a point F.

7C.5 HKCEE AM 1986 – I – 3

$f(x)=-kx^2+18x+4k$

$=-k\left[x^2-\frac{18}{k}x+\left(\frac{9}{k}\right)^2-\left(\frac{9}{k}\right)^2\right]+4k$

$=-k\left(x-\frac{9}{k}\right)^2+\frac{81}{k}+4k$

$\therefore \frac{81}{k}+4k=45$

$4k^2-45k+81=0 \Rightarrow k=\frac{9}{4}$ or 9

7C.6 HKCEE AM 1996 – I – 4

(a) $x^2-6x+11=(x-3)^2+2$

$\therefore a=-3,\ b=2$

(b) $x^2-6x+11\ge 2 \Rightarrow \frac{1}{x^2-6x+11}\le\frac{1}{2}$

$\therefore 0<\frac{1}{x^2-6x+11}\le\frac{1}{2}$

7C.7 HKDSE MA 2013 – I – 17

(a) $f(x)=-x^2+36x=-(x^2-36x+18^2-18^2)$

$=-(x-18)^2+324$

$\therefore$ Vertex $=(18,324)$

(b) (i) $A=x\left(\frac{108-3x}{2}\right)=\frac{3}{2}(36x-x^2)$

(ii) Max value of $A=\frac{3}{2}(324)$ (by (a))

$=486<500$

$\therefore$ NO.

7D Solving equations using graphs of functions

7D.1 HKCEE MA 1980(3) – I – 16

(a) $30=25x-x^3 \Rightarrow \begin{cases}y=25x-x^3\\ y=30\end{cases}$

Add $y=30 \Rightarrow x=1.3$ or 4.2

(b) (i) $AC^2=b^2+b^2=2b^2$

$5^2=h^2+\left(\frac{AC}{2}\right)^2$

$25=h^2+\frac{1}{2}b^2 \Rightarrow b=\sqrt{50-2h^2}$

$V=\frac{1}{3}b^2h=\frac{1}{3}(50-2h^2)h$

$=\frac{2}{3}(25h-h^3)$

(ii) $20=\frac{2}{3}(25h-h^3) \Rightarrow 30=25h-h^3$

From (a), $h=1.3$ or 4.2.

7D.2 HKCEE MA 1981(1) – I – 11

(a) One side $=x$ cm

The other side $=\frac{20-2x}{2}=10-x$ (cm)

$\therefore y=x(10-x)=10x-x^2$

(b) (i) $y=18.4$

(ii) Add $y=12 \Rightarrow x=1.4$ or 8.6

(iii) Greatest area $=y$-coordinate of vertex $=25$

7D.3 HKCEE MA 1983(A) – I – 14

(a) $V=x(7-2x)^2=4x^3-28x^2+49x$

(b) $4x^3-28x^2+49x=20 \Rightarrow \begin{cases}y=4x^3-28x^2+49x\\ y=20\end{cases}$

Add $y=20 \Rightarrow x=0.6$, 1.9 or 4.5

(c) $k=0.6$ or 1.9 or 4.5 (rejected)

7D.4 HKCEE MA 1985(A) – I – 12

(a) (i) $x^3+x-1=0 \Rightarrow \begin{cases}y=x^3+x\\ y=1\end{cases}$

Add $y=1 \Rightarrow x=0.7$

(b) (i) $(x+1)^4-(x-1)^4$

$=[(x+1)^2+(x-1)^2][(x+1)^2-(x-1)^2]$

$(2x^2+2)(4x)=8x^3+8x$

(ii) $8x^3+8x=8 \Rightarrow x^3+x-1=0$

By (a)(ii), $x=0.69$.

7D.5 HKCEE MA 1985(B) – I – 12

(a) Since ΔABC and thus ΔBPQ are right-angled isosceles,

$QR=(16-2x)$ cm.

$\therefore$ Area of $PQRS=x(16-2x)=2(8x-x^2)$ (cm^2)

(b) (i) The greatest area is attained when $x=4$.

(ii) $28=2(8x-x^2)$

$14=8x-x^2 \Rightarrow \begin{cases}y=8x-x^2\\ y=14\end{cases}$

Add $y=14 \Rightarrow x=2.6$ or 5.4.

7D.6 HKCEE MA 1986(B) – I – 14

(a) $c = y$-intercept $= 6$

Roots $= \ 2$ and $3 \Rightarrow \begin{cases} \frac{c}{a} = (-2)(3) \Rightarrow a = \ 1 \\ -\frac{b}{a} = (-2) + (3) \Rightarrow b = 1 \end{cases}$

(b) (i) $(x+2)(x-3) = -1 \Rightarrow \begin{cases} y = \ x^2 + x + 6 \\ y = -1 \end{cases}$

Add $y = \ 1 \Rightarrow x = \ 2.2$ or 3.2

7D.7 HKCEE MA 1987(A) – I – 14

(a) (i) $x^3 - 6x^2 + 9x - 1 = 0 \Rightarrow \begin{cases} y = x^3 - 6x^2 + 9x \\ y = 1 \end{cases}$

Add $y = 1 \Rightarrow x = 0.1, 2.3$ or 3.5

(c) $\begin{cases} y = x^3 \ \ 6x^2 + 9x \\ y = k \end{cases}$

To have 3 intersections, $0 < k < 4$.

7D.8 HKCEE MA 1997 – I – 13

(a) (i) 10

(ii) $1.8 < x \le 16 \Rightarrow 2 \le x \le 16$

(b) (i) Put $x = 3$ and $H = 144$: $144 = 3^2 - 51 + c$

$c = 186$

(iii) Total cost $= 10 \times \$20 + 6 \times 120 = \$520$

Total proceeds

$= 6 \times \$100 + 4 \times \$300 + 4 \times \$10 + 2 \times \$60$

$= \$1960$

$\therefore$ Gain $= 1960 - 520 = (\$)1440$

7D.9 HKCEE MA 2000 – I – 18

(a) Let $V = ah^2 + bh^3$.

$\begin{cases} \frac{29\pi}{3} = a + b \\ 81\pi = 9a + 27b \end{cases} \Rightarrow \begin{cases} a = 10\pi \\ b = -\frac{\pi}{3} \end{cases}$

$\therefore V = 10h^2 - \frac{\pi}{3}h^3$

(b) (i) Surface area = Surface area of original hemisphere

$= 2\pi(10)^2 = 200\pi$ (cm²)

(ii) $\frac{1}{2} \cdot \frac{4}{3}\pi(10)^3 - 2\left(10h^2 - \frac{\pi}{3}h^3\right) = \frac{1400}{3}\pi$

$\frac{2000}{3}\pi - 20h^2 + \frac{2\pi}{3}h^3 = \frac{1400}{3}\pi$

$h^3 - 30h^2 + 300 = 0$

(iii) $\begin{cases} y = x^3 - 30x^2 \\ y = -300 \end{cases}$

Add $y = -300$ to the graph $\Rightarrow h = 3.35$

7E Transformation of graphs of functions

7E.1 HKCEE MA 2010 – I – 16

(a) (i) $f(x) = \frac{-1}{144}(x^2 - 72x) - 6$

$= \frac{-1}{144}(x^2 - 72x + 36^2 - 36^2) - 6$

$= \frac{-1}{144}(x - 36)^2 + 3$

$\therefore$ Vertex $= (36, 3)$

(ii) $g(x) = f(x+4) + 5 = \frac{-1}{144}(x - 32)^2 + 8$

(iii) $h(x) = 2^{f(x+4)} + 5 = 2^{\frac{-1}{144}(x-32)^2 + 3} + 5$

(b) (i) When $u = 8$, $8 = 2^{f(s)}$

$3 = f(s) = \frac{-1}{144}(s - 36)^2 + 3$

$s = 36$

$\therefore$ The temperature is 36°C.

(ii) From the table, $\begin{cases} t = s - 4 \\ v = u + 5 \end{cases}$.

Hence, $u = 2^{f(s)}$ becomes: $v - 5 = 2^{f(t+4)}$

$\Rightarrow v = 2^{f(t+4)} + 5 = 2^{\frac{-1}{144}(t-32)^2 + 3} + 5$

7E.2 HKDSE MA 2015 – I – 18

(a) $\Delta = (-4k)^2 - 4(2)(3k^2 + 5) = -8k^2 - 40$

$\le -40 < 0$

$\because$ It does not cut the x-axis.

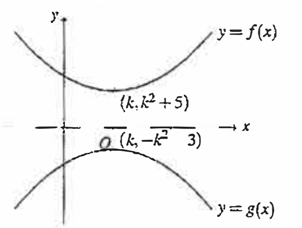
(b) $f(x) = 2x^2 \ \ 4kx + 3k^2 + 5$

$= 2(x^2 - 2kx + k^2 - k^2) + 3k^2 + 5$

$= 2(x - k)^2 + k^2 + 5$

$\therefore$ Vertex $= (k, k^2 + 5)$

(c)



S and T are nearest to each other when they are the vertices of the two parabolas respectively. Since $OS \ne OT$, ΔOST is not isosceles, and thus the x-axis is not the $\perp$ bisector of ST. NOT correct.

7E.3 HKDSE MA 2016 – I – 18

(a) $f(x) = -\frac{1}{3}(x^2 - 36x) - 121$

$= -\frac{1}{3}(x^2 - 36x + 18^2 - 18^2) - 121$

$= \ \frac{1}{3}(x \ \ 18)^2 \ \ 13$

$\therefore$ Vertex $= (18, -13)$

(b) $g(x) = f(x) + 13 = -\frac{1}{3}(x - 18)^2$

(c) $-\frac{1}{3}x^2 - 12x - 121 = f(-x)$

Hence, the transformation is a reflection in the y axis

7E.4 HKDSE MA 2018 – I – 18

(a) Let $f(x) = hx^2 + kx$.

$\begin{cases} 60 = f(2) = 4h + 2k \\ 99 = f(3) = 9h + 3k \end{cases} \Rightarrow \begin{cases} h = 3 \\ k = 24 \end{cases}$

$\therefore f(x) = 3x^2 + 24x$

(b) (i) $f(x) = 3(x^2 + 8x) = 3(x^2 + 8x + 16 - 16)$

$= 3(x + 4)^2 - 48$

$\therefore Q = (-4, -48)$

(ii) $R = (-4, 75)$

(iii) $QR = 75 - (-48) = 123$

$SQ = \sqrt{60^2 + 48^2} = \sqrt{5904}$

$RS = \sqrt{60^2 + 75^2} = \sqrt{9225}$

Hence, $QR^2 = SQ^2 + RS^2$. ΔQRS is right-$\angle$ed at S.

(converse of Pyth. thm)

$\therefore P$ is the mid-point of QR.

7E.5 HKDSE MA 2019 – I – 19

(a) $f(4) = \frac{1}{1+k}\left((4)^2 + (6k - 2)(4) + (9k + 25)\right)$

$= \frac{1}{1+k}(33 + 33k) = 33$

Hence, the graph passes through F.

(b) (i) $g(x) = f(-x) + 4$

$= \frac{1}{1+k}\left((-x)^2 + (6k - 2)(-x) + (9k + 25)\right) + 4$

$= \frac{1}{1+k}\left(x^2 - (6k - 2)x + (3k - 1)^2 - (3k - 1)^2 + (9k + 25)\right) + 4$

$= \frac{1}{1+k}\left((x - 3k + 1)^2 - 9k^2 + 3k + 24\right) + 4$

$= \frac{1}{1+k}\left((x - 3k + 1)^2 - 3(1 + k)(3k - 8)\right) + 4$

$= \frac{1}{1+k}(x - 3k + 1)^2 - 3(3k - 8) + 4$

$= \frac{1}{1+k}(x - 3k + 1)^2 + 28 - 9k$

$\therefore U = (3k - 1, 28 - 9k)$

8 Rate, Ratio and Variation

8A Rate and Ratio

8A.1 HKCEE MA 1980(1) I 8

A factory employs 10 skilled, 20 semi skilled, and 30 unskilled workers. The daily wages per worker of the three kinds are in the ratio $4:3:2$. If a skilled worker is paid \$120 a day, find the mean daily wage for the 60 workers.

8A.2 HKCEE MA 1981(1/2/3) I 9

Normally, a factory produces 400 radios in x days. If the factory were to produce 20 more radios each day, then it would take 10 days less to produce 400 radios. Calculate x.

8A.3 HKCEE MA 1983(A/B) – I 4

If $a:b=3:4$ and $a:c=2:5$, find

(a) $a:b:c$,

(b) the value of $\frac{ac}{a^2+b^2}$.

8A.4 HKCEE MA 1989 – I – 1

The monthly income of a man is increased from \$8000 to \$9000.

(a) Find the percentage increase.

(b) After the increase, the ratio of his savings to his expenditure is $3:7$ for each month. How much does he save each month?

8A.5 HKCEE MA 1989 – I 5

(a) Solve the simultaneous equations $\begin{cases} x+2y=5 \\ 5x-4y=4 \end{cases}$.

(b) Given that $\begin{cases} \frac{a}{c}+\frac{2b}{c}=5 \\ \frac{5a}{c}-\frac{4b}{c}=4 \end{cases}$, where a, b and c are non zero numbers, using the result of (a), find $a:b:c$.

8A.6 HKCEE MA 1991 I – 3

(Also as 2C.2.)

A man buys some British pounds (£) with 150 000 Hong Kong dollars (HK\$) at the rate £1 = HK\$15.00 and puts it on fixed deposit for 30 days. The rate of interest is 14.60% per annum.

(a) How much does he buy in British pounds?

(b) Find the amount in British pounds at the end of 30 days.
(Suppose 1 year = 365 days and the interest is calculated at simple interest.)

(c) If he sells the amount in (b) at the rate of £1 = HK\$14.50, how much does he get in Hong Kong dollars?

8A.7 HKCEE MA 1991 I – 4

Let $2a=3b=5c$.

(a) Find the ratio $a:b:c$.

(b) If $a-b+c=55$, find c.

8A.8 HKCEE MA 1995 – I 5

It is given that $x:(y+1)=4:5$.

(a) Express x in terms of y.

(b) If $2x+9y=97$, find the values of x and y.

8A.9 HKCEE MA 2005 – I 5

The ratio of the number of marbles owned by Susan to the number of marbles owned by Teresa is $5:2$. Susan has n marbles. If Susan gives 18 of her own marbles to Teresa, both of them will have the same number of marbles. Find n.

8A.10 HKCEE MA 2011 – I 6

In a summer camp, the ratio of the number of boys to the number of girls is $7:6$. If 17 boys and 4 girls leave the summer camp, then the number of boys and the number of girls are the same. Find the original number of girls in the summer camp.

8A.11 HKDSE MA PP I – 5

The ratio of the capacity of a bottle to that of a cup is $4:3$. The total capacity of 7 bottles and 9 cups is 11 litres. Find the capacity of a bottle.

8A.12 HKDSE MA 2018 I 9

A car travels from city P to city Q at an average speed of 72 km/h and then the car travels from city Q to city R at an average speed of 90 km/h. It is given that the car travels 210 km in 161 minutes for the whole journey. How long does the car take to travel from city P to city Q?

8A.13 HKDSE MA 2019 – I 7

In a playground, the ratio of the number of adults to the number of children is $13:6$. If 9 adults and 24 children enter the playground, then the ratio of the number of adults to the number of children is $8:7$. Find the original number of adults in the playground.

8A.14 HKDSE MA 2020 I 4

Let a, b and c be non-zero numbers such that $\frac{a}{b}=\frac{6}{7}$ and $3a=4c$. Find $\frac{b+2c}{a+2b}$.

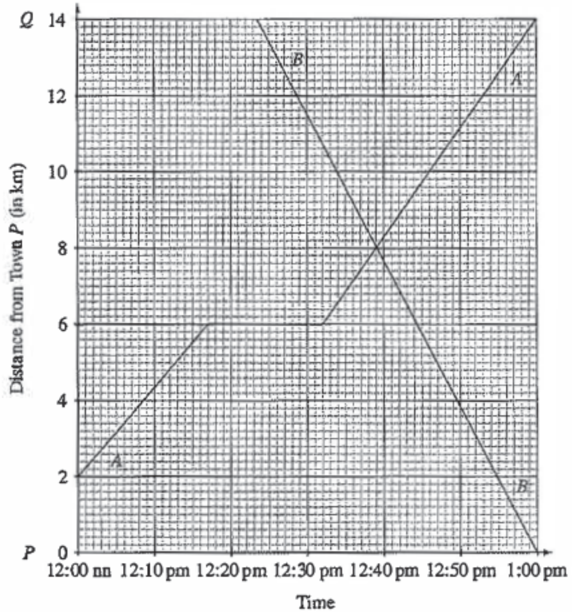
8B Travel graphs

8B.1 HKCEE MA 1984(B) – I – 3

The figure shows the travel graphs of two cyclists A and B travelling on the same road between towns P and Q, 14 km apart.

(a) For how many minutes does A rest during the journey?

(b) How many km away from P do A and B meet?



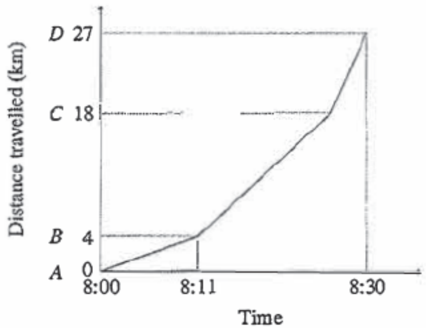
8B.2 HKDSE MA SP – I – 12

The figure shows the graph for John driving from town A to town D (via town B and town C) in a morning. The journey is divided into three parts: Part I (from A to B), Part II (from B to C) and Part III (from C to D).

(a) For which part of the journey is the average speed the lowest? Explain your answer.

(b) If the average speed for Part II of the journey is 56 km/h, when is John at C?

(c) Find the average speed for John driving from A to D in m/s.



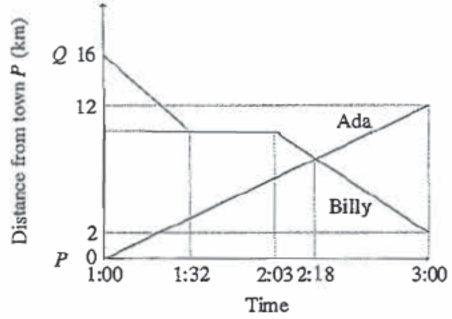
8B.3 HKDSE MA PP – I – 12

The figure shows the graphs for Ada and Billy running on the same straight road between town P and town Q during the period 1:00 to 3:00 in an afternoon. Ada runs at a constant speed. It is given that town P and town Q are 16 km apart.

(a) How long does Billy rest during the period?

(b) How far from town P do Ada and Billy meet during the period?

(c) Use average speed during the period to determine who runs faster. Explain your answer.



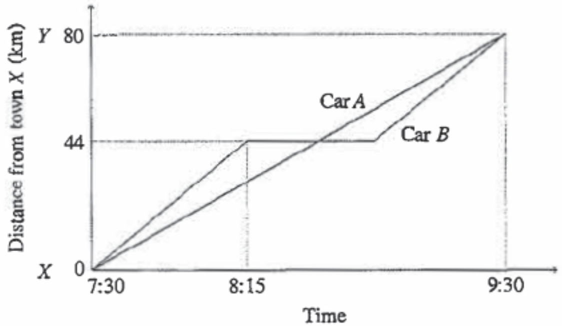
8B.4 HKDSE MA 2014 – I – 10

Town X and Town Y are 80 km apart. The figure shows the graphs for car A and car B travelling on the same straight road between town X and town Y during the period 7:30 to 9:30 in a morning. Car A travels at a constant speed during the period Car B comes to rest at 8:15 in the morning.

(a) Find the distance of car A from town X at 8:15 in the morning.

(b) At what time after 7:30 in the morning do car A and car B first meet?

(c) The driver of car B claims that the average speed of car B is higher than that of car A during the period 8:15 to 9:30 in the morning. Do you agree? Explain your answer.



8C Variation

8C.1 HKCEE MA 1982(1/2) – I 12

(To continue as 7C.2.)

The price of a certain monthly magazine is x dollars per copy. The total profit on the sale of the magazine is P dollars. It is given that $P=Y+Z$, where Y varies directly as x and Z varies directly as the square of x. When x is 20, P is 80 000; when x is 35, P is 87 500.

(a) Find P when $x=15$.

8C.2 HKCEE MA 1984(B) I 14

A school and a youth centre agree to share the total expenditure for a camp in the ratio 3 : 1. The total expenditure \$$E$ for the camp is the sum of two parts: one part is a constant \$$C$, and the other part varies directly as the number of participants N. If there are 300 participants, the school has to pay \$7500. If there are 500 participants, the school has to pay \$12000.

(a) Find the total expenditure for the camp, when the school has to pay \$7500.

(b) Find the value of C.

(c) Express E in terms of N.

(d) If the youth centre has to pay \$4750, find the number of participants.

8C.3 HKCEE MA 1986(B) I 5

It is given that z varies directly as x^2 and inversely as y. If $x=1$ and $y=2$, then $z=3$. Find z when $x=2$ and $y=3$.

8C.4 HKCEE MA 1987(B) I – 14

(To continue as 10C.3.)

Given $p=y+z$, where y varies directly as x, z varies inversely as x and x is positive. When $x=2$, $p=7$; when $x=3$, $p=8$.

(a) Find p when $x=4$.

8C.5 HKCEE MA 1988 – I – 10

(To continue as 7C.3.)

A variable quantity y is the sum of two parts. The first part varies directly as another variable x, while the second part varies directly as x^2. When $x=1$, $y=-5$; when $x=2$, $y=-8$.

(a) Express y in terms of x. Hence find the value of y when $x=6$.

8C.6 HKCEE MA 1991 – I 2

In a joint variation, x varies directly as y^2 and inversely as z. Given that $x=18$ when $y=3$, $z=2$,

(a) express x in terms of y and z,

(b) find x when $y=1$, $z=4$.

8C.7 HKCEE MA 1994 I 4

Suppose x varies directly as y^2 and inversely as z. When $y=3$ and $z=10$, $x=54$.

(a) Express x in terms of y and z.

(b) Find x when $y=5$ and $z=12$.

8C.8 HKCEE MA 1997 – I – 7

(Continued from 15C.5.)

The ratio of the volumes of two similar solid circular cones is 8 : 27.

(a) Find the ratio of the height of the smaller cone to the height of the larger cone.

(b) If the cost of painting a cone varies as its total surface area and the cost of painting the smaller cone is \$32, find the cost of painting the larger cone.

8C.9 HKCEE MA 1998 – I 12

The monthly service charge \$$S$ of mobile phone network A is partly constant and partly varies directly as the connection time t minutes. The monthly service charges are \$230 and \$284 when the connection times are 100 minutes and 130 minutes respectively.

(a) Express S in terms of t.

(b) The service charge of mobile phone network B only varies directly as the connection time. The charge is \$2.20 per minute. A man uses about 110 minutes connection time every month. Should he join network A or B in order to save money? Explain your answer.

8C.10 HKCEE MA 1999 I – 6

y varies partly as x and partly as x^2. When $x=2$, $y=20$ and when $x=3$, $y=39$. Express y in terms of x.

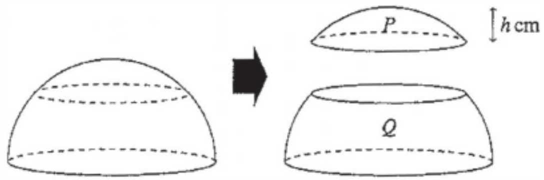
8C.11 HKCEE MA 2000 – I – 18

(To continue as 7D. 9.)

The figure shows a solid hemisphere of radius 10 cm. It is cut into two portions, P and Q, along a plane parallel to its base. The height and volume of P are h cm and V cm^3 respectively.

It is known that V is the sum of two parts. One part varies directly as h^2 and the other part varies directly as h^3. $V=\frac{29}{3}\pi$ when $h=1$ and $V=81\pi$ when $h=3$.

(a) Find V in terms of h and π.



8C.12 HKCEE MA 2001 – I 13

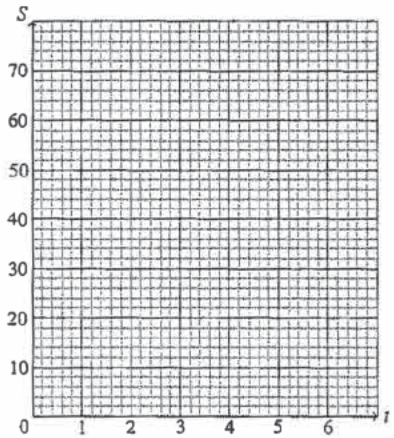
S is the sum of two parts. One part varies as t and the other part varies as the square of t. The table below shows certain pairs of the values of S and t.

S	0	33	56	69	72	65	48	21
t	0	1	2	3	4	5	6	7

(a) Express S in terms of t.

(b) Find the value(s) of t when $S=40$.

(c) Using the data given in the table, plot the graph of S against t for $0\le t\le 7$ in the following figure. Read from the graph the value of t when the value of S is greatest.



8C.13 HKCEE MA 2002 – I – 11 (To continue as **15C.8.**)

The area of a paper bookmark is A cm^2 and its perimeter is P cm. A is a function of P. It is known that A is the sum of two parts, one part varies as P and the other part varies as the square of P. When $P = 24$, $A = 36$ and when $P = 18$, $A = 9$.

(a) Express A in terms of P.

(b) (i) The best-selling paper bookmark has an area of $54\,\text{cm}^2$. Find the perimeter of this bookmark.

8C.14 HKCEE MA 2003 – I – 10 (To continue as **10C.5.**)

The speed of a solar-powered toy can is V cm/s and the length of its solar panel is L cm, where $5 \le L \le 25$. V is a function of L. It is known that V is the sum of two parts, one part varies as L and the other part varies as the square of L. When $L = 10$, $V = 30$ and when $L = 15$, $V = 75$.

(a) Express V in terms of L.

8C.15 HKCEE MA 2004 – I – 10 (To continue as **10C.6.**)

It is known that y is the sum of two parts, one part varies as x and the other part varies as the square of x. When $x = 3$, $y = 3$ and when $x = 4$, $y = 12$.

(a) Express y in terms of x.

8C.16 HKCEE MA 2005 – I – 10 (To continue as **4B.18.**)

It is known that $f(x)$ is the sum of two parts, one part varies as x^3 and the other part varies as x.
Suppose $f(2) = -6$ and $f(3) = 6$.

(a) Find $f(x)$.

8C.17 HKCEE MA 2006 – I – 15

The cost of a souvenir of surface area A cm^2 is \$$C$. It is given that C is the sum of two parts, one part varies directly as A while the other part varies directly as A^2 and inversely as n, where n is the number of souvenirs produced. When $A = 50$ and $n = 500$, $C = 350$; when $A = 20$ and $n = 400$, $C = 100$.

(a) Express C in terms of A and n.

(b) The selling price of a souvenir of surface area A cm^2 is \$$8A$ and the profit in selling the souvenir is \$$P$.

(i) Express P in terms of A and n.

(ii) Suppose $P : n = 5 : 32$. Find $A : n$.

(iii) Suppose $n = 500$. Can a profit of \$100 be made in selling a souvenir? Explain your answer.

(iv) Suppose $n = 400$. Using the method of completing the square, find the greatest profit in selling a souvenir.

8C.18 HKCEE MA 2007 – I – 14 (Continued from **4B.19.**)

(a) Let $f(x) = 4x^3 + kx^2 - 243$, where k is a constant. It is given that $x+3$ is a factor of $f(x)$.

(i) Find the value of k.

(ii) Factorize $f(x)$.

(b) Let \$$C$ be the cost of making a cubical handicraft with a side of length x cm. It is given that C is the sum of two parts, one part varies as x^3 and the other part varies as x^2. When $x = 5.5$, $C = 7381$ and when $x = 6$, $C = 9072$.

(i) Express C in terms of x.

(ii) If the cost of making a cubical handicraft is \$972, find the length of a side of the handicraft.

8C.19 HKCEE MA 2010 – I – 10

The cost of a tablecloth of perimeter x metres is \$$C$. It is given that C is the sum of two parts, one part varies as x and the other part varies as x^2. When $x = 4$, $C = 96$ and when $x = 5$, $C = 145$.

(a) Express C in terms of x.

(b) If the cost of a tablecloth is \$288, find its perimeter.

8C.20 HKCEE MA 2011 – I – 11 (To continue as **7B.9.**)

It is given that $f(x)$ is the sum of two parts, one part varies as x^2 and the other part varies as x. Suppose that $f(-2) = 28$ and $f(6) = -36$.

(a) Find $f(x)$.

8C.21 HKDSE MA SP – I – 11

In a factory, the production cost of a carpet of perimeter s metres is \$$C$. It is given that C is a sum of two parts, one part varies as s and the other part varies as the square of s. When $s = 2$, $C = 356$; when $s = 5$, $C = 1250$.

(a) Find the production cost of a carpet of perimeter 6 metres.

(b) If the production cost of a carpet is \$539, find the perimeter of the carpet.

8C.22 HKDSE MA PP – I – 11

Let \$$C$ be the cost of manufacturing a cubical carton of side x cm. It is given that C is partly constant and partly varies as the square of x. When $x = 20$, $C = 42$; when $x = 120$, $C = 112$.

(a) Find the cost of manufacturing a cubical carton of side 50 cm.

(b) If the cost of manufacturing a cubical carton is \$58, find the length of a side of the carton.

8C.23 HKDSE MA 2012 – I – 11 (To continue as **15C.14.**)

Let \$$C$ be the cost of painting a can of surface area A m^2. It is given that C is the sum of two parts, one part is a constant and the other part varies as A. When $A = 2$, $C = 62$; when $A = 6$, $C = 74$.

(a) Find the cost of painting a can of surface area 13 m^2.

8C.24 HKDSE MA 2013 – I – 11

The weight of a tray of perimeter ℓ metres is W grams. It is given that W is the sum of two parts, one part varies directly as ℓ and the other part varies directly as ℓ^2. When $\ell = 1$, $W = 181$ and when $\ell = 2$, $W = 402$.

(a) Find the weight of a tray of perimeter 1.2 metres.

(b) If the weight of a tray is 594 grams, find the perimeter of the tray.

8C.25 HKDSE MA 2014 – I – 13

It is given that $f(x)$ is the sum of two parts, one part varies as x^2 and the other part is a constant. Suppose that $f(2) = 59$ and $f(7) = -121$.

(a) Find $f(6)$.

(b) $A(6, a)$ and $B(-6, b)$ are points lying on the graph of $y = f(x)$. Find the area of $\triangle ABC$, where C is a point lying on the x axis.

8C.26 HKDSE MA 2015 – I – 10

When Susan sells n handbags in a month, her income in that month is $\$S$. It is given that S is a sum of two parts: one part is a constant and the other part varies as n. When $n = 10$, $S = 10600$; when $n = 6$, $S = 9000$.

(a) When Susan sells 20 handbags in a month, find her income in that month.

(b) Is it possible that when Susan sells a certain number of handbags in a month, her income in that month is \$18000? Explain your answer.

8C.27 HKDSE MA 2016 – I – 8

It is given that $f(x)$ is the sum of two parts, one part varies as x and the other part varies as x^2. Suppose that $f(3) = 48$ and $f(9) = 198$.

(a) Find $f(x)$.

(b) Solve the equation $f(x) = 90$.

8C.28 HKDSE MA 2017 – I – 8

It is given that y varies inversely as $\sqrt{x}$. When $x = 144$, $y = 81$.

(a) Express y in terms of x.

(b) If the value of x is increased from 144 to 324, find the change in the value of y.

8C.29 HKDSE MA 2018 – I – 18 (To continue as 7B.21.)

It is given that $f(x)$ partly varies as x^2 and partly varies as x. Suppose that $f(2) = 60$ and $f(3) = 99$.

(a) Find $f(x)$.

8C.30 HKDSE MA 2019 – I – 10

It is given that $h(x)$ is partly constant and partly varies as x. Suppose that $h(-2) = -96$ and $h(5) = 72$.

(a) Find $h(x)$.

(b) Solve the equation $h(x) = 3x^2$

8C.31 HKDSE MA 2020 – I – 10

The price of a brand X souvenir of height h cm is $\$P$. P is partly constant and partly varies as h^3. When $h = 3$, $P = 59$ and when $h = 7$, $P = 691$.

(a) Find the price of a brand X souvenir of height 4 cm. (4 marks)

(b) Someone claims that the price of a brand X souvenir of height 5 cm is higher than the total price of two brand X souvenirs of height 4 cm. Is the claim correct? Explain your answer. (2 marks)

8 Rate, Ratio and Variation

8A Rate and Ratio

8A.1 HKCEE MA 1980(1) – I – 8

Daily wage of a skilled worker $= \$120$

Daily wage of a semi-skilled worker $= \$120 \times \frac{3}{4} = \$90$

Daily wage of a unskilled worker $= \$120 \times \frac{2}{4} = \$60$

$\therefore$ Mean daily wage $= \frac{10 \times \$120 + 20 \times \$90 + 30 \times \$60}{10 + 20 + 30}$
$= \$80$

8A.2 HKCEE MA 1981(1/2/3) – I – 9

Original rate $= \frac{400}{x}$ radios/day

New rate $= \left(\frac{400}{x} + 20\right)$ radios/day

$\therefore \left(\frac{400}{x} + 20\right)(x \quad 10) = 400$
$(20 + x)(x \quad 10) = 20x$
$x^2 - 10x - 200 = 0 \Rightarrow x = 50$ or -40 (rejected)

8A.3 HKCEE MA 1983(A/B) – I – 4

(a) $\begin{cases} a:b \quad = 3:4 \quad = 6:8 \\ a: \quad c = 2: \quad 5 = 6: \quad 15 \end{cases} \Rightarrow a:b:c = 6:8:15$

(b) $\frac{ac}{a^2+b^2} = \frac{ac \times \frac{1}{a^2}}{(a^2+b^2) \times \frac{1}{a^2}} = \frac{\frac{c}{a}}{1+(\frac{b}{a})^2} = \frac{\frac{5}{2}}{1+(\frac{4}{3})^2} = \frac{9}{10}$

8A.4 HKCEE MA 1989 – I – 1

(a) % increase $= \frac{9000 - 8000}{8000} \times 100\% = 12.5\%$

(b) Amount saved $= \$9000 \times \frac{3}{3+7} = \$2700$

8A.5 HKCEE MA 1989 – I – 5

(a) $2(1) + (2) \Rightarrow 7x = 14 \Rightarrow x = 2 \Rightarrow y = \frac{3}{2}$

(b) From (a), $\frac{a}{c} = 2,\ \frac{b}{c} = \frac{3}{2}$.
i.e. $\begin{cases} a:c = 2:1 = 4:2 \\ b:c = 3:2 \end{cases} \Rightarrow a:b:c = 4:3:2$

8A.6 HKCEE MA 1991 – I – 3

(a) $£150000 \div 15 = £10000$

(b) Amount $= 10000 + 10000 \times 14.60\% \times \frac{30}{365} = (£)10120$

(c) $\$10120 \times 14.50 = \$146740$

8A.7 HKCEE MA 1991 – I – 4

(a) $2a = 3b \Rightarrow a:b \quad = 3:2$
$3b = 5c \Rightarrow \quad b:c = \quad 5:3$
$\therefore a:b:c = 15:10:6$

(b) Let $a = 15k, b = 10k, c = 6k$.
$a - b + c = 55$
$15k - 10k + 6k = 55 \Rightarrow k = 5$
$\therefore c = 6k = 30$

8A.8 HKCEE MA 1995 – I – 5

(a) $\frac{x}{y+1} = \frac{4}{5} \Rightarrow 5x = 4(y+1) \Rightarrow x = \frac{4}{5}(y+1)$

(b) $2x + 9y = 97$
$2 \cdot \frac{4}{5}(y+1) + 9y = 97 \Rightarrow \frac{53}{5}y = \frac{477}{5} \Rightarrow y = 9$
$\therefore x = \frac{4}{5}(9+1) = 8$

8A.9 HKCEE MA 2005 – I – 5

Teresa has $\frac{2}{5}n$ marbles.

$n - 18 = \frac{2}{5}n + 18 \Rightarrow \frac{3}{5}n = 36 \Rightarrow n = 60$

8A.10 HKCEE MA 2011 – I – 6

Let there be x girls and $\frac{7}{6}x$ boys originally.

$\frac{7}{6}x - 17 = x - 4 \Rightarrow x = 78$

$\therefore$ There were 78 girls originally.

8A.11 HKDSE MA PP – I – 5

Let the capacity of a bottle and a cup be x litres and $\frac{3}{4}x$ litres respectively.

$7x + 9\left(\frac{3}{4}x\right) = 11 \Rightarrow \frac{55}{4}x = 11 \Rightarrow x = 0.8$

$\therefore$ The capacity of a bottle is 0.8 litres.

8A.12 HKDSE MA 2018 – I – 9

Let x mins be the time taken from P to Q. Then the car took $(161 - x)$ mins from Q to R.

$72 \times \left(\frac{x}{60}\right) + 90 \times \left(\frac{161-x}{60}\right) = 210$
$\frac{483}{2} \quad \frac{3}{10}x = 210 \Rightarrow x = 105$

$\therefore$ The car takes 105 ~~mins from P to Q~~.

8A.13 HKDSE MA 2019 – I – 7

Let the original numbers of adults and children be $13k$ and $6k$ respectively.

$\frac{13k+9}{6k+24} = \frac{8}{7} \Rightarrow 91k - 48k = 192 - 63 \Rightarrow k = 3$

$\therefore$ Original number of adults was $13(3) = 39$.

8A.14 HKDSE MA 2020 – I – 4

$\frac{a}{b} = \frac{6}{7}$

$b = \frac{7}{6}a$

$3a = 4c$

$c = \frac{3}{4}a \quad \rightarrow \quad \frac{b+2c}{a+2b} = \frac{\frac{7}{6}a + 2\left(\frac{3}{4}a\right)}{a + 2\left(\frac{7}{6}a\right)}$

$= \frac{4}{5}$

8B Travel graphs

8B.1 HKCEE MA 1984(B) – I – 3

(a) Rested from 12:17 p.m. to 12:32 p.m. $\Rightarrow$ 15 min

(b) 8 km

8B.2 HKDSE MA SP – I – 12

(a) Part I since the slope of the graph is the smallest.

(b) Time for Part II $= (18 - 4) \div 56 = \frac{1}{4}$ (hours)
$\therefore$ The time at C is 8:26.

(c) Average speed $= \frac{27 \times 1000\text{ m}}{30 \times 60\text{ s}} = 15$ m/s

8B.3 HKDSE MA PP – I – 12

(a) Billy rested from 1:32 to 2:03 $\Rightarrow$ 31 min

(b) They meet at 2: 18.
$\because$ Speed of Ada $= \frac{12}{2} = 6$ (km/h)
$\therefore$ Dist. from P when they meet $= 6 \times \frac{60 + 18}{60} = 7.8$ (km)

(c) Average speed of Billy $= (16 - 2) \div 2 = 7$ (km/h)
> 6 km/h
$\therefore$ Billy runs faster.

8B.4 HKDSE MA 2014 – I 10

(a) Speed of $A = \frac{80}{2} = 40$ (km/h)
$\therefore$ Dist. from X at 8:15 $= 40 \times \frac{45}{60} = 30$ (km)

(b) They meet when A is 44 km from X.
Time taken by $A = \frac{44}{40} = 1.1$ (hour) $= 1$ hr 6 mins
$\therefore$ The time is 8:36.

(c) Dist. travelled by $B = 80 \quad 44 = 36$ (km)
Dist. travelled by $A = 80 - 30 = 50$ (km)
$\therefore$ A has a higher speed as the time taken is the same.
$\therefore$ NO

8C Variation

8C.1 HKCEE MA 1982(1/2) – I – 12

(a) Let $P = ax + bx^2$.
$\begin{cases} 80000 = 20a + 400b \Rightarrow a + 20b = 4000 \\ 87500 = 35a + 1225b \Rightarrow a + 35b = 2500 \end{cases}$
$\Rightarrow \begin{cases} a = 6000 \\ b = -100 \end{cases} \Rightarrow P = 6000x - 100x^2$
Hence, when $x = 15$, $P = 5000(15) - 100(15)^2 = 67500$.

8C.2 HKCEE MA 1984(B) – I – 14

(a) Total expenditure $= \$7500 \div \frac{3}{4} = \$10000$

(b) Let $E = C + kN$.
$\begin{cases} 7500 \div \frac{3}{4} = C + k(300) \Rightarrow C + 300k = 10000 \\ 12000 \div \frac{3}{4} = C + k(500) \Rightarrow C + 500k = 16000 \end{cases}$
$\Rightarrow \begin{cases} c = 1000 \\ k = 30 \end{cases} \Rightarrow E = 1000 + 30N$
i.e. $C = 1000$

(c) $E = 1000 + 30N$

(d) $4750 \div \frac{1}{4} = 1000 + 30N \Rightarrow N = 60$
$\therefore$ The number of participants is 60.

8C.3 HKCEE MA 1986(B) – I – 5

Let $z = \frac{kx^2}{y}$. Then $(3) \quad \frac{k(1)^2}{(2)} \Rightarrow k = 6$

$\therefore z = \frac{6x^2}{y}$

Hence, when $x = 2$ and $y = 3$, $z = \frac{6(2)^2}{(3)} = 8$.

8C.4 HKCEE MA 1987(B) – I – 14

(a) Let $p = ax + \frac{b}{x}$.
$\begin{cases} 7 = 2a + \frac{b}{2} \Rightarrow 4a + b = 14 \\ 8 = 3a + \frac{b}{3} \Rightarrow 9a + b = 24 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = 6 \end{cases}$
$\therefore p = 2x + \frac{6}{x}$.
When $x = 4$, $p = 2(4) + \frac{6}{(4)} = \frac{19}{2}$.

8C.5 HKCEE MA 1988 – I – 10

(a) Let $y = ax + bx^2$
$\begin{cases} 5 = a + b \\ -8 = 2a + 4b \end{cases} \Rightarrow \begin{cases} a = -6 \\ b = 1 \end{cases} \Rightarrow y = x^2 \quad 6x$
Hence, when $x = 6$, $y = (6)^2 - 6(6) = 0$

8C.6 HKCEE MA 1991 – I – 2

(a) Let $x = \frac{ky^2}{z} \Rightarrow 18 = \frac{k(3)^2}{2} \Rightarrow k = 4 \Rightarrow x = \frac{4y^2}{z}$

(b) $x = \frac{4(1)^2}{(4)} = 1$

8C.7 HKCEE MA 1994 – I – 4

(a) Let $x = \frac{ky^2}{z} \Rightarrow (54) = \frac{k(3)^2}{(10)} \Rightarrow k = 60$
$\therefore x = \frac{60y^2}{z}$

(b) $x = \frac{60(5)^2}{(12)} = 125$

8C.8 HKCEE MA 1997 – I 7

(a) Required ratio $= \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$

(b) Cost of painting larger cone $= \$32 \times \left(\frac{3}{2}\right)^2 = \$72$

8C.9 HKCEE MA 1998 – I – 12

(a) Let $S = a + bt$.
$$\begin{cases} 230 = a + 100b \\ 284 = a + 130b \end{cases} \Rightarrow \begin{cases} a = 50 \\ b = 1.8 \end{cases}$$
$\therefore\ S = 50 + 1.8t$

(b) Charge under $A = 50 + 1.8(110) = (\$)248$
Charge under $B = 2.20 \times 110 = (\$)232 < 248$
$\therefore$ He should join B to save money.

8C.10 HKCEE MA 1999 – I – 6

Let $y = ax + bx^2$.
$$\begin{cases} 20 = 2a + 4b \\ 39 = 3a + 9b \end{cases} \Rightarrow \begin{cases} a = 5 \\ k = b \end{cases} \Rightarrow y = 5x + 3x^2$$

8C.11 HKCEE MA 2000 – I – 18

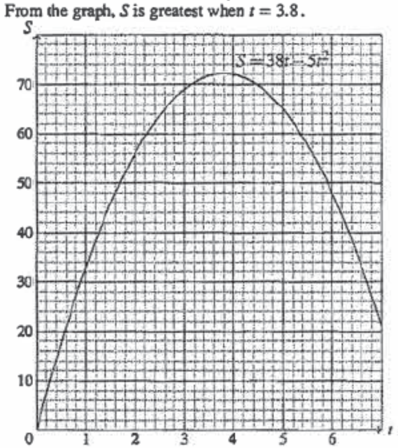
(a) Let $V = ah^2 + bh^3$.
$$\begin{cases} \frac{29\pi}{3} = a + b \\ 81\pi = 9a + 27b \end{cases} \Rightarrow \begin{cases} a = 10\pi \\ b = -\frac{\pi}{3} \end{cases}$$
$\therefore\ V = 10h^2 - \frac{\pi}{3}h^3$

8C.12 HKCEE MA 2001 – I – 13

(a) Let $S = ht + kt^2$.
$$\begin{cases} 33 = h + k \\ 56 = 2h + 4k \end{cases} \Rightarrow \begin{cases} h = 38 \\ k = -5 \end{cases} \Rightarrow S = 38t - 5t^2$$

(b) $40 = 38t\ \ 5t^2$
$5t^2 - 38t + 40 = 0$
$$t = \frac{38 \pm \sqrt{644}}{10} \left(= \frac{19 \pm \sqrt{161}}{5} \right)$$

(c) From the graph, S is greatest when $t = 3.8$.



8C.13 HKCEE MA 2002 – I – 11

(a) Let $A = hP + kP^2$.
$$\begin{cases} 36 = 24h + 576k \\ 9 = 18h + 324k \end{cases} \Rightarrow \begin{cases} h = -\frac{5}{2} \\ k = \frac{1}{6} \end{cases} \Rightarrow A = -\frac{5}{2}P + \frac{1}{6}P^2$$

(b) (i) $54 = -\frac{5}{2}P + \frac{1}{6}P^2$
$P^2 - 15P - 324 = 0 \Rightarrow P = 27$ or $\ 12$ (rejected)
$\therefore$ The perimeter is 27 cm.

8C.14 HKCEE MA 2003 – I – 10

(a) Let $V = hL + kL^2$.
$$\begin{cases} 30 = 10h + 100k \\ 75 = 15h + 225k \end{cases} \Rightarrow \begin{cases} h = \ 1 \\ k = 0.4 \end{cases} \Rightarrow V = 0.4L^2 - L$$

8C.15 HKCEE MA 2004 – I 10

(a) Let $y = hx + kx^2$.
$$\begin{cases} 3 = 3h + 9k \\ 12 = 4h + 16k \end{cases} \Rightarrow \begin{cases} h = -5 \\ k = 2 \end{cases} \Rightarrow y = 2x^2\ \ 5x$$

8C.16 HKCEE MA 2005 – I – 10

(a) Let $f(x) = hx^3 + kx$.
$$\begin{cases} -6 = f(2) = 8h + 2k \Rightarrow 4h + k = -3 \\ 6 = f(3) = 27h + 3k \Rightarrow 9h + k = 2 \end{cases} \Rightarrow \begin{cases} h = 1 \\ k = -7 \end{cases}$$
$\therefore\ f(x) = x^3\ \ 7x$

8C.17 HKCEE MA 2006 – I – 15

(a) Let $C = hA + \frac{kA^2}{n}$.
$$\begin{cases} 350 = 50h + \frac{k(50)^2}{500} \Rightarrow 10h + k = 70 \\ 100 = 20h + \frac{k(20)^2}{400} \Rightarrow 20h + k = 100 \end{cases}$$
$$\Rightarrow \begin{cases} h = 3 \\ k = 40 \end{cases} \Rightarrow C = 3A + \frac{40A^2}{n}$$

(b) (i) $P = 8A \quad C = 5A - \frac{40A^2}{n}$

(ii) $5A - \frac{40A^2}{n} = P$
$$5\left(\frac{A}{n}\right) - 40\left(\frac{A}{n}\right)^2 = \frac{P}{n} = \frac{5}{32} \text{ (both sides } \div n)$$
$$256\left(\frac{A}{n}\right)^2\ \ 32\left(\frac{A}{n}\right) + 1 = 0$$
$$\left[16\left(\frac{A}{n}\right) - 1\right]^2 = 0 \Rightarrow \frac{A}{n} = \frac{1}{16}$$

(iii) Put $n = 500$ and $P = 100$.
$100 = 5A - \frac{2}{25}A^2 \Rightarrow 2A^2\ \ 125A + 2500 = 0$
$\because\ \Delta = -4375 < 0$
$\therefore$ Not possible.

(iv) Put $n = 400$.
$$P = 5A\ \ \frac{1}{10}A^2 = \frac{-1}{10}(A^2 - 50A)$$
$$= \frac{-1}{10}(A^2 - 50A + 25^2\ \ 25^2)$$
$$= \frac{-1}{10}(A - 25)^2 + 62.5$$
$\therefore$ Greatest profit is \$62.5.

8C.18 HKCEE MA 2007 – I – 14

(a) (i) $0 = f(\ 3) = 4(-3)^3 + k(-3)^2 - 243 \Rightarrow k = 39$
(ii) $f(x) = (x+3)(4x^2 + 27x\ \ 81)$
$= (x+3)(4x\ \ 9)(x+9)$

(b) (i) Let $C = hx^3 + kx^2$.
$$\begin{cases} 7381 = h(5.5)^3 + k(5.5)^2 \\ 9077 = h(6)^3 + k(6)^2 \end{cases} \Rightarrow \begin{cases} h = 16 \\ k = 156 \end{cases}$$
$\therefore\ C = 16x^3 + 156x^2$

(ii) $972 = 16x^3 + 156x^2$
$4x^3 + 39x^2\ \ 243 = 0$
$x = -3$ (rej.) or -9 (rej.) or 2.25

8C.19 HKCEE MA 2010 – I – 10

(a) Let $C = hx + kx^2$.
$$\begin{cases} 96 = 4h + 16k \\ 145 = 5h + 25k \end{cases} \Rightarrow \begin{cases} h = 4 \\ k = 5 \end{cases} \Rightarrow C = 4x + 5x^2$$

(b) $4x + 5x^2 = 288$
$5x^2 + 4x\ \ 288 = 0 \Rightarrow x = 7.2$ or -8 (rejected)

8C.20 HKCEE MA 2011 – I – 11

(a) Let $f(x) = hx^2 + kx$.
$$\begin{cases} 28 = f(\ 2) = 4h - 2k \\ 36 = f(6) = 36h + 6k \end{cases} \Rightarrow \begin{cases} h = 1 \\ k = -12 \end{cases}$$
$\therefore\ f(x) = x^2 - 12x$

(b) (i) $f(x) = x^2\ \ 12x = (x-6)^2 - 36 \Rightarrow k = -36$
(ii) Put $x = 10$.
$y = 3(10-6)^2 - 36 = 2 \Rightarrow A = (10, 2)$
$y = (10)^2\ \ 12(10) = -20 \Rightarrow D = (10,\ \ 20)$
Since the graphs are symmetric about the common axis of symmetry $x = 6$,
$B = (6 - (10\ \ 6), 2)\ \ \ (2, 2)$
$C = (10\ \ (10-6), -20) = (2,\ \ 20)$
$\therefore$ Area of $ABCD = (2 - (\ \ 20))(10 - 2) = 176$

8C.21 HKDSE MA SP – I – 11

(a) Let $C = hs + ks^2$.
$$\begin{cases} 356 = 2h + 4k \\ 1250 = 5h + 25k \end{cases} \Rightarrow \begin{cases} h = 130 \\ k = 24 \end{cases} \Rightarrow C = 130s + 24s^2$$
$\therefore$ When $s = 6$, cost $= 130(6) + 24(6)^2 = (\$)1644$

(b) $130s + 24s^2 = 539$
$24s^2 + 130s\ \ 539 = 0 \Rightarrow s = \frac{11}{4}$ or $\frac{49}{6}$ (rejected)
$\therefore$ The perimeter is $\frac{11}{4}$ m.

8C.22 HKDSE MA PP – I – 11

(a) Let $C = h + kx^2$.
$$\begin{cases} 42 = h + 400k \\ 112 = h + 14400k \end{cases} \Rightarrow \begin{cases} h = 40 \\ k = 0.005 \end{cases} \Rightarrow C = 40 + 0.005x^2$$
$\therefore$ When $x = 50$, cost $= 40 + 0.005(50)^2 = (\$)52.5$.

(b) $40 + 0.005x^2 = 58$
$0.005x^2 = 18 \Rightarrow x = 60$
$\therefore$ The length of a side is 60 cm.

8C.23 HKDSE MA 2012 – I – 11

(a) Let $C = h + kA$.
$$\begin{cases} 62 = h + 2k \\ 74 = h + 6k \end{cases} \Rightarrow \begin{cases} h = 56 \\ k = 3 \end{cases} \Rightarrow C = 56 + 3A$$
$\therefore$ When $A = 13$, cost $= 56 + 3(13) = (\$)95$

8C.24 HKDSE MA 2013 – I – 11

(a) Let $W = h\ell + k\ell^2$.
$$\begin{cases} 181 = h + k \\ 402 = 2h + 4k \end{cases} \Rightarrow \begin{cases} h = 161 \\ k = 20 \end{cases} \Rightarrow W = 161\ell + 20\ell^2$$
$\therefore$ When $\ell = 1.2$, weight $= 161(1.2) + 20(1.2)^2 = 222$ (g)

(b) $161\ell + 20\ell^2 = 594$
$20\ell^2 + 161\ell\ \ 594 = 0 \Rightarrow \ell = \frac{11}{4}$ or $\ \frac{54}{5}$ (rejected)
$\therefore$ The perimeter is $\frac{11}{4}$ m.

8C.25 HKDSE MA 2014 – I – 13

(a) Let $f(x) = hx^2 + k$.
$$\begin{cases} 59 = f(2) = 4h + k \\ -121 = f(7) = 49h + k \end{cases} \Rightarrow \begin{cases} h = -4 \\ k = 75 \end{cases}$$
$\therefore\ f(x) = \ \ 4x^2 + 75$
$\therefore\ f(6) = \ \ 4(6)^3 + 75 = \ \ 69$

(b) From (a), $a = b = -69$.
$\therefore$ Area of $\Delta ABC = \frac{(6\ \ (-6))(69)}{2} = 414$

8C.26 HKDSE MA 2015 – I – 10

(a) Let $S = h + kn$.
$$\begin{cases} 16600 = h + 10k \\ 9000 = h + 6k \end{cases} \Rightarrow \begin{cases} h = \ \ 2400 \\ k = 1900 \end{cases}$$
$\therefore\ S = -2400 + 1900n$
$\therefore$ When $n = 20$, income $= \ \ 2400 + 1900(20) = (\$)35600$

(b) $18000 = -2400 + 1900n \Rightarrow n = \frac{204}{19}$, not an integer
$\therefore$ NOT possible

8C.27 HKDSE MA 2016 – I – 8

(a) Let $f(x) = hx + kx^2$.
$$\begin{cases} 48 = f(3) = 3h + 9k \\ 198 = f(9) = 9h + 81k \end{cases} \Rightarrow \begin{cases} h = 13 \\ k = 1 \end{cases}$$
$\therefore\ f(x) = 13x + x^2$

(b) $13x + x^2 = 90$
$x^2 + 13x\ \ 90 = 0 \Rightarrow x = 5$ or -18

8C.28 HKDSE MA 2017 – I 8

(a) Let $y = \frac{k}{\sqrt{x}} \Rightarrow 81 = \frac{k}{\sqrt{144}} \Rightarrow k = 972$
$\therefore\ y = \frac{972}{\sqrt{x}}$

(b) Change of $y = \frac{972}{\sqrt{(324)}}\ \ 81 = -27$

8C.29 HKDSE MA 2018 – I – 18

(a) Let $f(x) = hx^2 + kx$.
$$\begin{cases} 60 = f(2) = 4h + 2k \\ 99 = f(3) = 9h + 3k \end{cases} \Rightarrow \begin{cases} h = 3 \\ k = 24 \end{cases}$$
$\therefore\ f(x) = 3x^2 + 24x$

8C.30 HKDSE MA 2019 – I – 10

(a) Let $h(x) = a + bx$.

$$\begin{cases} -96 = h(-2) = a - 2b \\ 72 = h(5) = a + 5b \end{cases} \Rightarrow \begin{cases} a = \quad 48 \\ b = 24 \end{cases}$$

$\therefore\ h(x) = \quad 48 + 24x$

(b) $-48 + 24x = 3x^2 \Rightarrow x^2 - 8x + 16 = 0$

$\Rightarrow x = 4$ (repeated)

8C.31 HKDSE MA 2020 – I – 10

10a	Let $P = k_1 + k_2 h^3$, where k_1 and k_2 are non-zero constants.
	Sub. $h = 3$ and $P = 59$,
	$59 = k_1 + k_2(3)^3$
	$k_1 + 27k_2 = 59$ ------(1)
	Sub. $h = 7$ and $P = 691$,
	$691 = k_1 + k_2(7)^3$
	$k_1 + 343k_2 = 691$ ------(2)
	(2)–(1):
	$316k_2 = 632$
	$k_2 = 2$
	Sub. $k_2 = 2$ into (1).
	$k_1 + 27(2) \quad 59$
	$k_1 = 5$
	Therefore, $P = 5 + 2h^3$.
	When $h = 4$,
	$P = 5 + 2(4)^3$
	$= 133$
	Therefore, the price of a brand X souvenir is \$133.
b	When $h \quad 5$,
	$P = 5 + 2(5)^3$
	255
	< 266
	$= 2 \times 133$
	Hence, the price of a brand X souvenir of height 5 cm is lower than the total price of two brand X souvenirs of height 4 cm
	Consequently, the claim is not correct.

9 Arithmetic and Geometric Sequences

9A General terms and summations of sequences

9A.1 HKCEE MA 1980(1/1*/3) – I – 11

Let $k > 0$.

(a) (i) Find the common ratio of the geometric sequence $k, 10k, 100k$.
  (ii) Find the sum of the first n terms of the geometric sequence $k, 10k, 100k, \ldots$.

(b) (i) Show that $\log_{10}k, \log_{10}10k, \log_{10}100k$ is an arithmetic sequence.
  (ii) Find the sum of the first n terms of the arithmetic sequence $\log_{10}k, \log_{10}10k, \log_{10}100k, \ldots$. Also, if $n = 10$, what is the sum?

9A.2 HKCEE MA 1984(A/B) – I – 10

a and b are positive numbers. $a, -2, b$ is a geometric sequence and $2, b, a$ is an arithmetic sequence.

(a) Find the value of ab.

(b) Find the values of a and b.

(c) (i) Find the sum to infinity of the geometric sequence $a, -2, b, \ldots$.
  (ii) Find the sum to infinity of all the terms that are positive in the geometric sequence $a, -2, b, \ldots$.

9A.3 HKCEE MA 1986(A/B I) – B – 9

$2, -1, -4, \ldots$ form an arithmetic sequence.

(a) Find
  (i) the nth term,
  (ii) the sum of the first n terms,
  (iii) the sum of the sequence from the 21st term to the 30th term.

(b) If the sum of the first n terms of the sequence is less than -1000, find the least value of n.

9A.4 HKCEE MA 1989 – I – 9

The positive numbers $1, k, \frac{1}{2}, \ldots$ form a geometric sequence.

(a) Find the value of k, leaving your answer in surd form.

(b) Express the nth term $T(n)$ in terms of n.

(c) Find the sum to infinity, expressing your answer in the form $p + \sqrt{q}$, where p and q are integers.

(d) Express the product $T(1) \times T(3) \times T(5) \times \cdots \times T(2n-1)$ in terms of n.

9A.5 HKCEE MA 1995 – I – 3

(a) Find the sum of the first 20 terms of the arithmetic sequence $1, 5, 9, \ldots$.

(b) Find the sum to infinity of the geometric sequence $9, 3, 1, \ldots$.

9A.6 HKCEE MA 1996 – I – 3

The n-th term T_n of a sequence $T_1, T_2, T_3, \ldots$ is $7 - 3n$.

(a) Write down the first 4 terms of the sequence.

(b) Find the sum of the first 100 terms of the sequence.

9A.7 HKCEE MA 2003 – I – 7

Consider the arithmetic sequence $2, 5, 8, \ldots$. Find

(a) the 10th term of this sequence,

(b) the sum of the first 10 terms of this sequence.

9A.8 HKCEE MA 2005 – I – 7

The 1st term and the 2nd term of an arithmetic sequence are 5 and 8 respectively. If the sum of the first n terms of the sequence is 3925, find n.

9A.9 HKDSE MA 2015 – I – 17

For any positive integer n, let $A(n) = 4n - 5$ and $B(n) = 10^{4n-5}$.

(a) Express $A(1) + A(2) + A(3) + \cdots + A(n)$ in terms of n.

(b) Find the greatest value of n such that $\log(B(1)B(2)B(3)\ldots B(n)) \leq 8000$.

9A.10 HKDSE MA 2016 – I – 17

The 1st term and the 38th term of an arithmetic sequence are 666 and 555 respectively. Find

(a) the common difference of the sequence,

(b) the greatest value of n such that the sum of the first n terms of the sequence is positive.

9A.11 HKDSE MA 2018 – I – 16

The 3rd term and the 4th term of a geometric sequence are 720 and 864 respectively.

(a) Find the 1st term of the sequence.

(b) Find the greatest value of n such that the sum of the $(n+1)$th term and the $(2n+1)$th term is less than 5×10^{14}.

9A.12 HKDSE MA 2019 – I – 16

Let α and β be real numbers such that $\begin{cases} \beta = 5\alpha - 18 \\ \beta = \alpha^2 - 13\alpha + 63 \end{cases}$.

(a) Find α and β.

(b) The 1st term and the 2nd term of an arithmetic sequence are $\log\alpha$ and $\log\beta$ respectively. Find the least value of n such that the sum of the first n terms of the sequence is greater than 888.

9A.13 HKDSE MA 2020 – I – 16

The 3rd term and the 6th term of a geometric sequence are 144 and 486 respectively.

(a) Find the 1st term of the sequence. (2 marks)

(b) Find the least value of n such that the sum of the first n terms of the sequence is greater than 8×10^{18}. (3 marks)

9B Applications

9B.1 HKCEE MA 1981(1/2/3) – I – 10

In Figure (1), B_1C_1CD is a square inscribed in the right angled triangle ABC. $\angle C = 90°$, $BC = a$, $AC = 2a$, $B_1C_1 = b$.

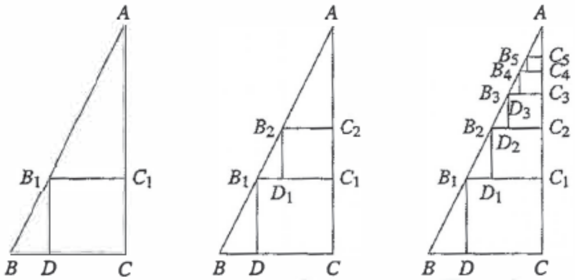


Figure (1) Figure (2) Figure (3)

(a) Express b in terms of a.

(b) $B_2C_2C_1D_1$ is a square inscribed in $\triangle AB_1C_1$ (see Figure (2)).

(i) Express B_2C_2 in terms of b.

(ii) Hence express B_2C_2 in terms of a.

(c) If squares $B_3C_3C_2D_2$, $B_4C_4C_3D_3$, $B_5C_5C_4D_4$, ... are drawn successively as indicated in Figure (3),

(i) write down the length of B_5C_5 in terms of a.

(ii) find, in terms of a, the sum of the areas of the infinitely many squares drawn in this way.

9B.2 HKCEE MA 1982(1/2/3) I 10

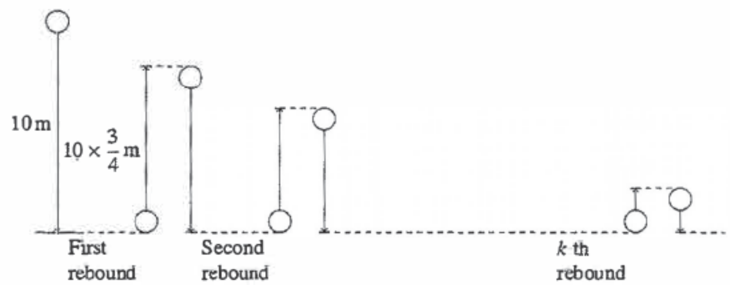
(a) (i) Find the sum of all the multiples of 3 from 1 to 1000.

(ii) Find the sum of all the multiples of 4 from 1 to 1000 (including 1000).

(b) Hence, or otherwise, find the sum of all the integers from 1 to 1000 (including 1 and 1000) which are neither multiples of 3 nor multiples of 4.

9B.3 HKCEE MA 1983(A/B) – I – 10

A ball is dropped vertically from a height of 10 m, and when it reaches the ground, it rebounds to a height of $10 \times \frac{3}{4}$ m. The ball continues to fall and rebound again, each time rebounding to $\frac{3}{4}$ of the height from which it previously fell (see the figure).



(a) Find the total distance travelled by the ball just before it makes its second rebound.

(b) Find, in terms of k, the total distance travelled by the ball just before it makes its $(k+1)$st rebound.

(c) Find the total distance travelled by the ball before it comes to rest.

9B.4 HKCEE MA 1985(A/B) – I 14

$\$P$ is deposited in a bank at the interest rate of $r\%$ per annum compounded annually. At the end of each year, $\frac{1}{3}$ of the amount in the account (including principal and interest) is drawn out and the remainder is redeposited at the same rate.

Let $\$Q_1$, $\$Q_2$, $\$Q_3$, ... denote respectively the sums of money drawn out at the end of the first year, second year, third year, ...

(a) (i) Express Q_1 and Q_2 in terms of P and r.

(ii) Show that $Q_3 = \frac{4}{27}P(1+r\%)^3$.

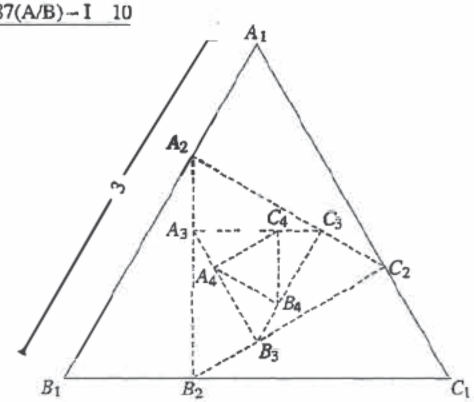
(b) Q_1, Q_2, Q_3, ... form a geometric sequence. Find the common ratio in terms of r.

(c) Suppose $Q_3 = \frac{27}{128}P$.

(i) Find the value of r.

(ii) If $P = 10000$, find $Q_1 + Q_2 + Q_3 + \cdots + Q_{10}$. (Give your answer correct to the nearest integer.)

9B.5 HKCEE MA 1987(A/B) – I 10



In this quesiton you should leave your answers in surd form.

In the figure, $A_1B_1C_1$ is an equilateral triangle of side 3 and area T_1.

(a) Find T_1.

(b) The points A_2, B_2 and C_2 divide internally the line segments A_1B_1, B_1C_1 and C_1A_1 respectively in the same ratio $1:2$. The area of $\triangle A_2B_2C_2$ is T_2.

(i) Find A_2B_2.

(ii) Find T_2.

(c) Triangles $A_3B_3C_3$, $A_4B_4C_4$, ... are constructed in a similar way. Their areas are T_3, T_4, ..., respectively. It is known that T_1, T_2, T_3, T_4, ... form a geometric sequence.

(i) Find the common ratio.

(ii) Find T_n.

(iii) Find the value of $T_1 + T_2 + \cdots + T_n$.

(iv) Find the sum to infinity of the geometric sequence.

9B.6 HKCEE MA 1988 – I – 9

(a) Write down the smallest and the largest multiples of 7 between 100 and 999.

(b) How many multiples of 7 are there between 100 and 999? Find the sum of these multiples.

(c) Find the sum of all positive three digit integers which are **NOT** divisible by 7.

9B.7 HKCEE MA 1990 I – 14

The positive integers $1, 2, 3 \ldots$ are divided into groups $G_1, G_2, G_3, \ldots$, so that the k^{th} group G_k consists of k consecutive integers as follows:

$$
\begin{array}{l}
G_1 : 1 \\
G_2 : 2, 3 \\
G_3 : 4, 5, 6 \\
\quad \cdots\cdots\cdots \\
\quad \cdots\cdots\cdots \\
\quad \cdots\cdots\cdots \\
G_{k-1} : u_1, u_2, \ldots, u_{k-1} \\
G_k : v_1, v_2, \ldots, v_{k-1}, v_k \\
\quad \cdots\cdots\cdots \\
\quad \cdots\cdots\cdots \\
\quad \cdots\cdots\cdots
\end{array}
$$

(a) (i) Write down all the integers in the 6^{th} group G_6.

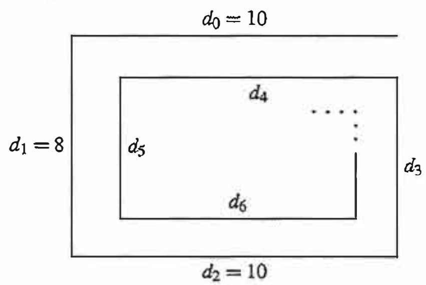
(ii) What is the total number of integers in the first 6 groups $G_1, G_2, \ldots, G_6$?

(b) Find, in terms of k,

(i) the last integer u_{k-1} in G_{k-1} and the first integer v_1 in G_k,

(ii) the sum of all the integers in G_k.

9B.8 HKCEE MA 1991 – I – 12



A maze is formed by line segments of lengths $d_0, d_1, d_2, \ldots, d_n, \ldots$, with adjacent line segments perpendicular to each other as shown in the figure. Let $d_0 = 10$, $d_1 = 8$, $d_2 = 10$ and $\frac{d_{n+2}}{d_n} = 0.9$ when $n \geq 1$,

i.e. $\frac{d_3}{d_1} = \frac{d_5}{d_3} = \cdots = 0.9$ and $\frac{d_4}{d_2} = \frac{d_6}{d_4} = \cdots = 0.9$.

(a) Find d_3 and d_5, and express d_{2n-1} in terms of n.

(b) Find d_6 and express d_{2n} in terms of n.

(c) Find, in terms of n, the sums

(i) $d_1 + d_3 + d_5 + \cdots + d_{2n-1}$,

(ii) $d_2 + d_4 + d_6 + \cdots + d_{2n}$.

(d) Find the value of the sum $d_0 + d_1 + d_2 + d_3 + \ldots$ to infinity.

9B.9 HKCEE MA 1992 – I – 14

(a) Given the geometric sequence $a^n, a^{n-1}b, a^{n-2}b^2, \ldots, a^2b^{n-2}, ab^{n-1}$, where a and b are unequal and non-zero real numbers, find the common ratio and the sum to n terms of the geometric sequence.

(b) A man joins a saving plan by depositing in his bank account a sum of money at the beginning of every year. At the beginning of the first year, he puts an initial deposit of \$$P$. Every year afterwards, he deposits 10% more than he does in the previous year. The bank pays interest at a rate of 8% p.a., compounded yearly.

(i) Find, in terms of P, an expression for the amount in his account at the end of

(1) the first year,

(2) the second year,

(3) the third year.

(Note: You need not simplify your expressions)

(ii) Using (a), or otherwise, show that the amount in his account at the end of the nth year is \$$54P(1.1^n - 1.08^n)$.

(c) A flat is worth \$1 080 000 at the beginning of a certain year and at the same time, a man joins the saving plan in (b) with an initial deposit \$$P$ = \$20 000. Suppose the value of the flat grows by 15% every year. Show that at the end of the nth year, the value of the flat is greater than the amount in the man's account.

9B.10 HKCEE MA 1993 – I – 10

Consider the food production and population problems of a certain country. In the 1st year, the country's annual food production was 8 million tonnes. At the end of the 1st year its population was 2 million. It is assumed that the annual food production increases by 1 million tonnes each year and the population increases by 6% each year.

(a) Find, in million tonnes, the annual food production of the country in

(i) the 3rd year,

(ii) the nth year.

(b) Find, in million tonnes, the total food production in the first 25 years.

(c) Find the population of the country at the end of

(i) the 3rd year,

(ii) the nth year.

(d) Starting from the end of the first year, find the minimum number of years it will take for the population to be doubled.

(e) If the 'annual food production per capita' (i.e. $\frac{\text{annual food production in a certain year}}{\text{population at the end of that year}}$) is less than 0.2 tonne, the country will face a food shortage problem. Determine whether the country will face a food shortage problem or not at the end of the 100th year.

9B.11 HKCEE MA 1994 – I – 15

Suppose the number of babies born in Hong Kong in 1994 is 70 000 and in subsequent years, the number of babies born each year increased by 2% of that of the previous year.

(a) Find the number of babies born in Hong Kong

(i) in the first year after 1994;

(ii) in the nth year after 1994.

(b) In which year will the number of babies born in Hong Kong first exceed 90 000?

(c) Find the total number of babies born in Hong Kong from 1997 to 2046 inclusive.

(d) It is known that from 1901 to 2099, a year is a leap year if its number is divisible by 4.

(i) Find the number of leap years between 1997 and 2046.

(ii) Find the total number of babies born in Hong Kong in the leap years between 1997 and 2046.

9B.12 HKCEE MA 1997 – I – 10

Suppose the population of a town grows by 2% each year and its population at the end of 1996 was 300 000.

(a) Find the population at the end of 1998.

(b) At the end of which year will the population just exceed 330 000?

9B.13 HKCEE MA 1997 – I – 15

As shown below, figure A_1 is a square of side ℓ. To the middle of each of three sides of figure A_1, a square of side $\frac{\ell}{3}$ is added to give figure A_2.

Following the same pattern, squares of side $\frac{\ell}{9}$ are added to figure A_2 to give figure A_3. The process is repeated indefinitely to give figures $A_4, A_5, \ldots, A_n, \ldots$.

(a) (i) Table 1 shows the numbers and the lengths of sides of the squares added when producing A_2 from A_1, A_3 from A_2 and A_4 from A_3. Complete Table 1.

(ii) Find the total area of all the squares in A_4.

(iii) As n increases indefinitely, the total area of all the squares in A_n tends to a constant k. Express k in terms of ℓ.

(b) The overlapping line segments in figures $A_1, A_2, A_3, \ldots, A_n, \ldots$ are removed to form figures $B_1, B_2, B_3, \ldots, B_n, \ldots$ as shown.

(i) Complete Table 2.

(ii) Write down the perimeter of B_n.
What would the perimeter of B_n become if n increases indefinitely?

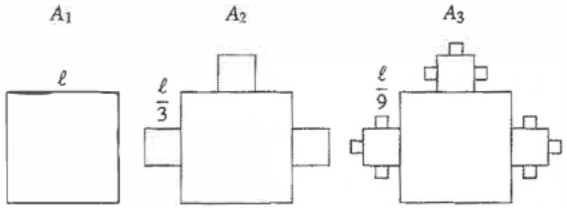


Table 1	$A_1 \to A_2$	$A_2 \to A_3$	$A_3 \to A_4$
Number of squares added	3	9	
Length of sides of the squares added	$\frac{\ell}{3}$	$\frac{\ell}{9}$	

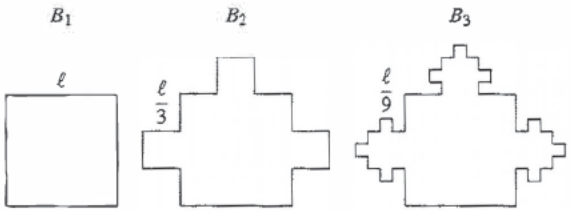


Table 2	B_1	B_2	B_3	B_4
Perimeter	4ℓ			

9B.14 HKCEE MA 1998 – I – 13

In Figure (1), $A_1B_1C_1D_1$ is a square of side 14 cm. A_2, B_2, C_2 and D_2 divide A_1B_1, B_1C_1, C_1D_1 and D_1A_1 respectively in the ratio 3 : 4 and form the square $A_2B_2C_2D_2$. Following the same pattern, A_3, B_3, C_3 and D_3 divide A_2B_2, B_2C_2, C_2D_2 and D_2A_2 respectively in the ratio 3 : 4 and form the square $A_3B_3C_3D_3$. The process is repeated indefinitely to give squares $A_4B_4C_4D_4$, $A_5B_5C_5D_5$, …, $A_nB_nC_nD_n$, ….

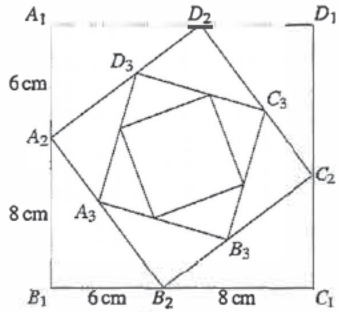


Figure (1)

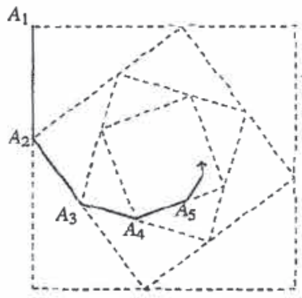


Figure (2)

(a) Find A_2B_2.

(b) Find $A_2A_3 : A_1A_2$.

(c) An ant starts at A_1 and crawls along the path $A_1A_2A_3 \ldots A_n \ldots$ as shown in Figure (2). Show that the total distance crawled by the ant cannot exceed 21 cm.

9B.15 HKCEE MA 1999 – I – 17

The manager of a factory estimated that in year 2000, the income of the factory will drop by $r\%$ each month from \$500000 in January to \$284 400 in December.

(a) Find r correct to the nearest integer.

(b) Suppose the factory's production cost is \$400000 in January 2000. The manager proposed to cut the cost by \$20 000 every month (i.e., the cost will be \$380000 in February and \$360 000 in March etc.) and claimed that it would not affect the monthly income.

(i) Using the value of r obtained in (a), show that the factory will still make a profit for the whole year.

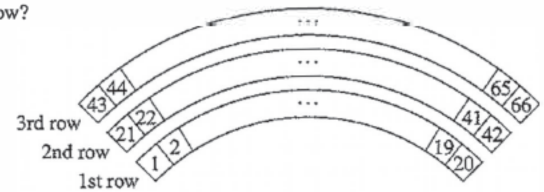
(ii) The factory will start a research project at the beginning of year 2000 on improving its production method. The cost of running the research project is \$300000 per month. The project will be stopped at the end of the k th month if the total cost spent in these k months on running the project exceeds the total production cost for the remaining months of the year.
Show that $k^2 - 71k + 348 < 0$. Hence determine how long the research project will last.

9B.16 HKCEE MA 2000 – I – 14

An auditorium has 50 rows of seats. All seats are numbered in numerical order from the first row to the last row, and from left to right, as shown in the figure. The first row has 20 seats. The second row has 22 seats. Each succeeding row has 2 more seats than the previous one.

(a) How many seats are there in the last row?

(b) Find the total number of seats in the first n rows. Hence determine in which row the seat numbered 2000 is located.



9B.17 HKCEE MA 2001 – I – 12

$F_1, F_2, F_3, \ldots, F_{40}$ as shown below are 40 similar figures. The perimeter of F_1 is 10 cm. The perimeter of each succeeding figure is 1 cm longer than that of the previous one.

(a) (i) Find the perimeter of F_{40}.
(ii) Find the sum of the perimeters of the 40 figures.

(b) It is known that the area of F_1 is $4\,\text{cm}^2$.
(i) Find the area of F_2.
(ii) Determine with justification whether the areas of $F_1, F_2, F_3, \ldots, F_{40}$ form an arithmetic sequence.

9B.18 HKCEE MA 2001 – I – 14

(a) *[Out of syllabus: The result* "The solution to the equation $x^5 \quad 6x+5=0$ is $x \approx 1.091$" *is obtained.]*

(b) From 1997 to 2000, Mr. Chan deposited \$1000 in a bank at the beginning of each year at an interest rate of $r\%$ per annum, compounded yearly. For the money deposited, the amount accumulated at the beginning of 2001 was \$5000. Using (a), find r correct to 1 decimal place.

9B.19 HKCEE MA 2002 – I – 13

A line segment AB of length 3 m is cut into three equal parts AC_1, C_1C_2 and C_2B as shown in Figure (1).

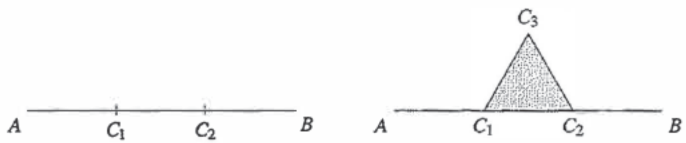


Figure (1) Figure (2)

On the middle part C_1C_2, an equilateral triangle $C_1C_2C_3$ is drawn as shown in Figure (2).

(a) Find, in surd form, the area of triangle $C_1C_2C_3$.

(b) Each of the line segments AC_1, C_1C_3, C_3C_2 and C_2B in Figure (2) is further divided into three equal parts. Similar to the previous process, four smaller equilateral triangles are drawn as shown in Figure (3). Find, in surd form, the total area of all the equilateral triangles.

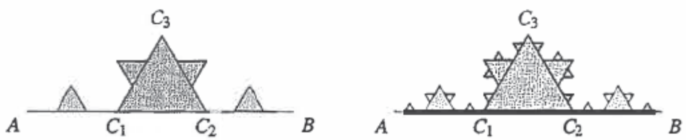


Figure (3) Figure (4)

(c) Figure (4) shows all the equilateral triangles so generated when the previous process is repeated again. What would the total area of all the equilateral triangles become if this process is repeated indefinitely? Give your answer in surd form.

9B.20 HKCEE MA 2003 – I – 15

Figure (1) shows an equilateral triangle $A_0B_0C_0$ of side 1 m. Another triangle $A_1B_1C_1$ is inscribed in triangle $A_0B_0C_0$ such that $\dfrac{A_0A_1}{A_0B_0} = \dfrac{B_0B_1}{B_0C_0} = \dfrac{C_0C_1}{C_0A_0} = k$, where $0 < k < 1$. Let $A_1B_1 = x$ m.

(a) (i) Express the area of triangle $A_1B_0B_1$ in terms of k.
(ii) Express x in terms of k.
(iii) Explain why $A_1B_1C_1$ is an equilateral triangle.

(b) Another equilateral triangle $A_2B_2C_2$ in inscribed in triangle $A_1B_1C_1$ such that $\dfrac{A_1A_2}{A_1B_1} = \dfrac{B_1B_2}{B_1C_1} = \dfrac{C_1C_2}{C_1A_1} = k$ as shown in Figure (2).
(i) Prove that the triangles $A_1B_0B_1$ and $A_2B_1B_2$ are similar.
(ii) The above process of inscribing triangles is repeated indefinitely to generate equilateral triangles $A_3B_3C_3, A_4B_4C_4, A_5B_5C_5, \ldots$. Find the total area of the triangles $A_1B_0B_1, A_2B_1B_2, A_3B_2B_3, \ldots$.

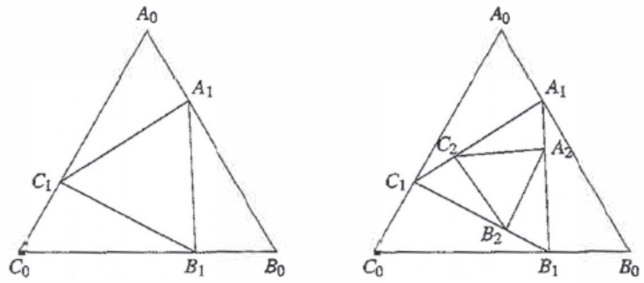


Figure (1) Figure (2)

9B.21 HKCEE MA 2004 – I – 15

In Figure (1), $F_1, F_2, F_3 \ldots$ are square frames. The perimeter of F_1 is 8 cm. Starting from F_2, the perimeter of each square frame is 4 cm longer than the perimeter of the previous frame.

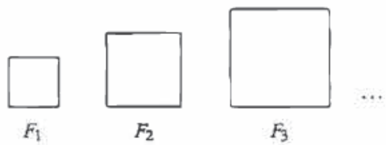


Figure (1)

(a) (i) Find the perimeter of F_{10}.
(ii) If a thin metal wire of length 1000 cm is cut into pieces and these pieces are then bent to form the above square frames, find the greatest number of distinct square frames that can be formed.

(b) Figure (2) shows three similar solid right pyramids S_1, S_2 and S_3. The total lengths of the four sides of the square bases of S_1, S_2 and S_3 are equal to the perimeters of F_1, F_2 and F_3 respectively.
(i) Do the volumes of S_1, S_2 and S_3 form a geometric sequence? Explain your answer.
(ii) When the length of the slant edge of S_1 is 5 cm, find the volume of S_3. Give the answer in surd form.

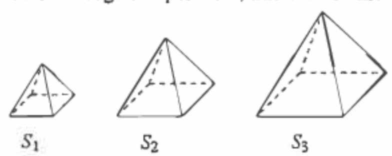


Figure (2)

9B.22 HKCEE MA 2005 – I – 16

Peter borrows a loan of \$200 000 from a bank at an interest rate of 6% per annum, compounded monthly. For each successive month after the day when the loan is taken, loan interest is calculated and then a monthly instalment of \$$x$ is immediately paid to the bank until the loan is fully repaid (the last instalment may be less than \$$x$), where $x < 200000$.

(a) (i) Find the loan interest for the 1st month.

(ii) Express, in terms of x, the amount that Peter still owes the bank after paying the 1st instalment.

(iii) Prove that if Peter has not yet fully repaid the loan after paying the nth instalment, he still owes the bank $\$\{200000(1.005)^n - 200x[(1.005)^n - 1]\}$.

(b) Suppose that Peter's monthly instalment is \$1 800 (the last instalment may be less than \$1 800).

(i) Find the number of months for Peter to fully repay the loan.

(ii) Peter wants to fully repay the loan with a smaller monthly instalment. He requests to pay a monthly instalment of \$900. However, the bank refuses his request. Why?

9B.23 HKCEE MA 2008 – I – 16

In the current financial year of a city, the amount of salaries tax charged for a citizen is calculated according to the following rules:

Net chargeable income (\$)	Rate
On the first 30 000	a%
On the next 30 000	10%
On the next 30 000	b%
Remainder	24%

The net chargeable income is equal to the net total income minus the sum of allowances. The salaries tax charged shall not exceed the standard rate of salaries tax applied to the net total income. The standard rate of salaries tax for the current financial year is 20%.

It is given that a, 10, b, 24 is an arithmetic sequence.

(a) Find a and b.

(b) Suppose that in the current financial year of the city, the sum of allowances of a citizen is \$172 000.

(i) Let \$$P$ be the net total income of the citizen. If the citizen has to pay salaries tax at the standard rate, express the amount of salaries tax charged for the citizen in terms of P.

(ii) Find the least net total income of the citizen so that the salaries tax is charged at the standard rate.

(c) Peter is a citizen in the city. In the current financial year, the net total income and the sum of allowances of Peter are \$1 400000 and \$172000 respectively. In order to pay his salaries tax, Peter begins to save money 12 months before the due day of paying salaries tax. A deposit of \$23000 is saved in a bank on the same day of each month at an interest rate of 3% per annum, compounded monthly. There are totally 12 deposits. Will Peter have enough money to pay his salaries tax on the due day? Explain your answer.

9B.24 HKCEE MA 2009 – I – 15

In a city, the taxi fare is charged according to the following table:

Distance travelled	Taxi fare
The first 2 km (under 2 km will be counted as 2 km)	\$30
Every 0.2km thereafter (under 0.2 km will be counted as 0.2 km)	\$2.4

Assume that there are no other extra fares.

(a) A hired taxi in the city travels a distance of x km, where $x \geq 2$.

(i) Suppose that x is a multiple of 0.2. Prove that the taxi fare is $\$(6 + 12x)$.

(ii) Suppose that x is not a multiple of 0.2. Is the taxi fare $\$(6+12x)$? Explain your answer.

(b) If a hired taxi in the city travels a distance of 3.1 km, find the taxi fare.

(c) In the city, a taxi is hired for 99 journeys. The 1st journey covers a distance of 3.1 km. Starting from the 2nd journey, the distance covered by each journey is 0.5 km longer than that covered by the previous journey. The taxi driver claims that the total taxi fare will not exceed \$33 000. Is the claim correct? Explain your answer.

9B.25 HKCEE MA 2010 – I – 17

Figure (1) shows the circle passing through the four vertices of the square $ABCD$. A rectangular coordinate system is introduced in Figure (1) so that the coordinates of A and B are $(0,0)$ and $(8,6)$ respectively.

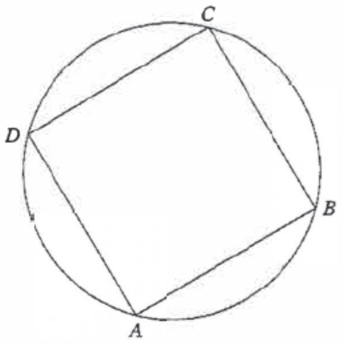


Figure (1)

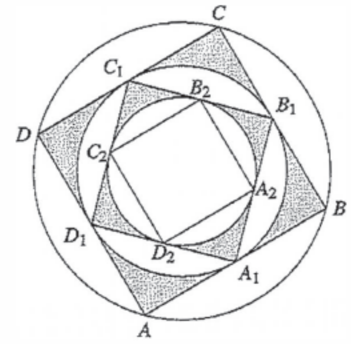


Figure (2)

(a) (i) Using a suitable transformation, or otherwise, write down the coordinates of D. Hence, or otherwise, find the coordinates of the centre of the circle $ABCD$.

(ii) Find the radius of the circle $ABCD$.

(b) A student uses the circle $ABCD$ of Figure (1) to design a logo the class association. The process of designing the logo starts by constructing the inscribed circle of the square $ABCD$ such that the inscribed circle touches AB, BC, CD and DA at A_1, B_1, C_1 and D_1 respectively. The region between the square $ABCD$ and its inscribed circle is shaded as shown in Figure (2). The inscribed circle of the square $A_1B_1C_1D_1$ is then constructed such that this inscribed circle touches A_1B_1, B_1C_1, C_1D_1 and D_1A_1 at A_2, B_2, C_2 and D_2 respectively. The region between the square $A_1B_1C_1D_1$ and its inscribed circle is also shaded. The process is carried in until the region between the square $A_9B_9C_9D_9$ and its inscribed circle is shaded.

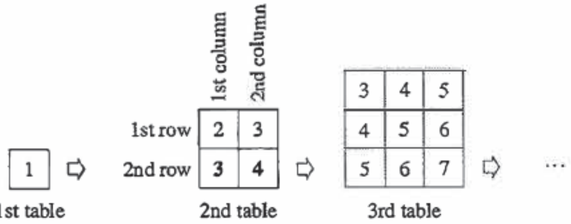
(i) Find the ratio of the area of the circle $A_1B_1C_1D_1$ to the area of the circle $ABCD$.

(ii) Suppose that the ratio of the total area of all the shaded regions to the area of the circle $ABCD$ is $p : 1$. The student thinks that the design of the logo is good when p lies between 0.2 and 0.3. According to the student, is the design of the logo good? Explain your answer.

9B.26 HKCEE MA 2011 – I – 15

The figure shows a sequence of tables filled with integers. The 1st table consists of 1 row and 1 column and 1 is assigned to the cell of the 1st table. For any integer $n > 1$, the nth table consists of n rows and n columns and the integers in the cells of the n table satisfy the following conditions:

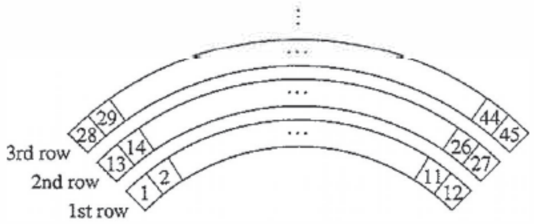
(1) The integer in the cell at the top left corner is n.
(2) In each row, the integer in the cell of the $(r+1)$th column is greater than that of the rth column by 1, where $1 \le r \le n-1$.
(3) In each column, the integer in the cell of the $(r+1)$th row is greater than that of the rth row by 1, where $1 \le r \le n-1$.



(a) Construct and complete the 4th table.
(b) Find the sum of all integers in the 1st row of the 99th table.
(c) Find the sum of all integers in the 99th table.
(d) Is there an odd number k such that the sum of all integers in the kth table is an even number? Explain your answer.

9B.27 HKDSE MA SP – I – 15

The seats in a theatre are numbered in numerical order from the first row to the last row, and from left to right, as shown in the figure. The first row has 12 seats. Each succeeding row has 3 more seats than the previous one. If the theatre cannot accommodate more than 930 seats, what is the greatest number of rows in the theatre?



9B.28 HKDSE MA PP – I – 19

The amount of investment of a commercial firm in the 1st year is \$4000000. The amount of investment in each successive year is r% less than the previous year. The amount of investment in the 4th year is \$1048576.

(a) Find r.
(b) The revenue made by the firm in the 1st year is \$2000000. The revenue made in each successive year is 20% less than the previous year.
  (i) Find the least number of years needed for the total revenue made by the firm to exceed \$9000000.
  (ii) Will the total revenue made by the firm exceed \$10000000? Explain your answer.
  (iii) The manager of the firm claims that the total revenue made by the firm will exceed the total amount of investment. Do you agree? Explain your answer.

9B.29 HKDSE MA 2012 – I – 19

In a city, the air cargo terminal X of an airport handles goods of weight $A(n)$ tonnes in the nth year since the start of its operation, where n is a positive integer. It is given that $A(n) = ab^{2n}$, where a and b are positive constants. It is found that the weights of the goods handled by X in the 1st year and the 2nd year since the start of its operation are 254 100 tonnes and 307 461 tonnes respectively.

(a) (i) Find a and b. Hence find the weight of the goods handled by X in the 4th year since the start of its operation.
  (ii) Express, in terms of n, the total weight of the goods handled by X in the first n year since the start of its operation.
(b) The air cargo terminal Y starts to operate since X has been operated for 4 years. Let $B(m)$ tonnes be the weight of the goods handled by Y in the mth year since the start of its operation, where m is a positive integer. It is given that $B(m) = 2ab^m$.
  (i) The manager of the airport claims that after Y has been operated, the weight of the goods handled by Y is less than that handled by X in each year. Do you agree? Explain your answer.
  (ii) The supervisor of the airport thinks that when the total weight of the goods handled by X and Y since the start of the operation of X exceeds 20000000 tonnes, new facilities should be installed to maintain the efficiency of the air cargo terminals. According to the supervisor, in which year since the start of the operation of X should the new facilities be installed?

9B.30 HKDSE MA 2013 – I – 19

The development of public housing in a city is under study. It is given that the total floor area of all public housing flats at the end of the 1st year is 9×10^6 m^2 and in subsequent years, the total floor area of public housing flats built each year is r% of the total floor area of all public housing flats at the end of the previous year, where r is a constant, and the total floor area of public housing flats pulled down each year is 3×10^5 m^2. It is found that the total floor area of all public housing flats at the end of the 3rd year is 1.026×10^7 m^2.

(a) (i) Express, in terms of r, the total floor area of all public housing flats at the end of the 2nd year.
  (ii) Find r.
(b) (i) Express, in terms of n, the total floor area of all public housing flats at the end of the nth year.
  (ii) At the end of which year will the total floor area of all public housing flats first exceed 4×10^7 m^2?
(c) It is assumed that the total floor area of public housing flats needed at the end of the nth year is $(a(1.21)^n + b)$ m^2, where a and b are constants. Some research results reveal the following information:

n	The total floor area of public housing flats needed at the end of the nth year (m^2)
1	1×10^7
2	1.063×10^7

A research assistant claims that based on the above assumption, the total floor area of all public housing flats will be greater than the total floor area of public housing flats needed at the end of a certain year. Is the claim correct? Explain your answer.

9B.31 HKDSE MA 2014 – I – 16

In the figure, the 1st pattern consists of 3 dots. For any positive integer n, the $(n+1)$st pattern is formed by adding 2 dots to the nth pattern. Find the least value of m such that the total number of dots in the first m patterns exceeds 6 888.

9B.32 HKDSE MA 2017 – I – 16

A city adopts a plan to import water from another city. It is given that the volume of water imported in the 1st year since the start of the plan is $1.5 \times 10^7\ \text{m}^3$ and in subsequent years, the volume of water imported each year is 10% less than the volume of water imported in the previous year.

(a) Find the total volume of water imported in the first 20 years since the start of the plan.

(b) Someone claims that the total volume of water imported since the start of the plan will not exceed $1.6 \times 10^8\ \text{m}^3$. Do you agree? Explain your answer.

9 Arithmetic and Geometric Sequences

9A General terms and summations of sequences

9A.1 HKCEE MA 1980(1/1*/3) – I – 11

(a) (i) Common ratio $= \frac{10k}{k} = 10$

(ii) Sum $= \frac{k(10^n \ \ 1)}{10-1} = \frac{k(10^n \ \ 1)}{9}$

(b) (i) $\log 10k - \log k = \log \frac{10k}{k} = 1$

$\log 100k - \log 10k = \log \frac{100k}{10k} = 1$

Since there is a common difference, it is an A.S.

(ii) Sum $= \frac{n}{2}[2(\log k) + (n-1)(1)]$

$= n\log k + 2n - 2$

When $n \ \ 10$,

Sum $= 10\log k + 20 - 2 = 10\log k + 18$

9A.2 HKCEE MA 1984(A/B) – I – 10

(a) $\because \frac{-2}{a} = \frac{b}{-2} =$ common ratio

$\therefore ab = (-2)^2 = 4$

(b) $a - b = b - (\ \ 2) \Rightarrow a = 2b + 2$

Put into (a): $(2b+2)(b) = 4$

$b^2 + b - 2 = 0$

$b = -2$ (rejected) or 1

$\therefore a = 4 \div 1 = 4$

(c) (i) Common ratio $= \frac{-2}{4} = \frac{-1}{2}$

$\therefore$ Sum to $\infty = \frac{4}{1-(\frac{-1}{2})} = \frac{8}{3}$

(ii) The positive terms are the 1st, 3rd, 5th, …, ones.

$\therefore$ Common ratio $= \left(\frac{-1}{2}\right)^2 = \frac{1}{4}$

$\Rightarrow$ Sum to $\infty = \frac{4}{1-\frac{1}{4}} = \frac{16}{3}$

9A.3 HKCEE MA 1986(A/B I) – B – 9

(a) (i) Common difference $= \ \ 1 - 2 = -3$

n-th term $= 2 + (n \ \ 1)(-3) = 5 - 3n$

(ii) Sum $= \frac{n}{2}[2 + (5-3n)] = \frac{7n-6n^2}{2}$

(iii) Required sum

$= \frac{7(30)-6(30)^2}{2} \ \ \frac{7(20) \ \ 6(20)^2}{2} = -1465$

(b) $\frac{7n-6n^2}{2} < -1000$

$6n^2 \ \ 7n - 2000 > 0$

$n < \frac{7-\sqrt{48049}}{12}$ or $n > \frac{7+\sqrt{48049}}{12}$

$n < -17.68$ or $n > 18.85$

$\therefore$ Least $n = 19$

9A.4 HKCEE MA 1989 – I – 9

(a) $\frac{k}{1} = \frac{\frac{1}{2}}{k} \Rightarrow k = \frac{1}{\sqrt{2}}$

(b) $T(n) = \left(\frac{1}{\sqrt{2}}\right)^{n-1} = 2^{\frac{1-n}{2}}$

(c) Sum to $\infty = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{1+\frac{1}{\sqrt{2}}}{(1)^2-(\frac{1}{\sqrt{2}})^2} = 2+\sqrt{2}$

(d) $T(1) \times T(3) \times T(5) \times \cdots \times T(2n-1)$

$= 2^{\frac{1-1}{2}} \cdot 2^{\frac{1-3}{2}} \cdot 2^{\frac{1-5}{2}} \cdot \ldots \cdot 2^{\frac{1 \ (2n-1)}{2}}$

$= 2^0 \cdot 2^{-1} \cdot 2^{-2} \cdot \ldots \cdot 2^{-(n-1)}$

$= 2^{-(1+2+\cdots+(n-1))} = 2^{\frac{-n(n-1)}{2}}$

9A.5 HKCEE MA 1995 – I – 3

(a) Sum $= \frac{20}{2}[2(1) + (20-1)(5-1)] = 780$

(b) Sum to $\infty = \frac{9}{1-(\frac{5}{9})} = \frac{27}{2}$

9A.6 HKCEE MA 1996 – I – 3

(a) 4, 1, −2, −5

(b) Sum $= \frac{100}{2}[2(4) + (100-1)(1-4)] = \ \ 14450$

9A.7 HKCEE MA 2003 – I – 7

(a) 10th term $= 2 + (10-1)(5-2) = 29$

(b) Sum $= \frac{(2+29)(10)}{2} = 155$

9A.8 HKCEE MA 2005 – I – 7

$\frac{n}{2}[2(5) + (n \ \ 1)(8 \ \ 5)] = 3925$

$3n^2 + 7n - 7850 = 0$

$n = 50$ or $\frac{-157}{3}$ (rejected)

9A.9 HKDSE MA 2015 – I – 17

(a) Common difference $= 4$

Sum $= \frac{n}{2}[2(4-5) + (n-1)(4)] = 2n^2 - 3n$

(b) Note that $\log B(n) = A(n)$. Hence

$\log(B(1)B(2)B(3). \ \ B(n)) \le 8000$

$A(1) + A(2) + A(3) + \cdots + A(n) \le 8000$

$2n^2 - 3n \le 8000$

$2n^2 - 3n - 8000 \le 0$

$-64 \le n \le 62.5$

$\therefore$ Greatest $n = 62$

9A.10 HKDSE MA 2016 – I – 17

(a) Common difference $= \frac{555-666}{38-1} = \ \ 3$

(b) $\frac{n}{2}[2(666) + (n-1)(\ \ 3)] > 0$

$n(1335 - 3n) > 0$

$0 < n < 445$

$\therefore$ Greatest $n = 444$

9A.11 HKDSE MA 2018 – I – 16

(a) Common ratio $= \frac{864}{720} = 1.2$

$\therefore$ 1st term $= 720 \div (1.2)^2 = 500$

(b) $500(1.2)^n + 500(1.2)^{2n} < 5 \times 10^{14}$

$(1.2^n)^2 + (1.2^n) - 1 \times 10^{12} < 0$

$-1000000.5 < 1.2^n < 999999.5$

$n < \frac{\log 999999.5}{\log 1.2} = 75.78$

$\therefore$ Least value of n is 75

9A.12 HKDSE MA 2019 – I – 16

(a)
$$5\alpha-18=\alpha^2-13\alpha+63$$
$$\Rightarrow\ \alpha^2-18\alpha+81=0$$
$$\Rightarrow\ \alpha=9\text{ (repeated)}\ \Rightarrow\ \beta=27$$

(b) First term $=\log 9$

Common difference $=\log 27-\log 9=\log 3$

$\therefore\ \dfrac{n}{2}[2\log 9+(n-1)\log 3]>888$

$4n\log 3+n^2\log 3-n\log 3>1776$

$(\log 3)n^2+(3\log 3)n-1776>0$

$n<-62.53$ or $n>59.53$

$\therefore$ The least n is 60.

9A.13 HKDSE MA 2020 – I – 16

16a Let a and r be the first term and the common ratio of the sequence respectively.

$$\begin{cases}ar^{3-1}=144\\ ar^{6-1}=486\end{cases}$$
$$\begin{cases}ar^{2}=144\text{------(1)}\\ ar^{5}=486\text{------(2)}\end{cases}$$

$(1)^5\div(2)^2$:

$$a^3=262144$$
$$a=64$$

Therefore, the 1st term of the sequence is 64.

b Sub. $a=64$ into (2),

$$64r^5=486$$
$$r^5=\frac{243}{32}$$
$$r=\frac{3}{2}$$

$$\frac{64\left[\left(\frac{3}{2}\right)^n-1\right]}{\frac{3}{2}-1}>8\times10^{18}$$
$$\left(\frac{3}{2}\right)^n>6.25\times10^{16}+1$$
$$n>\log_{\frac{3}{2}}\left(6.25\times10^{16}+1\right)\quad\left(\because\left(\frac{3}{2}\right)^x\text{ is strictly increasing}\right)$$
$$n>95.38167941$$

Therefore, the least value of n is 96.

9B Applications

9B.1 HKCEE MA 1981(1/2/3) – I – 10

(a) By similar triangles, $\dfrac{b}{a}=\dfrac{2a-b}{2a}$

$\dfrac{b}{a}=1-\dfrac{1}{2}\left(\dfrac{b}{a}\right)$

$\dfrac{3}{2}\cdot\dfrac{b}{a}=1\ \Rightarrow\ b=\dfrac{2}{3}a$

(b) (i) $B_2C_2=\dfrac{2}{3}b$

(ii) $B_2C_2=\dfrac{2}{3}\left(\dfrac{2}{3}a\right)=\dfrac{4}{9}a$

(c) (i) $B_5C_5=\left(\dfrac{2}{3}\right)^5a=\dfrac{32}{243}a$

(ii) $\text{Sum}=\dfrac{\left(\frac{2}{3}a\right)^2}{1-\left(\frac{2}{3}\right)}=\dfrac{4}{3}a^2$

9B.2 HKCEE MA 1982(1/2/3) – I – 10

(a) (i) $999=3(333)$

$\therefore$ Sum of all multiples of 3

$=3(1)+3(2)+3(3)+\cdots+3(333)$

$=\dfrac{(3+999)(333)}{2}=166833$

(ii) Sum of all multiples of 4

$=4(1)+4(2)+\cdots+4(250)$

$=\dfrac{(4+1000)(250)}{2}=125500$

(b) Required sum

$=$ Sum of all integers $-$ Sum in (a) $-$ Sum in (b) $+$ Sum of all multiples of 12

$=\dfrac{(1+1000)(1000)}{2}-166833-125500+\dfrac{(12+996)(83)}{2}$

$=249999$

9B.3 HKCEE MA 1983(A/B) – I – 10

(a) Required distance $=10+2\times\left(10\times\dfrac{3}{4}\right)=25$ (m)

(b) Required distance

$=10+2\left(10\times\dfrac{3}{4}\right)+2\left(10\times\left(\dfrac{3}{4}\right)^2\right)+\cdots+2\left(10\times\left(\dfrac{3}{4}\right)^k\right)$

$=10+\dfrac{2\left(10\times\frac{3}{4}\right)\left[1-\left(\frac{3}{4}\right)^k\right]}{1-\frac{3}{4}}$

$=10+60\left[1-\left(\dfrac{3}{4}\right)^k\right]=70-60\left(\dfrac{3}{4}\right)^k$ (m)

(c) Sum to $\infty=70$ m

9B.4 HKCEE MA 1985(A/B) – I – 14

(a) (i) $Q_1=P(1+r\%)\times\dfrac{1}{3}=\dfrac{1}{3}P(1+r\%)$

$Q_2=P(1+r\%)\times\dfrac{2}{3}\times(1+r\%)\times\dfrac{1}{3}$

$=\dfrac{2}{9}P(1+r\%)^2$

(ii) $Q_3=P(1+r\%)\times\dfrac{2}{3}\times(1+r\%)\times\dfrac{2}{3}\times(1+r\%)\times\dfrac{1}{3}$

$=\dfrac{4}{27}P(1+r\%)^3$

(b) Common ratio $=\dfrac{2}{3}(1+r\%)$

(c) (i) $\dfrac{27}{128}P=\dfrac{4}{27}P(1+r\%)^3$

$\dfrac{729}{512}=(1+r\%)^3\ \Rightarrow\ 1+r\%=\dfrac{9}{8}\ \Rightarrow\ r=12.5$

(ii) $Q_1+Q_2+Q_3+\cdots+Q_{10}$

$=\dfrac{\frac{1}{3}(10000)(1+12.5\%)\left(1-\left[\frac{2}{3}(1+12.5\%)\right]^{10}\right)}{1-\frac{2}{3}(1+12.5\%)}$

$=\dfrac{\frac{1}{3}(10000)\left(\frac{9}{8}\right)(1-0.75^{10})}{1-0.75}=(\$)14155$ (nrst int)

9B.5 HKCEE MA 1987(A/B) – I – 10

(a) $T_1=\dfrac{1}{2}(3)(3)\sin 60^\circ=\dfrac{9\sqrt{3}}{4}$

(b) (i) $A_2B_1=3\times\dfrac{2}{3}=2,\ B_1B_2=3\times\dfrac{1}{3}=1$

$\therefore\ A_2B_2=\sqrt{2^2+1^2-2(2)(1)\cos 60^\circ}=\sqrt{3}$

(ii) Ratio in length $=\dfrac{\sqrt{3}}{3}=\dfrac{1}{\sqrt{3}}$

$\Rightarrow$ Ratio in area $=\left(\dfrac{1}{\sqrt{3}}\right)^2=\dfrac{1}{3}$

$\therefore\ T_2=\dfrac{1}{3}T_1=\dfrac{3\sqrt{3}}{4}$

(c) (i) $\dfrac{1}{3}$

(ii) $T_n=\dfrac{9\sqrt{3}}{4}\cdot\left(\dfrac{1}{3}\right)^{n-1}=\dfrac{\sqrt{3}}{4\cdot3^{n-3}}$

(iii) $T_1+T_2+\cdots+T_n=\dfrac{\frac{9\sqrt{3}}{4}\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}}=\dfrac{27\sqrt{3}}{8}\left(1-\dfrac{1}{3^n}\right)$

(iv) Sum to $\infty=\dfrac{27\sqrt{3}}{8}$

9B.6 HKCEE MA 1988 – I – 9

(a) Smallest: 105, Largest: 994

(b) 128 multiples

Sum $=\dfrac{128}{2}(105+994)=70336$

(c) Sum $=\dfrac{900}{2}(100+999)-70336=424214$

9B.7 HKCEE MA 1990 – I – 14

(a) (i) G_6: 16, 17, 18, 19, 20, 21

(ii) Total number of integers $=1+2+3+4+5+6=21$

(b) (i) $u_{k-1}=1+2+3+\cdots+(k-1)=\dfrac{k(k-1)}{2}$

$v_1=\dfrac{k(k-1)}{2}+1$

(ii) $\text{Sum}=\dfrac{\left[\left(\frac{k(k-1)}{2}+1\right)+\left(\frac{k(k-1)}{2}+k\right)\right](k)}{2}$

$=\dfrac{[k(k-1)+1+k](k)}{2}=\dfrac{k(k^2+1)}{2}$

9B.8 HKCEE MA 1991 – I – 12

(a) $d_3=0.9d_1=7.2\quad d_5=0.9d_3=6.48$

$\therefore\ d_{2n-1}=0.9^{n-1}d_1=8\cdot0.9^{n-1}$

(b) $d_6=0.9d_4=0.9^2d_2=8.1$

$\therefore\ d_{2n}=0.9^{n-1}d_2=10\cdot0.9^{n-1}$

(c) (i) $d_1+d_3+\cdots+d_{2n-1}=\dfrac{8(1-0.9^n)}{1-0.9}=80(1-0.9^n)$

(ii) $d_2+d_4+\cdots+d_{2n}=\dfrac{10(1-0.9^n)}{1-0.9}=100(1-0.9^n)$

(d) $d_0+d_1+\cdots=d_0+(80)+(100)=190$

9B.9 HKCEE MA 1992 – I – 14

(a) Common ratio $=\dfrac{a^{n-1}b}{a^n}=\dfrac{b}{a}$

$\therefore$ Sum $=\dfrac{a^n\left[1-\left(\frac{b}{a}\right)^n\right]}{1-\frac{b}{a}}$

$=a^n\left(\dfrac{a^n-b^n}{a^n}\right)\cdot\dfrac{a}{a-b}=\dfrac{a(a^n-b^n)}{a-b}$

(b) (i) (1) $P(1+8\%)=1.08P$

(2) $(1.08P+1.1P)(1.08)=[(1.08)^2+(1.1)(1.08)]P$

(3) $\{[(1.08)^2+(1.1)(1.08)]P+(1.1)^2P\}(1.08)$

$=[(1.08)^3+(1.1)(1.08)^2+(1.1)^2(1.08)]P$

(ii) Take $a=1.08$ and $b=1.1$.

$\Rightarrow$ Amount

$=[(1.08)^n+(1.1)(1.08)^{n-1}+(1.1)^2(1.08)^{n-2}+\cdots+(1.1)^{n-1}(1.08)]P$

$=\dfrac{1.08(1.08^n-1.1^n)}{1.08-1.1}P$

$=(\$)54(1.1^n-1.08^n)P$

(c) Value of flat at the end of the nth year $=\$1080000(1.15)^n$

Amount in account $=\$54(20000)(1.1^n-1.08^n)$

$=\$1080000(1.1^n-1.08^n)$

$<\$1080000(1.1^n)$

$<\$1080000(1.15^n)=$ Value of flat

9B.10 HKCEE MA 1993 – I – 10

(a) (i) Food pdtn $=8+2(1)=10$ (mil. tonnes)

(ii) Food pdtn $=8+(n-1)(1)=7+n$ (mil. tonnes)

(b) Total $=\dfrac{25}{2}[2(8)+(25-1)(1)]=500$ (mil. tonnes)

(c) (i) Popln $=2(1+6\%)^2=2.2472$ (mil.)

(ii) Popln $=2(1+6\%)^{n-1}=2(1.06)^{n-1}$ (mil.)

(d) Let it take n years.

$(1.06)^n=2\ \Rightarrow\ n=\dfrac{\log 2}{\log 1.06}=11.896$

$\therefore$ At least 12 years

(e) At the end of the 100th year,

Anl food pdtn per capita $=\dfrac{7+100}{2(1.06)^{100-1}}=0.167<0.2$

$\therefore$ YES.

9B.11 HKCEE MA 1994 – I – 15

(a) (i) No. of babies $=70000(1+2\%)=71400$

(ii) No. of babies $=70000(1+2\%)^n=70000(1.02)^n$

(b) Let it happen in the kth year after 1994.

$70000(1.02)^k>90000$

$1.02^k>\dfrac{9}{7}\ \Rightarrow\ k>\dfrac{\log\frac{9}{7}}{\log 1.02}=12.69$

$\therefore$ It happens in 2007.

(c) No. of years $=50$

First term $=70000(1.02)^3$

$\therefore$ Total $=\dfrac{70000(1.02)^3(1.02^{50}-1)}{1.02-1}=6282944$ (nrst int.)

(d) (i) Leap years: 2000, 2004, 2008, …, 2044

$\Rightarrow$ No. of leap years $=\dfrac{2044-2000}{4}+1=12$

(ii) First term $=70000(1.02)^6$

Common ratio $=1.02^4$

$\therefore$ Total $=\dfrac{70000(1.02)^6[(1.02^4)^{12}-1]}{(1.02^4)-1}$

$=1517744$ (nearest integer)

9B.12 HKCEE MA 1997 – I – 10

(a) Population $= 300000 \times (1+2\%)^2 = 312120$

(b) Let it take n years.

$$300000(1+2\%)^n > 330000$$
$$1.02^n > 1.1$$
$$n\log 1.02 > \log 1.1$$
$$n > \frac{\log 1.1}{\log 1.02} = 4.81$$

$\therefore$ After 5 years, i.e. at the end of 2001.

9B.13 HKCEE MA 1997 – I – 15

(a) (i) Table 1

3	9	27
$\frac{\ell}{3}$	$\frac{\ell}{9}$	$\frac{\ell}{27}$

(ii) Total area $= \ell^2 + 3\left(\frac{\ell}{3}\right)^2 + 9\left(\frac{\ell}{9}\right)^2 + 27\left(\frac{\ell}{27}\right)^2$

$$= \frac{820}{729}\ell^2$$

(iii) $k = \ell^2 + 3\left(\frac{\ell}{3}\right)^2 + 9\left(\frac{\ell}{9}\right)^2 + 27\left(\frac{\ell}{27}\right)^2 + \dots$

$$= \ell^2 + \frac{\ell^2}{3} + \frac{\ell^2}{9} + \frac{\ell^2}{27} + \dots$$
$$= \frac{\ell^2}{1-\frac{1}{3}} = \frac{3}{2}\ell^2$$

(b) (i) Table 2

4ℓ	6ℓ	8ℓ	10ℓ

(ii) Perimeter of $B_n = 4\ell + (n-1)(2\ell) = 2\ell + 2\ell n$, which becomes infinitel ylarge!

9B.14 HKCEE MA 1998 – I – 13

(a) $A_2B_2 = \sqrt{8^2+6^2} = 10$ (cm)

(b) $A_2A_3 = \frac{3}{3+4}(10) = \frac{30}{7}$ (cm)

$\therefore A_2A_3 : A_1A_2 = \frac{30}{7} : 6 = 5 : 7$

(c) Total dist. $= A_1A_2 + A_2A_3 + A_3A_4 + \dots$

$$< \frac{6}{1-\frac{5}{7}} = 21 \text{ (cm)}$$

9B.15 HKCEE MA 1999 – I – 17

(a) $500000(1-r\%)^{11} = 254400$

$1 - r\% = 0.949999986 \Rightarrow r = 5$ (nrst int.)

(b) (i) Total income

$= 500000 + 500000(1-5\%) + 500000(1-5\%)^2 + \dots + 500000(1-5\%)^{11}$

$= \frac{500000(1-0.95^{12})}{1-0.95} = (\$)4596399$

Total cost

$= \frac{12}{2}[2(400000) + (12-1)(-20000)]$

$= (\$)3480000 < (\$)4596399$

Hence, there is still a profit.

(ii) $300000k > 3480000 - \frac{k}{2}[2(400000) + (k-1)(-20000)] + 10000k^2$

$300000k > 3480000 - 410000k + 10000k^2$

$0 > k^2 - 71k + 348$

$5.2965 < k < 65.7035$

The project will last for 5 months.

9B.16 HKCEE MA 2000 – I – 14

(a) Number of seats $= 20 + 49(2) = 118$

(b) Total number of seats in the first n rows

$= \frac{n}{2}[2(20) + (n-1)(2)] = 19n + n^2$

$\therefore 19n + n^2 \ge 2000$

$n^2 + 19n - 2000 \ge 0$

$n \le -14.28$ or $n \ge 36.22$

$\therefore$ Seat 2000 is in the 37th row.

9B.17 HKCEE MA 2001 – I – 12

(a) (i) Perimeter $= 10 + 39(1) = 49$ (cm)

(ii) Sum $= \frac{(10+49)(40)}{2} = 1180$ (cm)

(b) (i) $\frac{\text{Area of } F_2}{\text{Area of } F_1} = \left(\frac{\text{Perimeter of } F_2}{\text{Perimeter of } F_1}\right)^2$

Area of $F_2 = 4 \times \left(\frac{11}{10}\right)^2 = 4.84$ (cm^2)

(ii) Area of $F_3 = 4 \times \left(\frac{12}{10}\right)^2 = 5.76$ (cm^2)

$\because 4.84 - 4 = 0.84$

$5.76 - 4.84 = 0.92 \ne 0.84$

$\therefore$ They do not form an A.S.

9B.18 HKCEE MA 2001 – I – 14

(b) $1000(1+r\%)^4 + 1000(1+r\%)^3 + 1000(1+r\%)^2 + 1000(1+r\%) = 5000$

$$\frac{\cancel{1000}(1+r\%)[(1+r\%)^4-1]}{(1+r\%)-1} = \cancel{5000}$$
$$(1+r\%)^5 - (1+r\%) = 5(1+r\%) - 5$$
$$(1+r\%)^5 - 6(1+r\%) + 5 = 0$$

By (a), $1 + r\% = 1.091$

$r = 9.1$

9B.19 HKCEE MA 2002 – I – 13

(a) Area $= \frac{1}{2}(1)(1)\sin 60° = \frac{\sqrt{3}}{4}$ (m^2)

(b) Area of small $\triangle = \frac{\sqrt{3}}{4} \times \left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{4} \cdot \frac{1}{9}$

$\therefore$ Total area $= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{1}{9}$

$$= \frac{\sqrt{3}}{4} \cdot \frac{10}{9} = \frac{5\sqrt{3}}{18} \text{ (m}^2\text{)}$$

(c) Total area $= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{1}{9} + \left(\frac{\sqrt{3}}{4} \cdot \frac{1}{9}\right) \cdot \frac{1}{9} + \dots$

$$= \frac{\frac{\sqrt{3}}{4}}{1-\frac{1}{9}} = \frac{9\sqrt{3}}{32} \text{ (m}^2\text{)}$$

9B.20 HKCEE MA 2003 – I – 15

(a) (i) Area $= \frac{1}{2}(k)(1-k)\sin 60° = \frac{\sqrt{3}}{4}k(1-k)$ (m^2)

(ii) $x = \sqrt{k^2 + (1-k)^2 - 2(k)(1-k)\cos 60°}$

$= \sqrt{1 - 2k + 2k^2 - (k - k^2)} = \sqrt{1 - 3k + k^2}$

(iii) $\because \triangle A_1B_0B_1 \cong \triangle B_1C_0C_1 \cong \triangle C_1A_0A_1$

$\therefore A_1B_1 = B_1C_1 = C_1A_1$

(b) (i) In $\triangle A_1B_0B_1$ and $\triangle A_2B_1B_2$.

$\frac{A_1B_0}{B_0B_1} = \frac{1-k}{k}$ (given)

$\frac{A_2B_1}{B_1B_2} = \frac{1-k}{k}$ (given)

$\angle B_0 = \angle B_1$ 60° (property of equil. $\triangle$)

$\therefore \triangle A_1B_0B_1 \sim \triangle A_2B_1B_2$ (ratio of 2 sides, inc. $\angle$)

(ii) $\frac{\text{Area of } A_1B_1C_1}{\text{Area of } A_0B_0C_0} = \left(\frac{x}{1}\right)^2 = 1 - 3k + k^2$

$\therefore$ Total area $= \frac{\frac{\sqrt{3}}{4}k(1-k)}{1-(1-3k+k^2)}$

$$= \frac{\sqrt{3}k(1-k)}{4k(3-k)} = \frac{\sqrt{3}(1-k)}{4(3-k)}$$

9B.21 HKCEE MA 2004 – I – 15

(a) (i) Perimeter $= 8 + (10-1)(4) = 44$ (cm)

(ii) Let n frames can be formed.

$$\frac{n}{2}[2(8) + (n-1)(4)] \le 1000$$
$$10n + 2n^2 \le 1000$$
$$n^2 + 5n - 500 \le 0$$
$$-25 \le n \le 20$$

$\therefore$ 20 frames can be formed.

(b) (i) Vol of S_1 : Vol of S_2 : Vol of S_3

$= (\text{Peri of } S_1 : \text{Peri of } S_2 : \text{Peri of } S_3)^3$

$= (8 : 12 : 16)^3 = 8 : 27 : 81$

Since $8 : 27 \ne 27 : 81$, the volumes do not form a G.S.

(ii) For S_1, Diag of base $= \sqrt{2^2+2^2} = \sqrt{8}$ (cm)

Height $= \sqrt{5^2 - \left(\frac{\sqrt{8}}{2}\right)^2} = \sqrt{23}$ (cm)

Volume $= \frac{1}{3}(2)^2(\sqrt{23}) = \frac{4\sqrt{23}}{3}$ (cm^3)

$\therefore$ Vol of $S_3 = \frac{4\sqrt{23}}{3} \cdot \frac{81}{8} = \frac{27\sqrt{23}}{2}$ (cm^3)

9B.22 HKCEE MA 2005 – I – 16

(a) (i) Interest $= 200000\left(1 + \frac{6\%}{12}\right) - 200000$

$= 200000(1.005 - 1) = (\$)1000$

(ii) Amt owed $= \$(201000 - x)$

(iii) Amount owed after 2nd instalment

$= [200000(1.005) - x](1.005) - x$

$= 200000(1.005)^2 - x(1.005 + 1)$

Amount owed after 3rd instalment

$= [200000(1.005)^2 - x(1.005+1)](1.005) - x$

$= 200000(1.005)^3 - x(1.005^2 + 1.005 + 1)$

$\therefore$ Amount owed after nth instalment

$= 200000(1.005)^n - x(1.005^{n-1} + 1.005^{n-2} + \dots + 1.005 + 1)$

$= 200000(1.005)^n - x\left(\frac{1.005^n - 1}{1.005 - 1}\right)$

$= (\$)200000(1.005)^n - 200x[(1.005)^n - 1]$

(b) (i) Let the last instalment be the $(n+1)$st one.

$$200000(1.005)^n - 200(1800)(1.005^n - 1) < 1800$$
$$2000(1.005)^n - 3600(1.005)^n + 3600 < 18$$
$$1600(1.005)^n > 3582$$
$$1.005^n > 2.23875$$
$$n > \frac{\log 2.23875}{\log 1.005} = 161.586$$

$\therefore$ The last instalment is the 162nd one.

(ii) $200000(1.005)^n - 200(900)(1.005^n - 1) < 900$

$200(1.005)^n < -1791$

which has no solution.

i.e. Peter cannot fully repay the loan with $x = 900$.

9B.23 HKCEE MA 2008 – I – 16

(a) Common difference $= \frac{24-10}{2} = 7$

$\therefore a = 10 - 7 = 3$, $b = 10 + 7 = 17$

(b) (i) Tax $= (P - 172000) \times 20\% = (\$)0.2P - 34400$

(ii) $0.2P - 34400 = 30000 \times 3\% + 30000 \times 10\% + 30000 \times 17\% + (P - 172000) \times 24\%$

$= 9000 + 0.24P - 62880$

$\Rightarrow 19480 = 0.04P \Rightarrow P = 487000$

Hence, the least net total income is \$487000.

(c) Total amount in bank $= \frac{23000(1+\frac{3\%}{12})[(1+\frac{3\%}{12})^{12} - 1]}{(1+\frac{3\%}{12}) - 1}$

$= (\$)280526.37$

Tax payable $= (1400000 - 172000) \times 20\%$

$= (\$)245600 < (\$)280526.37$

$\therefore$ He will have enough.

9B.24 HKCEE MA 2009 – I – 15

(a) (i) Fare $= 30 + \frac{x-2}{0.2} \times 2.4 = (\$)6 + 12x$

(ii) The fare will be $6 + 2y$, where y is the least multiple of 0.2 which is larger than x.

$\therefore$ NO.

(b) Fare $= 6 + 12(3.2) = (\$)44.4$

(c) In the city, a taxi is hired for 99 journeys. The 1st journey covers a distance of 3.1 km. Starting from the 2nd journey, the distance covered by each journey is 0.5 km longer than that covered by the previous journey. The taxi driver claims that the total taxi fare will not exceed \$33000. Is the claim correct? Explain your answer.

9B.25 HKCEE MA 2010 – I – 17

(a) (i) Rotate B about A anticlockwise through 90°

$\Rightarrow D = (-6, 8)$

Centre $=$ mid-pt of $BD = \left(\frac{-6+8}{2}, \frac{8+6}{2}\right) = (1, 7)$

(ii) Radius $= \sqrt{(8-1)^2 + (6-7)^2} = \sqrt{50}$

(b) (i) Radius of circle $A_1B_1C_1D_1 = \frac{1}{2}AB = \frac{\sqrt{8^2+6^2}}{2} = 5$

$\therefore \frac{\text{Area of circle } A_1B_1C_1D_1}{\text{Area of circle } ABCD} = \left(\frac{\text{Radius of circle } A_1B_1C_1D_1}{\text{Radius of circle } ABCD}\right)^2 = \left(\frac{5}{\sqrt{50}}\right)^2 = \frac{1}{2}$

(ii) Shaded area between sq. $ABCD$ and cl. $A_1B_1C_1D_1$

$= 10^2 - \pi(5)^2 = 100 - 25\pi$

$\therefore$ Total shaded area

$= (100 - 25\pi) + \frac{100-25\pi}{2} + \frac{100-25\pi}{2^2} + \dots + \frac{100-25\pi}{2^9}$

$= \frac{(100-25\pi)[1 - (\frac{1}{2})^{10}]}{1 - \frac{1}{2}} = 42.87845$

$\therefore p = \frac{42.87845}{\pi(\sqrt{50})^2} = 0.27297$

which is indeed between 0.2 and 0.3.

Hence the design is good.

9B.26 HKCEE MA 2011 – I – 15

(a)

4	5	6	7
5	6	7	8
6	7	8	9
7	8	9	10

(b) The 1st row contains: 99, 100, … (99 integers) ….

$\Rightarrow \text{Sum} = \frac{99}{2}[2(99)+98\times 1] = 14652$

(c) Sum of all integers in the 2nd row

$= $ Sum of all integers in the 1st row $+ 99$

Sum of all integers in the 3rd row

$= $ Sum of all integers in the 1st row $+ 99\times 2$

Similarly, sum of all integers in the ith row

$= $ Sum of all integers in the 1st row $+ 99\times(i \quad 1)$

$\therefore$ Sum of all integers

$= $ Sum of all integers in the 1st row $\times 99$

$+(99+99\times 2+\cdot \quad +99\times 98)$

$= 14652\times 99+99\times\frac{(1+98)(98)}{2}$

$= 1930797$

(d) In the kth table, 1st row: $k, k+1, \ldots, k+(k \quad 1)$

$\Rightarrow \text{Sum} = \frac{[k+(2k-1)](k)}{2} = \frac{(3k-1)k}{2}$

$\therefore$ Sum of all integers

$= \frac{(3k-1)(k)}{2}\times k+[k+2k+3k+\cdot \quad +(k-1)k]$

$= \frac{(3k-1)k^2}{2}+k\times\frac{[1+(k-1)](k-1)}{2}$

$= \frac{(3k-1)k^2}{2}+\frac{k^2(k-1)}{2}$

$= \frac{k^2(3k \quad 1+k-1)}{2} = k^2(2k-1)$, which must be odd.

$\therefore$ NO.

9B.27 HKDSE MA SP I – 15

Let there be n rows.

$\frac{n}{2}[2(12)+(n-1)(3)] \le 930$

$n(21+3n) \le 930\times 2$

$n^2+7n \quad 620 \le 0$

$-28.64 \le n \le 21.64$

$\therefore$ Greatest number of rows is 21.

9B.28 HKDSE MA PP – I – 19

(a) $4000000(1-r\%)^3 = 1048576$

$1-r\% = 0.64 \Rightarrow r = 36$

(b) (i) Let n be the number of years.

$2000000+2000000(0.8)+$

$+\cdots+2000000(0.8)^{n-1} > 9000000$

$\frac{1-0.8^n}{1 \quad 0.8} > \frac{9000000}{2000000}$

$0.8^n > 0.1$

$n\log 0.8 > \log 0.1$

$n > \frac{\log 0.1}{\log 0.8} = 10.319$

$\therefore$ The least number of years is 11.

(ii) Total revenue $< \frac{2000000}{1 \quad 0.8} = 10000000$

$\therefore$ No.

(iii) In n years, total revenue $= \frac{2000000(1 \quad 0.8^n)}{1 \quad 0.8}$

$= 10000000(1 \quad 0.8^n)$

Total investment $= \frac{4000000(1-0.64^n)}{1-0.64}$

$\frac{100000000}{9}(1 \quad 0.64^n)$

$\therefore$ Total revenue – Total investment

$= \frac{10000000}{9}[9(1 \quad 0.8^n)-10(1 \quad 0.64^n)]$

$= \frac{10000000}{9}[10(0.8^2)^n-9(0.8^n)-1]$

$= \frac{10000000}{9}[10(0.8^n)^2 \quad 9(0.8^n)-1]$

$= \frac{10000000}{9}[10(0.8^n)+1][(0.8^n)-1]$

$< 0 \quad (\because 0.8^n < 1 \text{ for any } n > 0)$

Hence, Total revenue < Total investment

Thus the claim is disagreed.

9B.29 HKDSE MA 2012 – I – 19

(a) (i) $\begin{cases} ab^2 \quad 254100 \\ ab^4 = 307461 \end{cases}$

$\Rightarrow b^2 = \frac{307461}{254100} \Rightarrow b = 1.1 \Rightarrow a = 210000$

$\therefore$ Required weight $= (210000)(1.1)^{2(4)}$

$= 450000$ (tonnes, 3 s.f.)

(ii) Total weight $= \frac{210000(1.1)^2[(1.1^2)^n \quad 1]}{1.1^2-1}$

$= 1210000(1.21^n-1)$ (tonnes)

(b) (i) In the mth year, $n = m+4$.

Then, $A(m+4) = ab^{2(m+4)}$ and $B(m) = 2ab^{2n}$

$\Rightarrow \frac{A(m+4)}{B(m)} = \frac{ab^{2m}b^8}{2ab^m}$

$= \frac{b^8}{2}b^m$

$= 1.072(1.1)^m > 1$

$\therefore A(m+4) > B(m)$, and the claim is agreed.

(ii) Total weight by Y in the first $n \quad 4$ years

$= \frac{2(210000)(1.1)(1.1^{n \quad 4} \quad 1)}{1.1-1}$

$= 4620000(1.1^{n-4} \quad 1)$

$1210000(1.21^n \quad 1)$

$+4620000(1.1^{n-4} \quad 1) > 20000000$

$121[(1.1^n)^2 \quad 1]+462\left(\frac{1.1^n}{1.1^4}-1\right) > 2000$

$177.1561[(1.1^n)^2-1]+462(1.1^n-1.4641) > 2928.2$

$177.1561(1.1^n)^2+462(1.1^n)-3781.7703 > 0$

$1.1^n < \quad 6.1047$ (rejected) or $1.1^n > 3.4968$

$\therefore n > \frac{\log 3.4968}{\log 1.1} = 13.13$

$\therefore$ The 14th year since the start of X.

9B.30 HKDSE MA 2013 – I – 19

(a) (i) Total floor area $\quad 9\times 10^6(1+r\%)-3\times 10^5$

$= 9\times 10^6+9r\times 10^4-3\times 10^5$

$= (870+9r)\times 10^4$ (m^2)

(ii) $[9\times 10^6(1+r\%)-3\times 10^5](1+r\%)$

$-3\times 10^5 = 1.026\times 10^7$

$150(1+r\%)^2 \quad 5(1+r\%)-176 = 0$

$1+r\% = \frac{11}{10}$ or $\frac{-16}{15}$ (rej)

$r = 10$

(b) (i) Required area

$= 9\times 10^6(1.1)^{n-1}-3\times 10^5(1.1)^{n-2}$

$-3\times 10^5(1.1)^{n-3} \quad \cdots \quad 3\times 10^5$

$= 9\times 10^6(1.1)^{n-1}-3\times 10^5\cdot\frac{(1.1)^{n-1}-1}{1.1-1}$

$= 9\times 10^6(1.1)^{n-1}-3\times 10^6(1.1^{n-1} \quad 1)$

$= [6(1.1)^{n \quad 1}+3]\times 10^6$ (m^2)

(ii) $[6(1.1)^{n \quad 1}+3]\times 10^6 > 4\times 10^7$

$1.1^{n-1} > \frac{37}{6}$

$n-1 > \frac{\log\frac{37}{6}}{\log 1.1} \Rightarrow n > 20.0867$

$\therefore$ At the end of the 21st year.

(c) $\begin{cases} a(1.21)^1+b = 1\times 10^7 \\ a(1.21)^2+b = 1.063\times 10^7 \end{cases}$

$\Rightarrow (1.4641 \quad 1.21)a = (1.063-1)\times 10^7$

$\Rightarrow a = \frac{3\times 10^8}{121} \Rightarrow b = 7\times 10^6$

If the claim happens at the end of the nth year,

$[6(1.1)^{n-1}+3]\times 10^6 > \frac{3\times 10^8}{121}(1.21)^n+7\times 10^6$

$\frac{6(1.1^n)}{1.1}+3 > \frac{300}{121}(1.1^n)^2+7$

$300(1.1^n)^2-660(1.1^n)+484 < 0$

Since the inequality has no solution, the claim is wrong.

9B.31 HKDSE MA 2014 I – 16

$\frac{m}{2}[2(3)+(m-1)(2)] > 6888$

$m(2+m) > 6888$

$m^2+2m-6888 > 0$

$(m+84)(m \quad 82) > 0$

$m < -84$ (rejected) or $m > 82$

$\therefore$ Least value of m is 83.

9B.32 HKDSE MA 2017 – I – 16

(a) Total volume $= \frac{1.5\times 10^7(1-0.9^{20})}{1-0.9} = 131763501\ 8$ (m^3)

(b) Total volume $< \frac{1.5\times 10^7}{1-0.9}$

$= 1.5\times 10^7 < 1.6\times 10^8$

$\therefore$ The claim is agreed.

10 Inequalities and Linear Programming

10A Linear inequalities in one unknown

10A.1 HKCEE MA 1989 I – 2

Consider $x+1>\frac{1}{5}(3x+2)$.

(a) Solve the inequality.

(b) In addition, if $-4\le x\le 4$, find the range of x.

10A.2 HKCEE MA 1995 I – 1(a)

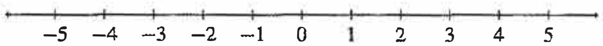
Solve the inequality $3x+1\ge 7$.

10A.3 HKCEE MA 1999 – I 3

Find the range of values of x which satisfy both $3x-4>2$ $(x$ $1)$ and $x<6$.

10A.4 HKCEE MA 2000 – I – 5

Solve $\frac{11-2x}{5}<1$ and represent the solution in the figure.



10A.5 HKCEE MA 2002 – I 7

(a) Solve the inequality $3x+6\ge 4+x$.

(b) Find all integers which satisfy both the inequalities $3x+6\ge 4+x$ and $2x-5<0$.

10A.6 HKCEE MA 2003 I – 2

Find the range of values of x which satisfy both $\frac{3-5x}{4}\ge 2$ x and $x+8>0$.

10A.7 HKCEE MA 2005 I – 4

Solve the inequality $\frac{-3x+1}{4}>x-5$.

Also write down all integers which satisfy both the inequalities $\frac{-3\ \ x+1}{4}>x$ 5 and 2 $x+1\ge 0$.

10A.8 HKCEE MA 2006 – I – 2

(a) Solve the inequality $x+1<\frac{x+25}{6}$.

(b) Write down the greatest integer satisfying the inequality $x+1<\frac{x+25}{6}$.

10A.9 HKCEE MA 2008 I – 2

(a) Solve the inequality $\frac{14x}{5}\ge 2$ $x+7$.

(b) Write down the least integer satisfying the inequality $\frac{14x}{5}\ge 2x+7$.

10A.10 HKCEE MA 2010 – I – 2

(a) Solve the inequality $\frac{29x\ \ 22}{7}\le 3x$.

(b) Write down the greatest integer satisfying the inequality in (a).

10A.11 HKDSE MA 2012 I – 6

(a) Find the range of values of x which satisfy both $\frac{4x+6}{7}>2(x$ $3)$ and $2x-10\le 10$.

(b) How many positive integers satisfy both the inequalities in (a)?

10A.12 HKDSE MA 2013 – I – 5

(a) Solve the inequality $\frac{19-7x}{3}>23-5x$.

(b) Find all integers satisfying both the inequalities $\frac{19-7x}{3}>2$ 3- $5x$ and 18 $2x\ge 0$.

10A.13 HKDSE MA 2015 I – 5

(a) Find the range of values of x whichsatisfy both $\frac{7-3x}{5}\le 2(x+2)$ and $4x$ $13>0$.

(b) Write down the least integer which satisfies both inequalities in (a).

10A.14 HKDSE MA 2016 – I – 6

Consider the compound inequality $x+6<6(x+11)$ or $x\le$ 5 ......(*).

(a) Solve (*).

(b) Write down the greatest negative integer satisfying (*).

10A.15 HKDSE MA 2017 I – 5

(a) Find the range of values of x which satisfy both $7(x-2)\le\frac{11x+8}{3}$ and 6 $x<5$.

(b) How many integers satisfy both inequalities in (a)?

10A.16 HKDSE MA 2018 – I 6

(a) Find the range of values of x which satisfy both $\frac{3-x}{2}>2x+7$ and $x+8\ge 0$.

(b) Write down the greatest integer satisfying both inequalities in (a).

10A.17 HKDSE MA 2019 – I – 6

(a) Solve the inequality $\frac{7x+26}{4}\le 2(3x-1)$.

(b) Find the number of integers satisfying both inequalities $\frac{7x+26}{4}\le 2(3x-1)$ and 45 $5x\ge 0$.

10A.18 HKDSE MA 2020 I 6

Consider the compound inequality

3 $x>\frac{7-x}{2}$ or $5+x>4$ .... (*).

(a) Solve (*).

(b) Write down the greatest negative integer satisfying (*).

(4 marks)

10B Quadratic inequalities in one unknown

10B.1 HKCEE MA 1982(1/2/3) I – 3

Solve $2x^2 - x < 36$.

10B.2 HKCEE MA 1988 – I 3

Solve the inequality $2x^2 \geq 5x$.

10B.3 HKCEE MA 1990 – I – 4

(a) Solve the following inequalities:
 (i) $6x+1 \geq 2x-3$,
 (ii) $(2-x)(x+3) > 0$.
(b) Using (a), find the values of x which satisfy both $6x+1 \geq 2x-3$ and $(2-x)(x+3) > 0$.

10B.4 HKCEE MA 1993 – I – 4

Solve the inequality $x^2 - x - 2 < 0$.
Hence solve the inequality $(y-100)^2 - (y-100) - 2 < 0$.

10B.5 HKCEE MA 1996 – I – 5

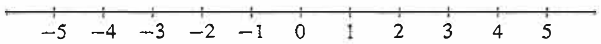
Solve (i) $\frac{x+5}{2} > 4$; (ii) $x^2 - 6x + 8 < 0$.
Hence write down the range of values of x which satisfy both the inequalities in (i) and (ii).

10B.6 HKCEE MA 1997 I – 4

Solve (i) $2x - 17 > 0$, (ii) $x^2 - 16x + 63 > 0$.
Hence write down the range of values of x which satisfy both the inequalities in (i) and (ii).

10B.7 HKCEE MA 2001 – I 4

Solve $x^2 + x - 6 > 0$ and represent the solution in the figure.



10B.8 HKCEE AM 1985 – I – 3

Solve the inequality $x^2 - ax - 4 \leq 0$, where a is real.

If, among the possible values of x satisfying the above inequality, the greatest is 4, find the least.

10B.9 HKCEE AM 1986 I 7

Solve $x > \frac{3}{x} + 2$ for each of the following cases:
(a) $x > 0$;
(b) $x < 0$.

10B.10 (HKCEE AM 1994 – I – 1)

Solve the inequality $\frac{2(x+1)}{x-2} \geq 1$ for each of the following cases:
(a) $x > 2$;
(b) $x < 2$.

10B.11 HKCEE AM 1995 – I – 4

Solve the inequality $x - \frac{5}{x} > 4$ for each of the following cases:
(a) $x > 0$;
(b) $x < 0$.

10B.12 (HKCEE AM 1996 – I – 3)

Solve the inequality $\frac{2x-3}{x+1} \leq 1$ for each of the following cases:
(a) $x > -1$;
(b) $x < -1$.

10B.13 HKCEE AM 1998 – I 6(a)

Solve $x^2 - 6x - 16 > 0$.

10B.14 (HKCEE AM 1999 – I 2)

Solve the inequality $\frac{x}{x-1} > 2$ for each of the following cases:
(a) $x > 1$;
(b) $x < 1$.

10B.15 (HKCEE AM 2000 I 1)

Solve the inequality $\frac{1}{x} \geq 1$ for each of the following cases:
(a) $x > 0$;
(b) $x < 0$.

10B.16 HKCEE AM 2011 – 3

Solve the following inequalities:
(a) $5x - 3 > 2x + 9$;
(b) $x(x-8) \leq 20$;
(c) $5x - 3 > 2x + 9$ or $x(x-8) \leq 20$.

10C Problems leading to quadratic inequalities in one unknown

10C.1 HKCEE MA 1983(B) – I – 14 (Continued from **6C.3**.)

α and β are the roots of the quadratic equation $x^2-2mx+n=0$, where m and n are real numbers.

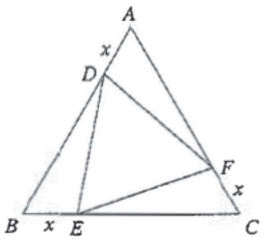
(a) Find, in terms of m and n,
  (i) $(m-\alpha)+(m-\beta)$,
  (ii) $(m-\alpha)(m-\beta)$.

(b) Find, in terms of m and n, the quadratic equation having roots $m-\alpha$ and $m-\beta$.

(c) If $n=4$, find the range of values of m such that the equation $x^2-2mx+n=0$ has real roots.

10C.2 HKCEE MA 1985(A/B) – I – 13 (Continued from **7C.1**.)

In the figure, ABC is an equilateral triangle. $AB=2$. D, E, F are points on AB, BC, CA respectively such that $AD=BE=CF=x$.



(a) By using the cosine formula or otherwise, express DE^2 in terms of x.

(b) Show that the area of $\triangle DEF=\dfrac{\sqrt{3}}{4}(3x^2-6x+4)$.
Hence, by using the method of completing the square, find the value of x such that the area of $\triangle DEF$ is smallest.

(c) If the area of $\triangle DEF\le\dfrac{\sqrt{3}}{3}$, find the range of the values of x.

10C.3 HKCEE MA 1987(B) – I – 14 (Continued from **8C.4**.)

Given $p=y+z$, where y varies directly as x, z varies inversely as x and x is positive. When $x=2$, $p=7$; when $x=3$, $p=8$.

(a) Find p when $x=4$.

(b) Find the range of values of x such that p is less than 13.

10C.4 HKCEE MA 1992 – I – 6

Find the range of values of k so that the quadratic equation $x^2+2kx+(k+6)=0$ has two distinct real roots.

10C.5 HKCEE MA 2003 – I – 10 (Continued from **8C.14**.)

The speed of a solar powered toy can is V cm/s and the length of its solar panel is L cm, where $5\le L\le 25$. V is a function of L. It is known that V is the sum of two parts, one part varies as L and the other part varies as the square of L. When $L=10$, $V=30$ and when $L=15$, $V=75$.

(a) Express V in terms of L.

(b) Find the range of values of L when $V\ge 30$.

10C.6 HKCEE MA 2004 – I – 10 (Continued from **8C.15**.)

It is known that y is the sum of two parts, one part varies as x and the other part varies as the square of x. When $x=3$, $y=3$ and when $x=4$, $y=12$.

(a) Express y in terms of x.

(b) If x is an integer and $y<42$, find all possible value(s) of x.

10C.7 HKCEE AM 1983 – I – 1

Determine the range of values of λ for which the equation $x^2+4x+2+\lambda(2x+1)=0$ has no real roots.

10C.8 HKCEE AM 1988 – I – 5

Let $f(x)=x^2+4mx+4m+15$, where m is a constant. Find the discriminant of the equation $f(x)=0$.
Hence, or otherwise, find the range of values of m so that $f(x)>0$ for all real values of x.

10C.9 HKCEE AM 1988 – I – 10 (Continued from **7B.10**.)

Let $f(x)=x^2+2x-1$ and $g(x)=-x^2+2kx-k^2+6$ (where k is a constant.)

(a) Suppose the graph of $y=f(x)$ cuts the x-axis at the points P and Q, and the graph of $y=g(x)$ cuts the x axis at the points R and S.
  (i) Find the lengths of PQ and RS.
  (ii) Find, in terms of k, the x-coordinate of the mid-point of RS.
  If the mid points of PQ and RS coincide with each other, find the value of k.

(b) If the graphs of $y=f(x)$ and $y=g(x)$ intersect at only one point, find the possible values of k; and for each value of k, find the point of intersection.

(c) Find the range of values of k such that $f(x)>g(x)$ for any real value of x.

10C.10 HKCEE AM 1991 – I – 7 (Continued from **6C.17**.)

p, q and k are real numbers satisfying the following conditions: $\begin{cases}p+q+k=2,\\ pq+qk+kp=1.\end{cases}$

(a) Express pq in terms of k.

(b) Find a quadratic equation, with coefficients in terms of k, whose roots are p and q.
Hence find the range of possible values of k.

10C.11 HKCEE AM 1991 – I – 9 (Continued from **7B.11**.)

Let $f(x)=x^2+2x-2$ and $g(x)=-2x^2-12x-23$.

(a) Express $g(x)$ in the form $a(x+b)^2+c$, where a, b and c are real constants.
Hence show that $g(x)<0$ for all real values of x.

(b) Let k_1 and k_2 ($k_1>k_2$) be the two values of k such that the equation $f(x)+kg(x)=0$ has equal roots.
  (i) Find k_1 and k_2.
  (ii) Show that $f(x)+k_1g(x)\le 0$ and $f(x)+k_2g(x)\ge 0$ for all real values of x.

(c) Using (a) and (b), or otherwise, find the greatest and least values of $\dfrac{f(x)}{g(x)}$.

10C.12 HKCEE AM 1995 – I – 1

Let $f(x)=x^2+(1-m)x+2m-5$, where m is a constant. Find the discriminant of the equation $f(x)=0$.
Hence find the range of values of m so that $f(x)>0$ for all real values of x.

10C.13 (HKCEE AM 1995 – I – 10) ***[Difficult]*** (Continued from **6C.20**.)

Let $f(x)=12x^2+2px-q$ and $g(x)=12x^2+2qx-p$, where p, q are distinct real numbers. α, β are the roots of the equation $f(x)=0$ and α, γ are the roots of the equation $g(x)=0$.

(a) Using the fact that $f(\alpha)=g(\alpha)$, find the value of α. Hence show that $p+q=3$.

(b) Express β and γ in terms of p.

(c) Suppose $-\dfrac{7}{24}<\beta^3+\gamma^3<\dfrac{7}{24}$.
  (i) Find the range of possible values of p.
  (ii) Furthermore, if $p>q$, write down the possible integral values of p and q.

10C.14 (HKCEE AM 1996 I 8)

The graph of $y = x^2 - (k-2)x + k + 1$ intersects the x-axis at two distinct points $(\alpha, 0)$ and $(\beta, 0)$, where k is real.

(a) Find the range of possible values of k.

(b) Furthermore, if $-5 < \alpha + \beta < 5$, find the range of possible values of k.

10C.15 (HKCEE AM 1997 – I 8)

Let α and β be the roots of the equation $x^2 + (k+2)x + 2(k-1) = 0$, where k is real.

(a) Show that α and β are real and distinct.

(b) If the difference between α and β is larger than 3, find the range of possible values of k.

10C.16 HKCEE AM 1999 – I – 4

Let $f(x) = 2x^2 + 2(k-4)x + k$, where k is real.

(a) Find the discriminant of the equation $f(x) = 0$.

(b) If the graph of $y = f(x)$ lies above the x axis for all values of x, find the range of possible values of k.

10C.17 HKCEE AM 2005 – 5

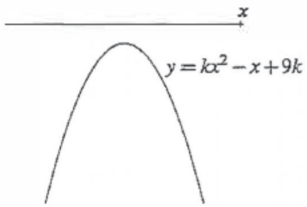
Find the range of values of k such that $x^2 - x - 1 > k(x-2)$ for all real values of x.

10C.18 HKCEE AM 2006 – 4

If $kx^2 + x + k > 0$ for all real values of x, where $k \neq 0$, find the range of possible values of k.

10C.19 HKCEE AM 2008 – 4

The graph of $y = kx^2 - x + 9k$ lies below the x axis, where $k \neq 0$ (see the figure). Find the range of possible values of k.



10C.20 HKCEE AM 2010 – 4

It is given that $(k-1)x^2 + kx + k \geq 0$ for all real values of x. Find the range of possible values of k.

10D Linear programming (with given region)

10D.1 HKCEE MA 1984(A/B) – I 8

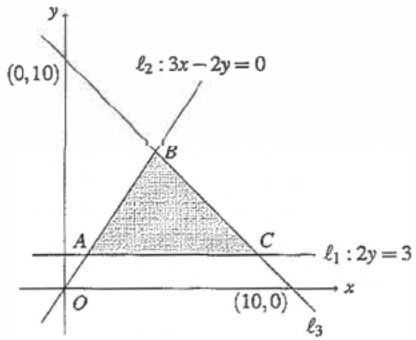
In the figure, $\ell_1 : 2y = 3$, $\ell_2 : 3x - 2y = 0$. The line ℓ_3 passes through $(0, 10)$ and $(10, 0)$.

(a) Find the equation of ℓ_3.

(b) Find the coordinates of the points A, B and C.

(c) In the figure, the shaded region, including the boundary, is determined by three inequalities. Write down these inequalities.

(d) (x, y) is any point in the shaded region, including the boundary, and $P = x + 2y - 5$. Find the maximum and minimum values of P.



10D.2 HKCEE MA 1988 – I – 12

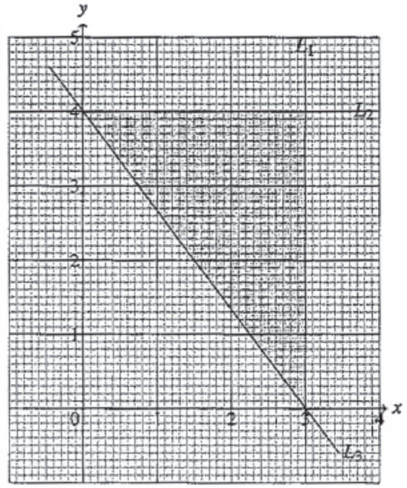
In the figure, L_1 is the line $x = 3$ and L_2 is the line $y = 4$. L_3 is the line passing through the points $(3, 0)$ and $(0, 4)$.

(a) Find the equation of L_3 in the form $ax + by = c$, where a, b and c are integers.

(b) Write down the three constraints which determine the shaded region, including the boundary.

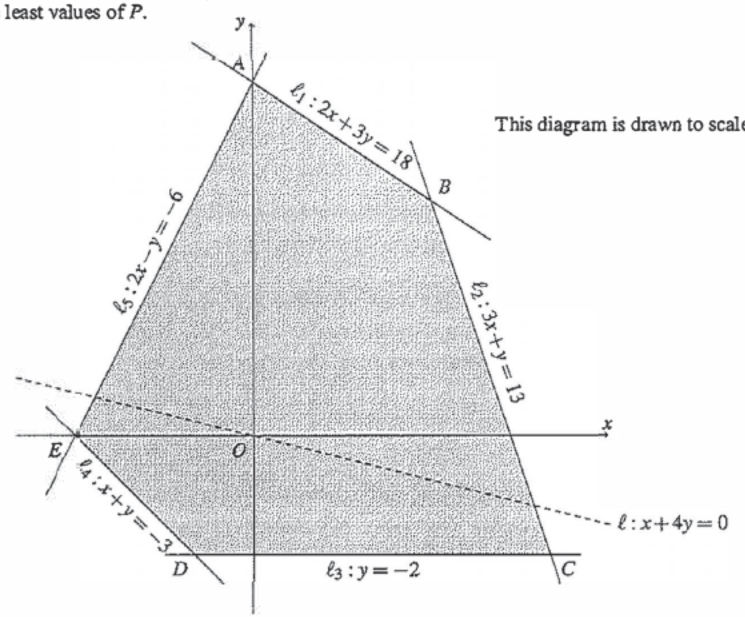
(c) Let $P = x + 4y$. If (x, y) is any point satisfying all the constraints in (b), find the greatest and the least values of P.

(d) If one more constraint $2x - 3y + 3 \leq 0$ is added, shade in the figure the new region satisfying all the four constraints.
For any point (x, y) lying in the new region, find the least value of P defined in (c).



10D.3 HKCEE MA 1990 I 5

In the figure, the shaded region $ABCDE$ is bounded by the five given lines ℓ_1, ℓ_2, ℓ_3, ℓ_4 and ℓ_5. The line $\ell: x+4y=0$ passes through the origin O.
Let $P=x+4y-2$, where (x,y) is any point in the shaded region including the boundary. Find the greatest and the least values of P.



10D.4 HKCEE MA 1991 – I – 8

In the figure, L_1 is the line $x=4$, L_2 is the line passing through the point $(0,2)$ with slope 1, and L_3 is the line passing through the points $(5,0)$ and $(0,5)$.

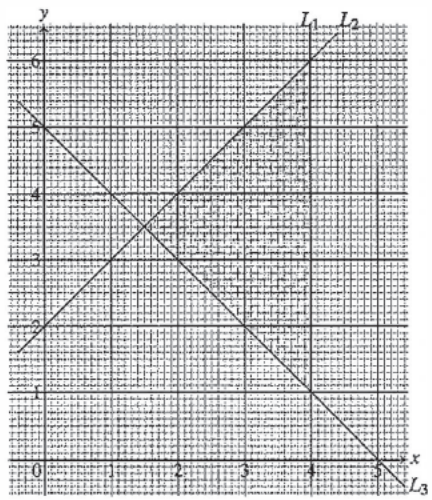
(a) Find the equations of L_2 and L_3.

(b) Write down the three inequalities which determine the shaded region, including the boundary.

(c) Suppose $P=x+2y-3$ and (x,y) is any point satisfying all the inequalities in (b).

(i) Find the point (x,y) at which P is a minimum. What is this minimum value of P?

(ii) If $P \geq 7$, by adding a suitable straight line to the figure, find the range of possible values of x.



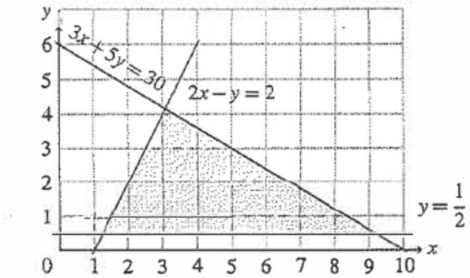
10D.5 HKCEE MA 1992 – I – 3

In this question, working steps are not required and you need to given the answers only.

In the figure, the shaded region, including the boundary, is determined by three inequalities.

(a) Write down the three inequalities.

(b) How many points (x,y), where x and y are both integers, satisfy the three inequalities in (a)?



10D.6 HKCEE MA 1993 I 1(d)

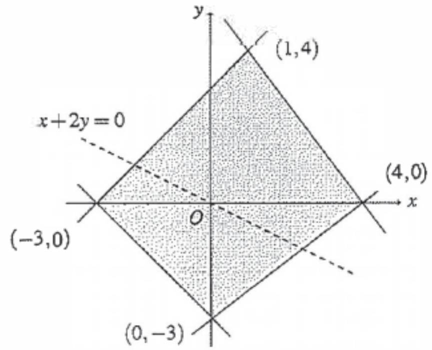
In this question, working steps are not required and you need to give the answers only.

In the figure, find a point (x,y) in the shaded region (including the boundary) at which the value of $x+2y$ is

(i) greatest,

(ii) least.

What are these greatest and least values?



10D.7 HKCEE MA 1995 – I – 12

A box of Brand X chocolates costs \$25 and contains 20 chocolates. A box of Brand Y chocolates costs \$37.50 and contains 40 chocolates.

Mrs. Chiu wants to spend not more than \$300 to buy at least 240 chocolates for her students. She wants to buy at least 3 boxes of each brand of chocolates but not more than 10 boxes altogether.

(a) If Mrs. Chiu buys x boxes of Brand X chocolates and y boxes of Brand Y chocolates, then x, y are integers such that $x \geq 3$ and $y \geq 3$. Write down the inequalities in terms of x and y which say

  (i) the total number of chocolates is at least 240;

  (ii) the total cost is not more than \$300;

  (iii) the total number of boxes is not more than 10.

(b) The points representing the ordered pairs (x, y) satisfying all the constraints in (a) are contained in the shaded region in the graph below. List all these ordered pairs (x, y).

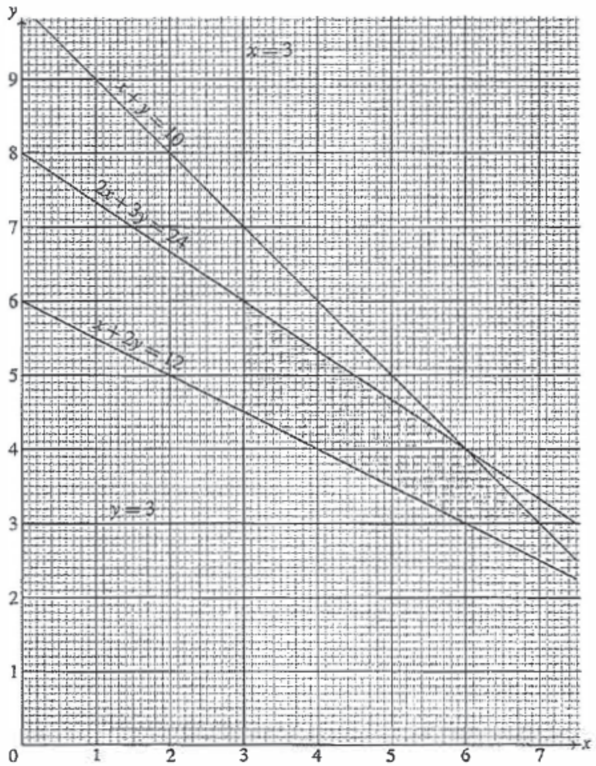
(c) Find the least amount Mrs. Chiu has to pay in buying chocolates for her students.

(d) Mrs. Chiu goes to a shop to buy the chocolates. She finds that she can get a free gift for every purchase of \$300. In order to get the free gift, she decides to spend exactly \$300 on buying the chocolates. Find

  (i) all possible combinations (x, y) of the numbers of boxes of Brand X and Brand Y chocolates, and

  (ii) the greatest number of chocolates

  Mrs. Chiu can buy.



10D.8 HKCEE MA 1996 – I – 9

In the figure, $\mathscr{R}$ is the region (including the boundary) bounded by the three straight lines

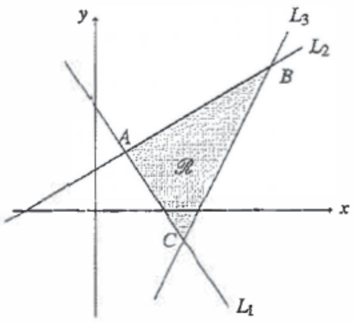
$$L_1 : 3x + 2y - 7 = 0,$$
$$L_2 : 3x - 5y + 7 = 0$$
$$\text{and } L_3 : 2x - y - 7 = 0.$$

L_1 and L_2 intersect at $A(1, 2)$. L_2 and L_3 intersect at $B(6, 5)$.

(a) Find the coordinates of C at which L_1 and L_3 intersect.

(b) Write down the three inequalities which define the region $\mathscr{R}$.

(c) Find the maximum value of $2x - 2y - 7$, where (x, y) is any point in the region $\mathscr{R}$.



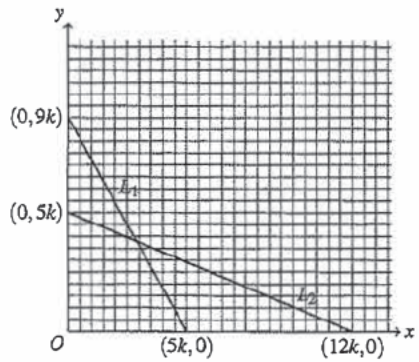
10D.9 HKCEE MA 2002 – I – 17

(a) The figure shows two straight lines L_1 and L_2. L_1 cuts the coordinate axes at the points $(5k, 0)$ and $(0, 9k)$ while L_2 cuts the coordinate axes at the points $(12k, 0)$ and $(0, 5k)$, where k is a positive integer. Find the equations of L_1 and L_2.

(b) A factory has two production lines A and B. Line A requires 45 man-hours to produce an article and the production of each article discharges 50 units of pollutants. To produce the same article, line B required 25 man hours and discharges 120 units of pollutants. The profit yielded by each article produced by the production line A is \$3000 and the profit yielded by each article produced by the production line B is \$2000.

  (i) The factory has 225 man hours available and the total amount of pollutants discharged must not exceed 600 units. Let the number of articles produced by the production lines A and B be x and y respectively. Write down the appropriate inequalities and by putting $k = 1$ in the figure, find the greatest possible profit of the factory.

  (ii) Suppose now the factory has 450 man hours available and the total amount of pollutants discharged must not exceed 1200 units. Using the figure, find the greatest possible profit.



10D.10 HKCEE MA 2009 – I – 16

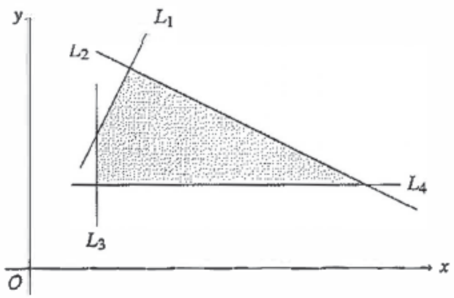
(a) In the figure, the straight lines L_1 and L_2 are perpendicular to each other. The equations of the straight lines L_3 and L_4 are $x = 8$ and $y = 10$ respectively. It is given that L_1 and L_2 intersect at the point $(12, 24)$ while L_1 and L_3 intersect at the point $(8, 16)$.

(i) Find the equations of L_1 and L_2.

(ii) In the figure, the shaded region (including the boundary) represents the solution of a system of inequalities. Write down the system of inequalities.

(b) There are two kinds of dining tables placed in a restaurant: square tables and round tables. The manager of the restaurant wants to place at least 8 square tables and 10 round tables. Moreover, the number of round tables placed is not more than 2 times that of the square tables placed. Each square table occupies a floor area of $4\,\text{m}^2$ and each round tables occupies a floor area of $8\,\text{m}^2$. The floor area occupied by the dining tables in the restaurant is at most $240\,\text{m}^2$. On a certain day, the profits on a square table and a round table at \$4000 and \$6000 respectively.

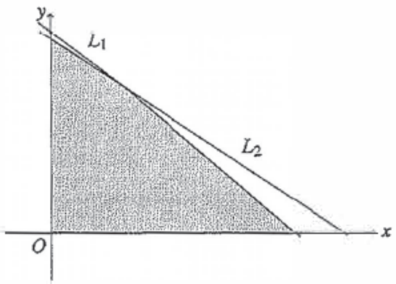
The manager claims that the total profit on the dining tables can exceed \$230 000 that day. Do you agree? Explain your answer.



10D.11 HKDSE MA 2014 – I – 18

(a) In the figure, the equation of the straight line L_1 is $6x + 7y = 900$ and the x intercept of the straight line L_2 is 180. L_1 and L_2 intersect at the point $(45, 90)$. The shaded region (including the boundary) represents the solution of a system of inequalities. Find the system of inequalities.

(b) A factory produces two types of wardrobes, X and Y. Each wardrobe X requires 6 man-hours for assembly and 2 man-hours for packing while each wardrobe Y requires 7 man-hours for assembly and 3 man hours for packing. In a certain month, the factory has 900 man hours available for assembly and 360 man hours available for packing. The profits for producing a wardrobe X and a wardrobe Y are \$440 and \$665 respectively. A worker claims that the total profit can exceed \$80 000 that month. Do you agree? Explain your answer.



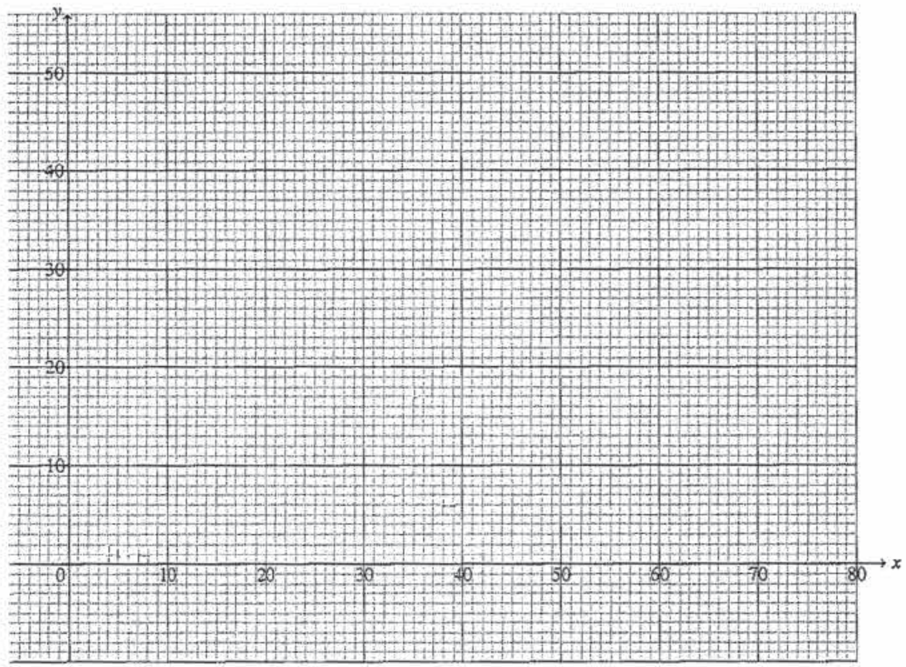
10E Linear programming (without given region)

10E.1 HKCEE MA 1980(1/1*/3) – I – 12

An airline company has a small passenger plane with a luggage capacity of 720 kg, and a floor area of $60\,\text{m}^2$ for installing passenger seats. An economy class seat takes up $1\,\text{m}^2$ of floor area while a first class seat takes up $1.5\,\text{m}^2$. The company requires that the number of first class seats should not exceed the number of economy class seats. An economy class passenger cannot carry more than 10 kg of luggage while a first-class passenger cannot carry more than 30 kg of luggage.

The profit from selling a first class ticket is double that from selling an economy-class ticket. If all tickets are sold out in every flight, find graphically how many economy-class seats and how many first class seats should be installed to give the company the maximum profit.

(Let x be the number of economy-class seats installed, y be the number of first-class seats installed.)



10E.2 HKCEE MA 1981(1/2/3) I – 8

An association plans to build a hostel with x single rooms and y double rooms satisfying the following conditions:

(1) The hostel will accommodate at least 48 persons.

(2) Each single room will occupy an area of $10\,\text{m}^2$, each double room will occupy an area of $15\,\text{m}^2$ and the total available floor area for the rooms is $450\,\text{m}^2$.

(3) The number of double rooms should not exceed the number of single rooms.

If the profits on a single room and a double room are \$300 and \$400 per month respectively, find graphically the values of x and y so that the total profit will be a maximum.

10E.3 HKCEE MA 1983(A/B) I 12

(a) On the graph paper provided below, draw the following straight lines:

$$y = 2x, \quad x + y = 30, \quad 2x + 3y = 120.$$

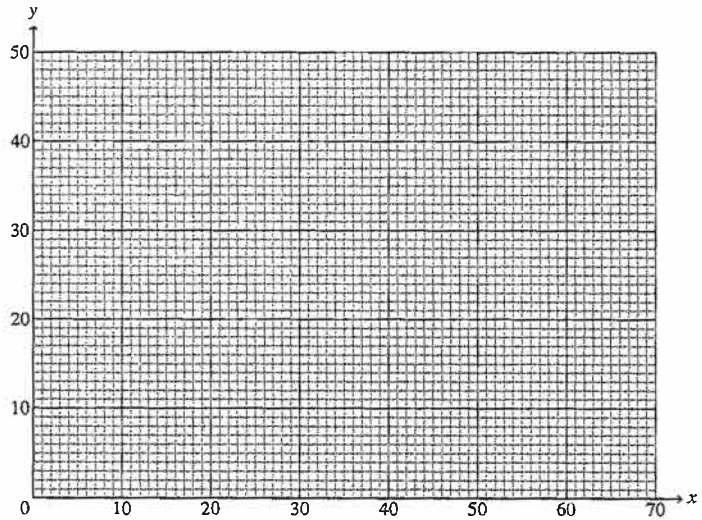
(b) On the same graph paper, shade the region that satisfies all the following inequalities:

$$\begin{cases} y \geq 0, \\ y \leq 2x, \\ x + y \geq 30, \\ 2x + 3y \leq 120. \end{cases}$$

(c) It is given that $P = 3x + 2y$. Under the constraints given by the inequalities in (b),

(i) find the maximum and minimum values of P, and

(ii) find the maximum and minimum values of P if there is the additional constraint $x \leq 45$.



10E.4 HKCEE MA 1986(A/B) – I 11

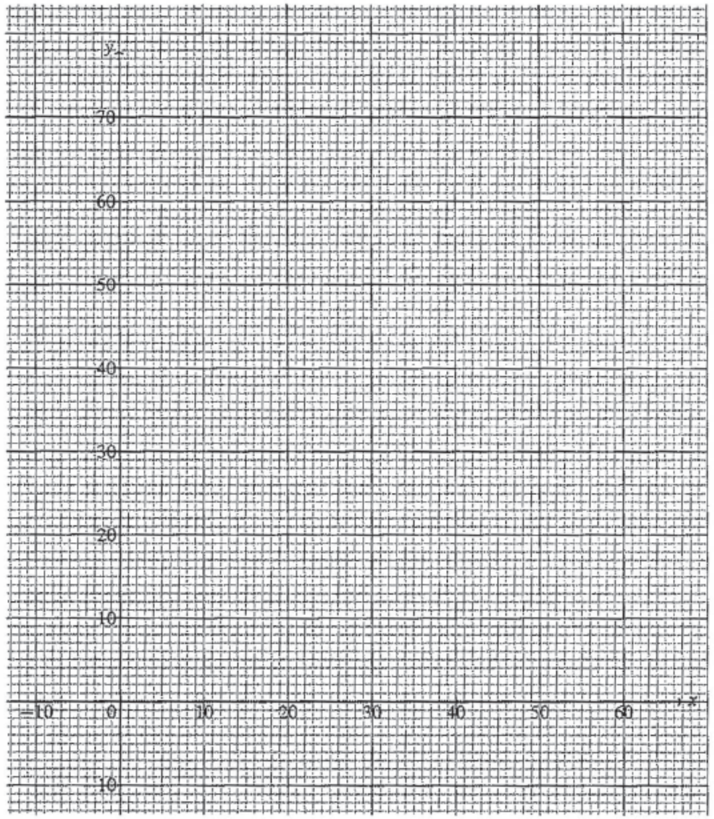
(a) (i) On the graph paper provided, draw the following straight lines:

$$x+y=40,\quad x+3y=60,\quad 7x+2y=140.$$

(ii) On the same graph, paper, shade the region that satisfies all the following constraints:

$$x\geq 0,\quad y\geq 0,\quad x+y\geq 40,\quad x+3y\geq 60,\quad 7x+2y\geq 140.$$

(b) A company has two workshops A and B. Workshop A produces 1 cabinet, 1 table and 7 chairs each day. Workshop B produces 1 cabinet, 3 tables and 2 chairs each day. The company gets an order for 40 cabinets, 60 tables and 140 chairs. The expenditures to operate Workshop A and Workshop B are respectively \$1000 and \$2000 each day. Use the result of (a)(ii) to find the number of days each workshop should operate to meet the order if the total expenditure in operating the workshops is to be kept to a minimum.
(Denote the number of days that Workshops A and B should operate by x and y respectively.)



10E.5 HKCEE MA 1987(A/B) – I 12

A factory produces three products A, B and C from two materials M and N.

Each tonne of M produces 4000 pieces of A, 20 000 pieces of B and 6000 pieces of C.

Each tonne of N produces 6000 pieces of A, 5000 pieces of B and 3000 pieces of C.

The factory has received an order for 24 000 pieces of A, 60 000 pieces of B and 24 000 pieces of C. The costs of M and N are respectively \$4000 and \$3000 per tonne. By following the steps below, determine the least cost of the materials used so as the meet the order.

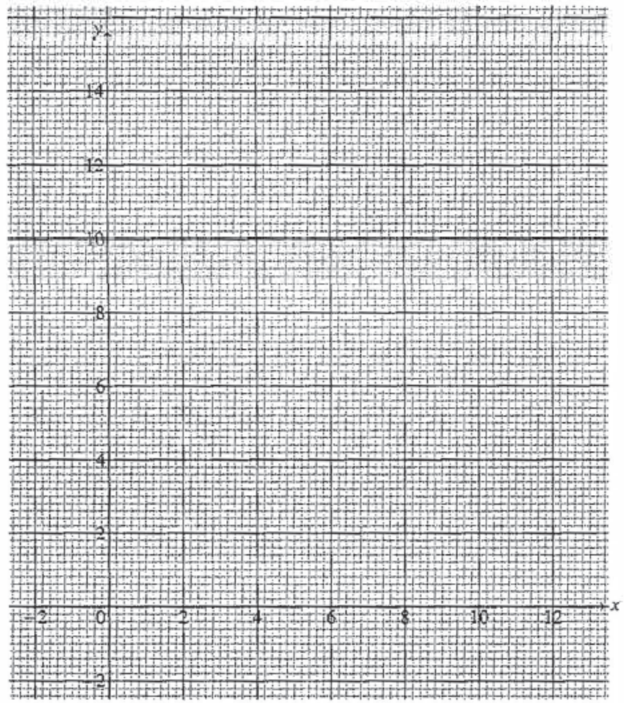
(a) Suppose x tonnes of M and y tonnes of N were used. By considering the requirement of A, B and C of the order, five constraints could be obtained. Three of them are:

$$x\geq 0,\quad y\geq 0,\quad 4000x+6000y\geq 24000.$$

Write down the other two constraints on x and y.

(b) On the graph paper provided, draw and shade the region which satisfies the five constraints in (a).

(c) Express the cost of materials in terms of x and y.
Hence use the graph in (b) to find the least cost of materials used to meet the order.



10E.6 HKCEE MA 1989 – I – 14

(a) In the figure, draw and shade the region that satisfies the following inequalities:

$$\begin{cases} y \geq 20 \\ 2x \quad y \geq 40 \\ x + y \leq 100 \end{cases}$$

(b) The vitamin content and the cost of three types of food X, Y and Z are shown in the following table:

	Food X	Food Y	Food Z
Vitamin A (units/kg)	400	600	400
Vitamin B (units/kg)	800	200	400
Cost (dollars/kg)	6	5	4

A man wants to produce 100 kg of a mixture by mixing these three types of food. Let the amount of food X, food Y and food Z used by x, y and z kilograms respectively.

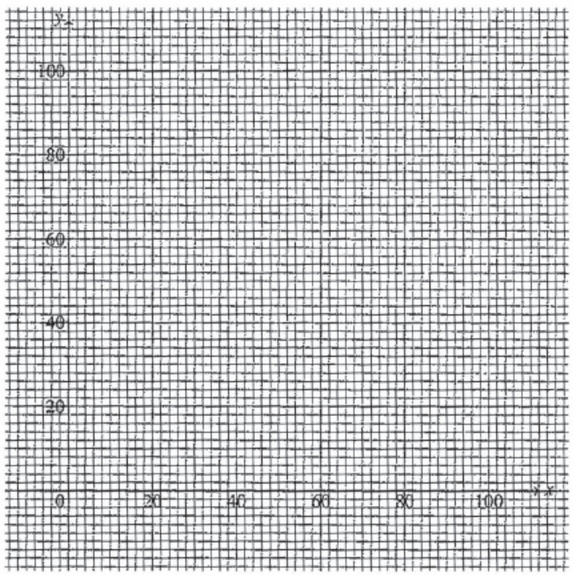
(i) Express z in terms of x and y.

(ii) Express the cost of the mixture in terms of x and y.

(iii) Suppose the mixture must contain at least 44 000 units of vitamin A and 48 000 units of vitamin B.

Show that $\begin{cases} y \geq 20 \\ 2x - y \geq 40 \\ x + y \leq 100 \end{cases}$

(iv) Using the result in (a), determine the values of x, y and z so that the cost is the least.

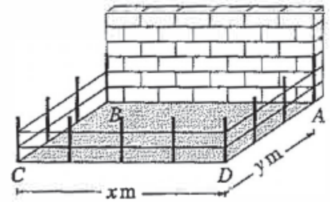


10E.7 HKCEE MA 1994 – I – 11

(a) Draw the following straight lines on the graph paper provided:

$x + y = 10$, $x + 2y = 12$, $2x = 3y$.

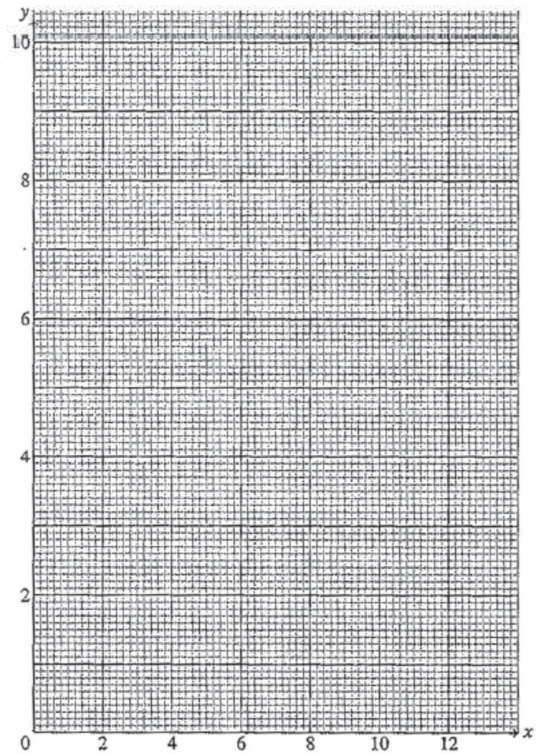
(b) Mr. Chan intends to employ a contractor to build a rectangular flower bed $ABCD$ with length AB equal to x metres and width BC equal to y metres. This project includes building a wall of length x metres along the side AB and fences along the other three sides as shown in the figure.



Mr. Chan wishes to have the total length of the four sides of the flower bed not less than 20 metres, and he also adds the condition that twice the length of the flower bed should not less than three times its width. However, no contractor will build the fences if their total length is less than 12 metres.

(i) Write down all the above constraints for x and y.

(ii) Mr. Chan has to pay the contractor \$500 per metre for building the wall and \$300 per metre for building the fences. Find the length and width of the flower bed so that the total payment for building the wall and fences is the minimum.
Find also the minimum total payment.



10E.8 HKCEE MA 1998 – I – 18

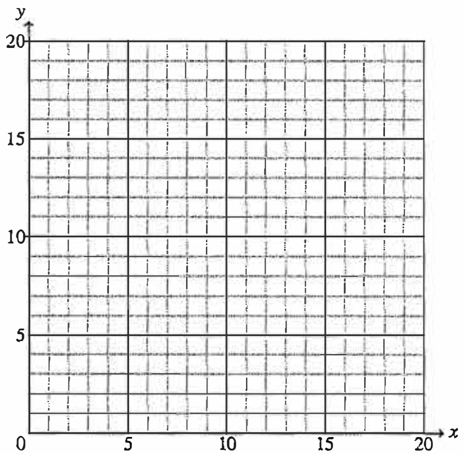
Miss Chan makes cookies and cakes for a school fair. The ingredients needed to make a tray of cookies and a tray of cakes are shown in the table.

	Flour	Sugar	Eggs
Cookies	0.32 kg	0.24 kg	2
Cakes	0.28 kg	0.36 kg	10

Miss Chan has 4.48 kg of flour, 4.32 kg of sugar and 100 eggs, from which she makes x trays of cookies and y trays of cakes.

(a) Write down the inequalities that represent the constraints on x and y. Let $\mathscr{R}$ be the region of points representing ordered pairs (x, y) which satisfy these inequalities. Draw and shade the region $\mathscr{R}$ in the figure below.

(b) The profit from selling a tray of cookies is \$90, and that from selling a tray of cakes is \$120. If x and y are integers, find the maximum possible profit.

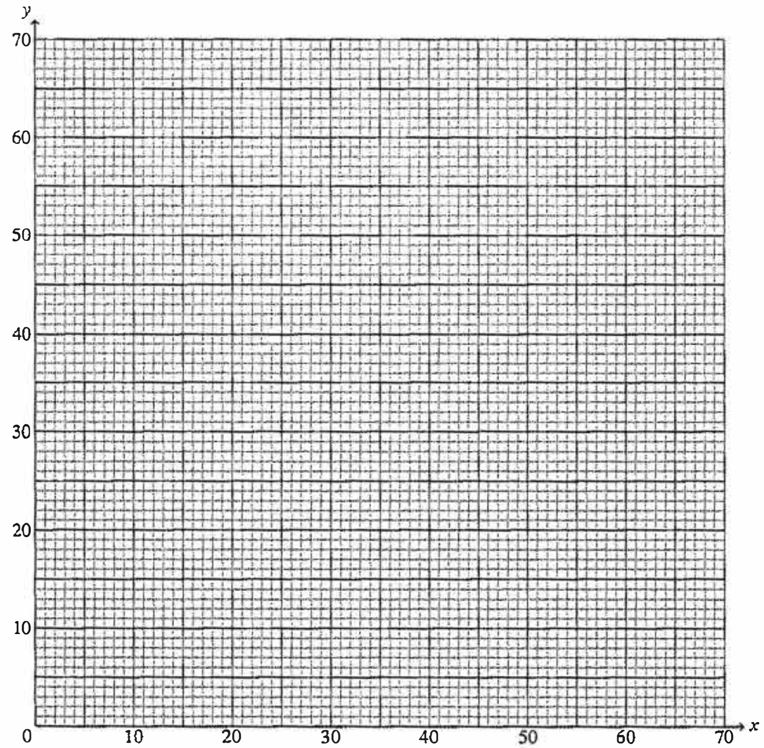


10E.9 HKCEE MA 2000 – I – 15

A company produces two brands, A and B, of mixed nuts by putting peanuts and almonds together. A packet of brand A mixed nuts contains 40 g of peanuts and 10 g of almonds. A packet of brand B mixed nuts contains 30 g of peanuts and 25 g of almonds. The company has 2400 kg of peanuts, 1200 kg of almonds and 70 carton boxes. Each carton box can pack 1000 brand A packets or 800 brand B packets.
The profits generated by a box of brand A mixed nuts and a box of brand B mixed nuts are \$800 and \$1000 respectively. Suppose x boxes of brand A mixed nuts and y boxes of brand B mixed nuts are produced.

(a) Using the graph paper provided, find x and y so that the profit is the greatest.

(b) If the number of boxes of brand B mixed nuts is to be smaller than the number of boxes of brand A mixed nuts, find the greatest profit.



10E.10 HKCEE MA 2001 – I – 15

(a) In Figure (1), shade the region that represents the solution to the following constraints: $\begin{cases} 1 \le x \le 9, \\ 0 \le y \le 9, \\ 5x - 2y > 15. \end{cases}$

(b) A restaurant has 90 tables. Figure (2) shows its floor plan where a circle represents a table. Each table is assigned a 2 digit number from 10 to 99. A rectangular coordinate system is introduced to the floor plan such that the table numbered $10x + y$ is located at (x, y) where x is the tens digit and y is the units digit of the table number. The table numbered 42 has been marked in the figure as an illustration.
The restaurant is partitioned into two areas, one smoking and one non smoking. Only those tables with the digits of the table numbers satisfying the constraints in (a) are in the smoking area.

(i) In Figure (2), shade all the circles which represent the tables in the smoking area.

(ii) *[Probability]*
Two tables are randomly selected, one after another and without replacement from the 90 tables. Find the probability that

(1) the first selected table is in the smoking area;

(2) of the two selected tables, one is in the smoking area, and the other is in the non smoking area and its number is a multiple of 3.

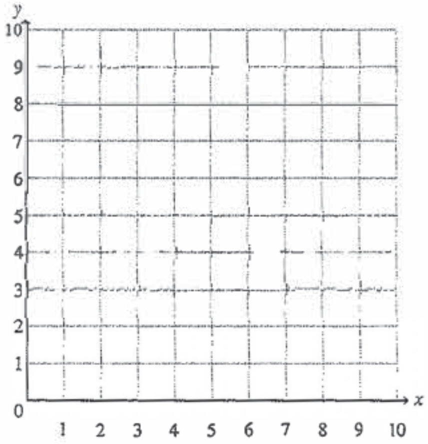


Figure (1)

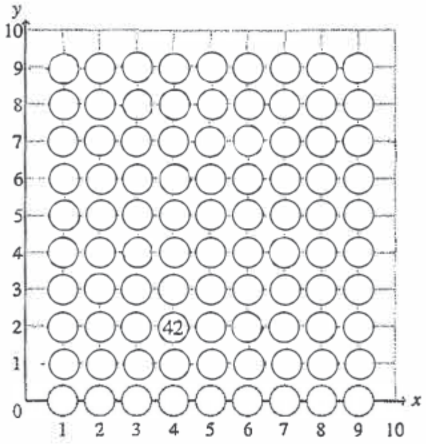


Figure (2)

10 Inequalities and Linear Programming

10A Linear inequalities in one unknown

10A.1 HKCEE MA 1989 – I – 2

(a) $5x+5>3x+2 \Rightarrow 2x> \;3 \Rightarrow x>\frac{-3}{2}$

(b) $\frac{\;3}{2}<x\leq 4$

10A.2 HKCEE MA 1995 – I – 1(a)

$3x+1\geq 7 \Rightarrow 3x\geq 6 \Rightarrow x\geq 2$

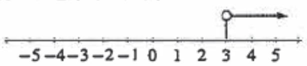
10A.3 HKCEE MA 1999 – I – 3

$3x-4>2(x-1) \Rightarrow 3x-4>2x \;\; 2 \Rightarrow x>2$

'And' with $x<6$: $2<x<6$

10A.4 HKCEE MA 2000 – I – 5

$11-2x<5 \Rightarrow 2x>6 \Rightarrow x>3$



10A.5 HKCEE MA 2002 – I – 7

(a) $3x+6\geq 4+x \Rightarrow 2x\geq -2 \Rightarrow x\geq -1$

(b) $2x-5<0 \Rightarrow x<\frac{5}{2}$

$\therefore$ 'And': $1\leq x<\frac{5}{2}$

10A.6 HKCEE MA 2003 – I – 2

$\frac{3\;\;5x}{4}\geq 2-x \Rightarrow 3-5x\geq 8-4x \Rightarrow x\leq -5$

$x+8>0 \Rightarrow x>-8$

$\therefore$ 'And': $-8<x\leq \;5$

10A.7 HKCEE MA 2005 – I – 4

$-3x+1>4x-20 \Rightarrow 7x<21 \Rightarrow x<3$

$2x+1\geq 0 \Rightarrow x\geq\frac{-1}{2}$

$\therefore$ 'And': $\frac{-1}{2}\leq x<3$

10A.8 HKCEE MA 2006 – I – 2

(a) $6x+6<x+25 \Rightarrow 5x<19 \Rightarrow x<\frac{19}{5}$

(b) 3

10A.9 HKCEE MA 2008 – I – 2

(a) $14x\geq 10x+35 \Rightarrow 4x\geq 35 \Rightarrow x\geq\frac{35}{4}$

(b) 9

10A.10 HKCEE MA 2010 – I – 2

(a) $29x-22\leq 21x \Rightarrow 8x\leq 22 \Rightarrow x\leq\frac{11}{4}$

(b) 2

10A.11 HKDSE MA 2012 – I – 6

(a) $\frac{4x+6}{7}>2(x\;\;3) \Rightarrow 4x+6>14x-42 \Rightarrow x<\frac{24}{5}$

$2x\;\;10\leq 10 \Rightarrow x\leq 10$

'And': $x<\frac{24}{5}$

(b) 4 (1, 2, 3 and 4)

10A.12 HKDSE MA 2013 – I – 5

(a) $\frac{19-7x}{3}>23-5x \Rightarrow 19-7x>69-15x \Rightarrow x>\frac{25}{4}$

(b) $18\;\;2x\geq 0 \Rightarrow x\leq 9$

$\therefore$ Integers satisfying both: 7, 8 and 9

10A.13 HKDSE MA 2015 – I – 5

(a) $\frac{7-3x}{5}\leq 2(x+2) \Rightarrow 7-3x\leq 10x+20 \Rightarrow x\geq -1$

$4x-13>0 \Rightarrow x>\frac{13}{4}$

$\therefore$ 'And'. $x>\frac{13}{4}$

(b) 4

10A.14 HKDSE MA 2016 – I – 6

(a) $x+6<6(x+11) \Rightarrow x>-12$

$\therefore$ 'Or': $x>-12$

(b) -1

10A.15 HKDSE MA 2017 – I – 5

(a) $7(x\;\;2)\leq\frac{11x+8}{3} \Rightarrow 21x\;\;42\leq 11x+8 \Rightarrow x\leq 5$

$6\;\;x<5 \Rightarrow x>1$

$\therefore$ 'And': $1<x\leq 5$

(b) 4 (2, 3, 4 and 5)

10A.16 HKDSE MA 2018 – I – 6

(a) $\frac{3-x}{2}>2x+7 \Rightarrow 3-x>4x+14 \Rightarrow x<\frac{-11}{5}$

$x+8\geq 0 \Rightarrow x\geq -8$

$\therefore$ 'And': $-8\leq x<\frac{-11}{5}$

(b) -3

10A.17 HKDSE MA 2019 – I – 6

(a) $\frac{7x+26}{4}\leq 2(3x\;\;1) \Rightarrow 7x+26\leq 24x\;\;8 \Rightarrow x\geq 2$

(b) $45-5x\geq 0 \Rightarrow x\leq 9$

$\therefore$ 'And': $2\leq x\leq 9$

$\therefore$ 8 (2. 3, 4, 5, 6, 7, 8, 9)

10A.18 HKDSE MA 2020 – I – 6

6a

$3-x>\frac{7-x}{2}$ or $5+x>4$

$6-2x>7-x$ or $x>-1$

$x<-1$ or $x>-1$

Therefore, x can be any real numbers except -1.

b -2

10B Quadratic inequalities in one unknown

10B.1 HKCEE MA 1982(1/2/3) – I – 3

$2x^2\;\;x-36<0$

$(2x-9)(x+4)<0 \Rightarrow \;4<x<\frac{9}{2}$

10B.2 HKCEE MA 1988 – I – 3

$2x^2-5x\geq 0$

$x(2x-5)\geq 0 \Rightarrow x\leq 0$ or $x\geq\frac{5}{2}$

10B.3 HKCEE MA 1990 – I – 4

(a) (i) $6x+1\geq 2x-3 \Rightarrow 4x\geq \;4 \Rightarrow x\geq -1$

(ii) $(2\;\;x)(x+3)>0 \Rightarrow -3<x<2$

(b) $\;1\leq x<2$

10B.4 HKCEE MA 1993 – I – 4

$x^2\;\;x\;\;2<0 \Rightarrow (x+1)(x\;\;2)<0 \Rightarrow -1<x<2$

Hence, $\;1<y-100<2 \Rightarrow 99<y<102$

10B.5 HKCEE MA 1996 – I – 5

(i) $x+5>8 \Rightarrow x>3$

(ii) $(x\;\;2)(x-4)<0 \Rightarrow 2<x<4$

Hence, $3<x<4$

10B.6 HKCEE MA 1997 – I – 4

(i) $2x>17 \Rightarrow x>\frac{17}{2}$

(ii) $(x\;\;9)(x\;\;7)>0 \Rightarrow x<7$ or $x>9$

Hence, $x>9$

10B.7 HKCEE MA 2001 – I – 4

$x^2+x-6>0 \Rightarrow (x+3)(x\;\;2)>0 \Rightarrow x<-3$ or $x>2$

10B.8 HKCEE AM 1985 – I – 3

$x^2-ax-4\leq 0 \Rightarrow \frac{a\;\;\sqrt{a^2+16}}{2}\leq x\leq\frac{a+\sqrt{a^2+16}}{2}$

$\therefore \frac{a+\sqrt{a^2+16}}{2}=4 \Rightarrow a^2+16=(8-a)^2 \Rightarrow a=3$

$\Rightarrow$ Least possible value of $x=\frac{(3)-\sqrt{(3)^2+16}}{2}=\;\;1$

10B.9 HKCEE AM 1986 – I – 7

(a) $x>\frac{3}{x}+2 \Rightarrow x^2>3+2x$

$\Rightarrow x^2-2x-3>0 \Rightarrow x<-1$ or $x>3$

$\because x>0$

$\therefore x>3$

(b) $x>\frac{3}{x}+2 \Rightarrow x^2<3+2x$

$\Rightarrow x^2-2x-3<0 \Rightarrow -1<x<3$

$\because x<0$

$\therefore -1<x<0$

10B.10 (HKCEE AM 1994 – I – 1)

(a) $\frac{2(x+1)}{x-2}\geq 1 \Rightarrow 2x+2\geq x\;\;2 \Rightarrow x\geq -4$

$\because x>2$

$\therefore x\geq -4$ 'and' $x>2 \Rightarrow x>2$

(b) $\frac{2(x+1)}{x\;\;2}>1 \Rightarrow 2x+2\leq x-2 \Rightarrow x\leq -4$

$\because x<2$

$\therefore x\leq\;\;4$ 'and' $x<2 \Rightarrow x\leq -4$

10B.11 HKCEE AM 1995 – I – 4

Solve the inequality $x-\frac{5}{x}>4$ for each of the following cases:

(a) $x-\frac{5}{x}>4 \Rightarrow x^2-5>4x$

$\Rightarrow x^2-4x-5>0 \Rightarrow x<\;\;1$ or $x>5$

$\because x>0$

$\therefore x>5$

(b) $x-\frac{5}{x}>4 \Rightarrow x^2\;\;5<4x$

$\Rightarrow x^2-4x\;\;5<0 \Rightarrow -1<x<5$

$\because x<0$

$\therefore -1<x<0$

10B.12 (HKCEE AM 1996 – I – 3)

(a) $\frac{2x-3}{x+1}<1 \Rightarrow 2x-3\leq x+1 \Rightarrow x\leq 4$

$\because x>-1$

$\therefore \;\;1<x\leq 4$

(b) $\frac{2x\;\;3}{x+1}<1 \Rightarrow 2x-3\geq x+1 \Rightarrow x\geq 4$

$\because x<-1$

$\therefore$ No solution

10B.13 HKCEE AM 1998 – I – 6(a)

$x^2-6x-16>0 \Rightarrow (x\;\;8)(x+2)>0 \Rightarrow x<-2$ or $x>8$

10B.14 (HKCEE AM 1999 – I – 2)

(a) $\frac{x}{x-1}>2 \Rightarrow x>2(x-1) \Rightarrow x<2$

$\because x>1$

$\therefore 1<x<2$

(b) $\frac{x}{x\;\;1}>2 \Rightarrow x<2(x-1) \Rightarrow x>2$

$\because x<1$

$\therefore$ No solution

10B.15 (HKCEE AM 2000 – I – 1)

Solve the inequality $\frac{1}{x}\geq 1$ for each of the following cases:

(a) $\frac{1}{x}>1 \Rightarrow 1\geq x \Rightarrow x\leq 1$

$\because x>0$

$\therefore 0<x\leq 1$

(b) $\frac{1}{x}\geq 1 \Rightarrow 1\leq x \Rightarrow x\geq 1$

$\because x<0$

$\therefore$ No solution

10B.16 HKCEE AM 2011 – 3

Solve the following inequalities:

(a) $5x-3>2x+9 \Rightarrow 3x>12 \Rightarrow x>4$

(b) $x(x-8)\leq 20 \Rightarrow x^2-8x-20\leq 0 \Rightarrow -2\leq x\leq 10$

(c) 'Or': $x\geq -2$

10C Problems leading to quadratic inequalities in one unknown

10C.1 HKCEE MA 1983(B) – I – 14

(a) $\begin{cases}\alpha+\beta=2m\\ \alpha\beta=n\end{cases}$

(i) $(m-\alpha)+(m-\beta)=2m-(\alpha+\beta)=2m-(2m)=0$

(ii) $(m-\alpha)(m-\beta)=m^2-(\alpha+\beta)m+\alpha\beta$
$=m^2-(2m)m+(n)=n-m^2$

(b) By (a), the equation is
$x^2-(\text{sum})x+(\text{product})=0$
$x^2-(0)x+(n-m^2)=0 \Rightarrow x^2+n-m^2=0$

(c) $x^2-2mx+4=0$
Real roots $\Rightarrow \Delta\geq 0$
$(2m)^2-4(4)\geq 0$
$m^2\geq 4 \Rightarrow m\leq -2$ or $m\geq 2$

10C.2 HKCEE MA 1985(A/B) – I – 13

(a) $DE^2=BD^2+BE^2-2\cdot BD\cdot BE\cos\angle B$
$=(2-x)^2+x^2-2(2-x)(x)\cos 60^\circ$
$=3x^2-6x+4$

(b) Area of $\triangle DEF=\frac{1}{2}DE\cdot DE\sin 60^\circ$
$=\frac{1}{2}(3x^2-6x+4)\cdot\frac{\sqrt{3}}{2}$
$=\frac{\sqrt{3}}{4}(3x^2-6x+4)$
$=\frac{3\sqrt{3}}{4}\left(x^2-2x+\frac{4}{3}\right)$
$=\frac{3\sqrt{3}}{4}\left(x^2-2x+1+\frac{1}{3}\right)$
$=\frac{3\sqrt{3}}{4}(x-1)^2+\frac{\sqrt{3}}{4}$
$\therefore$ Minimum area is attained when $x=1$.

(c) $\frac{3\sqrt{3}}{4}(x-1)^2+\frac{\sqrt{3}}{4}\leq\frac{\sqrt{3}}{3}$
$(x-1)^2\leq\frac{1}{9}$
$\frac{-1}{3}\leq x-1\leq\frac{1}{3} \Rightarrow \frac{2}{3}\leq x\leq\frac{4}{3}$

10C.3 HKCEE MA 1987(B) – I – 14

(a) Let $p=ax+\frac{b}{x}$.
$\begin{cases}7=2a+\frac{b}{2} \Rightarrow 4a+b=14\\ 8=3a+\frac{b}{3} \Rightarrow 9a+b=24\end{cases} \Rightarrow \begin{cases}a=2\\ b=6\end{cases}$
$\therefore\ p=2x+\frac{6}{x}$.
When $x=4$, $p=2(4)+\frac{6}{(4)}=\frac{19}{2}$.

(b) $2x+\frac{6}{x}<13$
$2x^2+6<13x \quad (\because \text{ given } x>0)$
$2x^2-13x+6<0 \Rightarrow \frac{1}{2}<x<6$

10C.4 HKCEE MA 1992 – I – 6

$\Delta>0$
$(2k)^2+4(k+6)>0$
$(k+2)(k+3)>0 \Rightarrow k<-3$ or $k>-2$

10C.5 HKCEE MA 2003 – I – 10

(a) Let $V=hL+kL^2$.
$\begin{cases}30=10h+100k\\ 75=15h+225k\end{cases} \Rightarrow \begin{cases}h=-1\\ k=0.4\end{cases} \Rightarrow V=0.4L^2-L$

(b) $0.4L^2-L\geq 30$
$2L^2-5L-150\geq 0 \Rightarrow L\leq\frac{-15}{2}$ or $L\geq 10$
Since $5\leq L\leq 25$, the solution is $10\leq L\leq 25$.

10C.6 HKCEE MA 2004 – I – 10

(a) Let $y=hx+kx^2$.
$\begin{cases}3=3h+9k\\ 12=4h+16k\end{cases} \Rightarrow \begin{cases}h=-5\\ k=2\end{cases} \Rightarrow y=2x^2-5x$

(b) $2x^2-5x<42 \Rightarrow 2x^2-5x-42<0 \Rightarrow -\frac{7}{2}<x<6$
$\therefore$ Possible values of x are -3, -2, -1, 0, 1, 2, 3, 4 and 5.

10C.7 HKCEE AM 1983 – I – 1

$x^2+4x+2+\lambda(2x+1)=0 \Rightarrow x^2+2(2+\lambda)x+(2+\lambda)=0$
No real roots $\Rightarrow \Delta<0$
$4(2+\lambda)^2-4(2+\lambda)<0$
$\lambda^2+3\lambda+2<0 \Rightarrow -2<\lambda<-1$

10C.8 HKCEE AM 1988 – I – 5

$\Delta=(4m)^2-4(4m+15)=16m^2-16m-60$
If $f(x)>0$ for all real x, $\Delta<0$
$4(4m^2-4m-15)<0$
$(2m+3)(2m-5)<0 \Rightarrow -\frac{3}{2}<m<\frac{5}{2}$

10C.9 HKCEE AM 1988 – I – 10

(a) (i) For $f(x)$, $\begin{cases}\text{Sum of rts}=-2\\ \text{Prod of rts}=-1\end{cases}$

For $g(x)$, $\begin{cases}\text{Sum of rts}=2k\\ \text{Prod of rts}=k^2-6\end{cases}$

$PQ=$ Difference of rts of $f(x)$
$=\sqrt{(-2)^2-4(-1)}=\sqrt{8}$
$RS=$ Difference of rts of $g(x)$
$=\sqrt{(2k)^2-4(k^2-6)}=\sqrt{24}$

(ii) Mid-pt of $RS=\left(\frac{\text{Sum of rts}}{2},0\right)=(k,0)$

If this is also the mid-point of PQ, $k=\frac{-2}{2}=-1$.

(b) $\begin{cases}y=f(x)\\ y=g(x)\end{cases} \Rightarrow x^2+2x-1=-x^2+2kx-k^2+6$
$2x^2+2(1-k)x+k^2-7=0 \quad \dots (*)$
$\Delta=0$
$4(1-k)^2-8(k^2-7)=0$
$k^2+2k-15=0 \Rightarrow k=-5$ or 3
For $k=-5$, $(*)$ becomes $2x^2+12x+18=0$
$2(x+3)^2=0$
$x=-3$
$\Rightarrow$ Intersection $=(-3,(-3)^2+2(-3)-1)=(-3,2)$
For $k=3$, $(*)$ becomes $2x^2-4x+2=0$
$2(x-1)^2=0$
$x=1$
$\Rightarrow$ Intersection $=(1,1^2+2(1)-1)=(1,2)$

(c) $f(x)>g(x)$
$2x^2+2(1-k)x+k^2-7>0$
If this is true for all real x, $\Delta<0$
$k^2+2k-15>0$
$k<-5$ or $k>3$

10C.10 HKCEE AM 1991 – I – 7

(a) From the first equation, $p+q=2-k$
From the second equation, $pq+k(p+q)=1$
$pq=1-k(2-k)$
$=(k+1)^2$

(b) Sum of roots $=p+q=2-k$
Product of roots $=(k+1)^2$
$\therefore$ Required equation: $x^2-(2-k)x+(k+1)^2=0$
Hence, $\Delta\geq 0$
$(k-2)^2-4(k+1)^2\geq 0$
$3k^2+4k\leq 0 \Rightarrow \frac{-4}{3}\leq k\leq 0$

10C.11 HKCEE AM 1991 – I – 9

(a) $g(x)=-2x^2-12x-23=-2(x^2+6x+9-9)-25$
$=-2(x+3)^2-5$
$\leq -5<0$

(b) (i) $f(x)+kg(x)=0$
$(x^2+2x-2)+k(-2x^2-12x-23)=0$
$(1-2k)x^2+2(1-6k)x-(2+23k)=0$
Equal rts $\Rightarrow \Delta=0$
$4(1-6k)^2+4(1-2k)(2+23k)=0$
$10k^2-7k-3=0$
$k=1$ or $\frac{-3}{10}$
$\therefore\ k_1=1,\ k_2=\frac{-3}{10}$

(ii) $f(x)+k_1g(x)$
$=(x^2+2x-2)-(2x^2+12x+23)$
$=-x^2-10x-25=-(x+5)^2\leq 0$
$f(x)+k_2g(x)$
$=(x^2+2x-2)+\frac{3}{10}(2x^2+12x+23)$
$=\frac{8}{5}\left(x^2+\frac{7}{2}x+\frac{49}{16}\right)=\frac{8}{5}\left(x+\frac{7}{4}\right)^2\geq 0$

(c) $f(x)+k_1g(x)\leq 0$
$f(x)\leq -g(x)$
$\frac{f(x)}{g(x)}\geq -1 \quad (\because g(x)<0 \text{ by (a)})$
$\therefore$ Least value $=-1$
(attained when $f(x)+k_1g(x)=0 \Leftrightarrow x=-5$)
$f(x)+k_2g(x)\geq 0$
$f(x)\geq\frac{3}{10}g(x)$
$\frac{f(x)}{g(x)}<\frac{3}{10}$
$\therefore$ Greatest value $=\frac{3}{10}$
(attained when $\left(x+\frac{7}{4}\right)^2=0 \Leftrightarrow x=-\frac{7}{4}$)

10C.12 HKCEE AM 1995 – I – 1

$\Delta=(1-m)^2-4(2m-5)=m^2-10m+21$
If $f(x)>0$ for all real x, $\Delta<0$
$m^2-10m+21<0$
$(m-3)(m-7)<0 \Rightarrow 3<m<7$

10C.13 (HKCEE AM 1995 – I – 10)

(a) $f(\alpha)=g(\alpha)$
$12\alpha^2+2p\alpha-q=12\alpha^2+2q\alpha-p$
$2\alpha\cancel{(p-q)}=-\cancel{(p-q)} \quad (\because p,q \text{ are distinct})$
$2\alpha=-1 \Rightarrow \alpha=\frac{-1}{2}$

(b) $\alpha+\beta=-\frac{2p}{12} \Rightarrow \beta=\frac{-p}{6}+\frac{1}{2}$
$\alpha\gamma=\frac{-p}{12} \Rightarrow \gamma=\frac{-p}{12}\div\frac{-1}{2}=\frac{p}{6}$

(c) (i) $\beta^3+\gamma^3=(\beta+\gamma)(\beta^2-\beta\gamma+\gamma^2)$
$=\left(\frac{1}{2}\right)\left[\frac{p^2}{36}-\frac{p}{6}+\frac{1}{4}-\frac{p}{6}\left(\frac{-p}{6}+\frac{1}{2}\right)+\frac{p^2}{36}\right]$
$=\frac{1}{2}\left(\frac{p^2}{12}-\frac{p}{4}+\frac{1}{4}\right)$
Thus, the given inequality becomes
$-\frac{7}{24}<\frac{p^2}{24}-\frac{p}{8}+\frac{1}{8}<\frac{7}{24}$
$\Rightarrow -7<p^2-3p+3<7$
$\Rightarrow \begin{cases}p^2-3p-4<0\\ p^2-3p+10>0\end{cases}$
$\Rightarrow \begin{cases}-1<p<4\\ \text{All real nos}\end{cases} \Rightarrow -1<p<4$

(ii) $p=3$ and $q=0$
$p=2$ and $q=1$ (since $p+q=3$)

10C.14 (HKCEE AM 1996 – I – 8)

The graph of $y=x^2-(k-2)x+k+1$ intersects the x-axis at two distinct points $(\alpha,0)$ and $(\beta,0)$, where k is real.

(a) Two distinct roots $\Rightarrow \Delta>0$
$(k-2)^2-4(k+1)>0$
$k^2-8k>0 \Rightarrow k<0$ or $k>8$

(b) $-5<\alpha+\beta<5 \Rightarrow -5<k-2<5 \Rightarrow -3<k<7$
$\therefore$ 'And': $-3<k<0$

10C.15 (HKCEE AM 1997 – I – 8)

(a) $\Delta=(k+2)^2-8(k-1)=k^2-4k+12=(k-2)^2+8$
$\geq 8>0$
$\therefore$ The roots are real and distinct.

(b) $\begin{cases}\alpha+\beta=-(k+2)\\ \alpha\beta=2(k-1)\end{cases}$
$(\alpha-\beta)^2>3^2$
$(\alpha+\beta)^2-4\alpha\beta>9$
$(k+2)^2-8(k-1)>9$
$(k-2)^2+8>9$
$(k-2)^2>1 \Rightarrow k-2<-1$ or $k-2>1$
$\Rightarrow k<1$ or $k>3$

10C.16 HKCEE AM 1999 – I – 4

Let $f(x)=2x^2+2(k-4)x+k$, where k is real.

(a) $\Delta=4(k-4)^2-8k=4k^2-40k+64$

(b) No intersection with x-axis $\Rightarrow \Delta<0$
$4(k^2-10k+16)<0$
$(k-2)(k-8)<0 \Rightarrow 2<k<8$

10C.17 HKCEE AM 2005 – 5

$x^2\ \ x\ \ 1>k(x-2)\ \Rightarrow\ x^2-(1+k)x+(2k-1)>0$

If this is true for all real x, $\Delta<0$

$(1+k)^2-4(2k-1)<0$

$k^2-6k+5<0\ \Rightarrow\ 1<k<5$

10C.18 HKCEE AM 2006 – 4

If $kx^2+x+k>0$ is true for all real x,

$\Delta<0$ and $k>0$

$1^2-4k^2<0$

$k^2>\frac{1}{4}\ \Rightarrow\ k<\frac{-1}{2}$ or $k>\frac{1}{2}$

$\therefore\ k>\frac{1}{2}$

10C.19 HKCEE AM 2008 – 4

$\Delta<0$

$(-1)^2-4(k)(9k)<0$

$1-36k^2<0$

$k^2>\frac{1}{36}\ \Rightarrow\ k<\frac{-1}{6}$ or $k>\frac{1}{6}$ (rejected)

10C.20 HKCEE AM 2010 – 4

$k-1>0$ and $\Delta\le 0$

$k^2-4k(k\ \ 1)\le 0$

$3k^2-4k\ge 0\ \Rightarrow\ k\le 0$ or $k\ge\frac{4}{3}$

$\Rightarrow\ k\ge\frac{4}{3}$

10D Linear programming (with given region)

10D.1 HKCEE MA 1984(A/B) – I – 8

(a) (Two-pt form) $\ell_3:\frac{y\ \ 0}{x-10}=\frac{10-0}{0-10}\ \Rightarrow\ y=\ \ x+10$

(Or intercept form) $\ell_3:\frac{x}{10}+\frac{y}{10}=1\ \Rightarrow\ y=-x+10$

(b) $A:\begin{cases}\ell_1:2y=3\\ \ell_2:3x-2y=0\end{cases}\Rightarrow\begin{cases}x=1\\ y=\frac{3}{2}\end{cases}\Rightarrow A=\left(1,\frac{3}{2}\right)$

$B:\begin{cases}\ell_2:3x\ \ 2y=0\\ \ell_3:y=-x+10\end{cases}\Rightarrow\begin{cases}x=4\\ y=6\end{cases}\Rightarrow B=(4,6)$

$C:\begin{cases}\ell_1:2y=3\\ \ell_3:y=-x+10\end{cases}\Rightarrow\begin{cases}x=\frac{17}{2}\\ y=\frac{3}{2}\end{cases}\Rightarrow A=\left(\frac{17}{2},\frac{3}{2}\right)$

(c) $\begin{cases}2y\ge 3\\ 3x-2y\ge 0\\ y\le\ \ x+10\end{cases}$

(d) At A, $P=(1)+2\left(\frac{3}{2}\right)-5=-1$

At B, $P=(4)+2(6)-5=11$

At C, $P=\left(\frac{17}{2}\right)+2\left(\frac{3}{2}\right)-5=\frac{13}{2}$

$\therefore$ Max of $P=11$, min of $P=\ \ 1$

10D.2 HKCEE MA 1988 – I – 12

(a) (Two-pt form) $L_3:\frac{y-4}{x-0}=\frac{0-4}{3-0}\ \Rightarrow\ 4x+3y=12$

(Or intercept form) $L_3:\frac{x}{3}+\frac{y}{4}=1\ \Rightarrow\ 4x+3y=12$

(b) $\begin{cases}x\le 3\\ y\le 4\\ 4x+3y\ge 12\end{cases}$

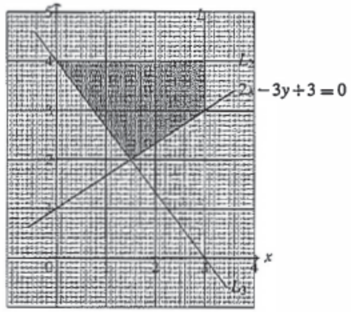
(c) At $(0,4)$, $P=(0)+4(4)=16$

At $(3,4)$, $P=(3)+4(4)=19$

At $(3,0)$, $P=(3)+4(0)=3$

$\therefore$ Greatest $P=19$, least $P=3$

(d)



At $(0,4)$, $P=(0)+4(4)=16$

At $(3,4)$, $P=(3)+4(4)=19$

At $(3,3)$, $P=(3)+4(3)=15$

At $(1.5,2)$, $P=(1.5)+4(2)=9.5$

$\therefore$ Least $P=9.5$

10D.3 HKCEE MA 1990 – I – 5

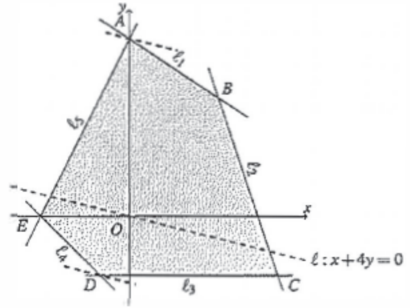
By sliding the dashed line, P attains its greatest value at A and least value at D.

$A:\begin{cases}\ell_1:2x+3y=18\\ \ell_5:2x\ \ y=-6\end{cases}\Rightarrow\begin{cases}x=0\\ y=6\end{cases}$

$\therefore$ Greatest $P=(0)+4(6)-2=22$

$D:\begin{cases}\ell_3:y=\ \ 2\\ \ell_4:x+y=-3\end{cases}\Rightarrow\begin{cases}x=\ \ 1\\ y=-2\end{cases}$

$\therefore$ Least $P=(-1)+4(-2)-2=-11$



10D.4 HKCEE MA 1991 – I – 8

(a) (Slope-int form) $L_2:y=x+2$

(Two-pt form) $L_3:\frac{y\ \ 0}{x-5}=\frac{5-0}{0-5}\ \Rightarrow\ y=-x+5$

(Or intercept form) $L_3:\frac{x}{5}+\frac{y}{5}=1\ \Rightarrow\ y=\ \ x+5$

(b) $\begin{cases}x\le 4\\ y\le x+2\\ y\ge -x+5\end{cases}$

(c) (i) At $(4,6)$, $P=(4)+2(6)-3=13$

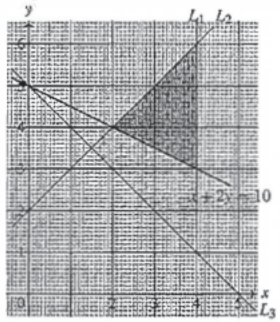
At $(4,1)$, $P=(4)+2(1)-3=3$

At $(1.5,3.5)$, $P=(1.5)+2(3.5)-3=5.5$

$\therefore$ Min of $P=3$, attained at $(4,1)$

(ii) $P=x+2y-3\ge 7\ \Rightarrow\ x+2y\ge 10$

Draw it into the diagram:

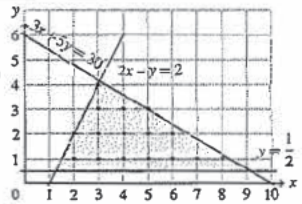


The range of x that covers the new feasible region is $2\le x\le 4$.

10D.5 HKCEE MA 1992 – I – 3

(a) $\begin{cases}y\ge\frac{1}{2}\\ 2x\ \ y\ge 2\\ 3x+5y\le 30\end{cases}$

(b) 16

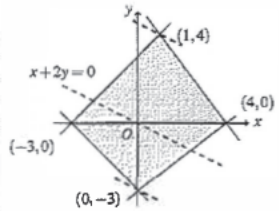


10D.6 HKCEE MA 1993 – I – 1(d)

By sliding the given line,

(i) Greatest value $=(1)+2(4)=9$, at $(1,4)$

(ii) Least value $=(0)+2(-3)=-6$, at $(0,-3)$



10D.7 HKCEE MA 1995 – I – 12

(a) (i) $20x+40y\ge 240\ \Rightarrow\ x+2y\ge 12$

(ii) $25x+37.5y\le 300\ \Rightarrow\ 2x+3y\le 24$

(iii) $x+y\le 10$

(b) (x and y must be integers!)

$(3,5)$, $(3,6)$, $(4,4)$, $(4,5)$, $(5,4)$, $(6,3)$, $(6,4)$, $(7,3)$

(c) Cost $=25x+37.5y$

By sliding the line $25x+37.5y=0\ \Rightarrow\ 2x+3y=0$, the least cost is attained at $(4,4)$.

Least cost $=25(4)+37.5(4)=(\$)250$.

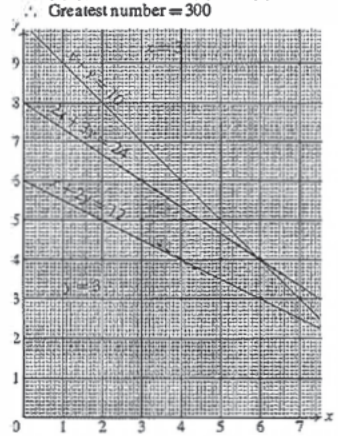
(d) (i) As Cost $=300$, the only two points lying on the line $25x+37.5y=300$ are $(x,y)=(3,6)$ and $(6,4)$.

(ii) Number of chocolates $=20x+40y$

At $(3,6)$, Number $=20(3)+40(6)=300$

At $(6,4)$, Number $=20(6)+40(4)=280$

$\therefore$ Greatest number $=300$



10D.8 HKCEE MA 1996 – I – 9

(a) $C:\begin{cases}L_1: 3x+2y-7=0\\ L_3: 2x-y-7=0\end{cases} \Rightarrow \begin{cases}x=3\\ y=-1\end{cases} \Rightarrow (3,-1)$

(b) $\begin{cases}3x+2y-7\ge 0\\ 3x-5y+7\ge 0\\ 2x-y-7\le 0\end{cases}$

(c) At A, $2(1)-2(2)-7=-9$
At B, $2(6)-2(5)-7=-5$
At C, $2(3)-2(-1)-7=1 \quad\Rightarrow\quad$ Max value $=1$

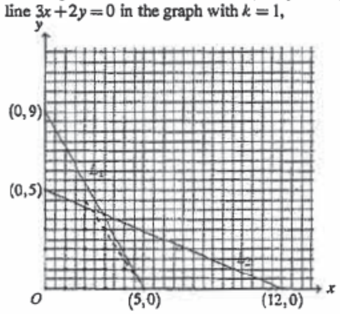
10D.9 HKCEE MA 2002 – I – 17

(a) $L_1: \dfrac{x}{5k}+\dfrac{y}{9k}=1 \Rightarrow 9x+5y=45k$

$L_2: \dfrac{x}{12k}+\dfrac{y}{5k}=1 \Rightarrow 5x+12y=60k$

(b) (i) $\begin{cases}45x+25y\le 225 \Rightarrow 9x+5y\le 45\\ 50x+120y\le 600 \Rightarrow 5x+12y\le 60\\ x \text{ and } y \text{ are non-negative integers.}\end{cases}$

Let the profit be $P=3000x+2000y$. By sliding the line $3x+2y=0$ in the graph with $k=1$,

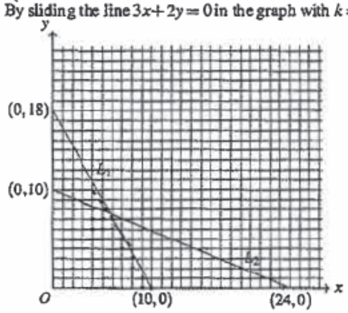


the greatest possible profit is attained at (3,3) and (5,0)

$\therefore$ Greatest profit $=3000(5)+0=(\$)15000$

(ii) $\begin{cases}45x+25y\le 450 \Rightarrow 9x+5y\le 90\\ 50x+120y\le 1200 \Rightarrow 5x+12y\le 120\\ x \text{ and } y \text{ are non-negative integers.}\end{cases}$

By sliding the line $3x+2y=0$ in the graph with $k=2$,



the greatest possible profit is attained at (6,7)

$\therefore$ Greatest profit $=3000(6)+2000(7)=(\$)32000$

10D.10 HKCEE MA 2009 – I – 16

(a) (i) $L_1: \dfrac{y-24}{x-12}=\dfrac{24-16}{12-8}=2 \Rightarrow y=2x$

$L_2: y-24=\dfrac{-1}{2}(x-12) \Rightarrow x+2y-60=0$

(ii) $\begin{cases}y\le 2x\\ x+2y\le 60\\ x\ge 8\\ y\ge 10\end{cases}$

(b) The constraints are $\begin{cases}x\ge 8\\ y\ge 10\\ y\le 2x\\ 4x+8y\le 240 \Rightarrow x+2y\le 60\\ x \text{ and } y \text{ are integers.}\end{cases}$

Let the profit be $P=4000x+6000y$.
At $(8,16)$, $P=4000(8)+6000(16)=128000$
At $(12,24)$, $P=4000(12)+6000(24)=192000$
At $(8,10)$, $P=4000(8)+6000(10)=92000$
At $(40,10)$, $P=4000(40)+6000(10)=220000$
$\because$ Max profit $=\$220000<\$230000$
$\therefore$ NO.

10D.11 HKDSE MA 2014 – I – 18

(a) $L_2: \dfrac{y-90}{x-45}=\dfrac{90-0}{45-180}=\dfrac{-2}{3} \Rightarrow 2x+3y-360=0$

$\therefore$ The constraints are $\begin{cases}6x+7y\le 900\\ 2x+3y\le 360\\ x\ge 0\\ y\ge 0\end{cases}$.

(b) The constraints are $\begin{cases}6x+7y\le 900\\ 2x+3y\le 360\\ x \text{ and } y \text{ are non-negative integers.}\end{cases}$

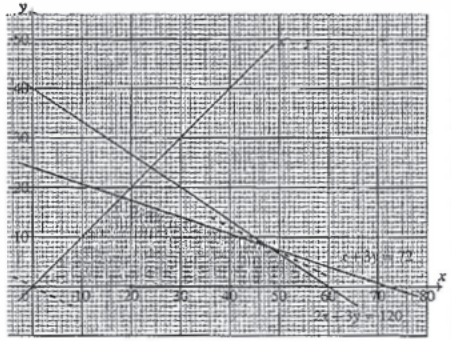
Let the profit be $P=440x+665y$.
At $(0,0)$, $P=440(0)+665(0)=0$
At $(0,120)$, $P=440(0)+665(120)=79800$
At $(45,90)$, $P=440(45)+665(90)=79650$
At $(150,0)$, $P=440(150)+665(0)=66000$
$\because$ Max profit $=\$79800$
$\therefore$ NO.

10E Linear Programming (without given region)

10E.1 HKCEE MA 1980(1/1*/3) – I – 12

$\begin{cases}10x+30y\le 720 \Rightarrow x+3y\le 72\\ x+1.5y\le 60 \Rightarrow 2x+3y\le 120\\ x\ge y\\ x \text{ and } y \text{ are non-negative integers.}\end{cases}$

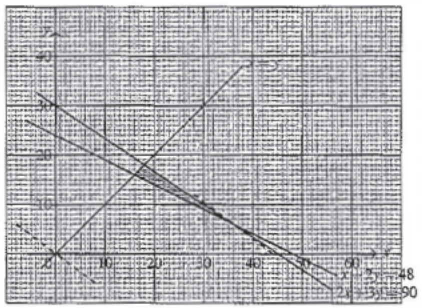
Let the profit be $P=kx+2ky$. By sliding the line $P=0$, the maximum is attained at $(48,8)$.
$\therefore$ 48 economy- and 8 first-class seats respectively



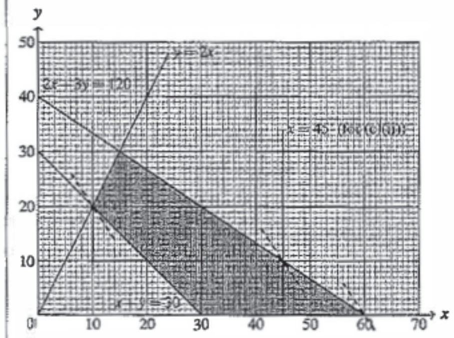
10E.2 HKCEE MA 1981(1/2/3) – I – 8

$\begin{cases}x+2y\ge 48\\ 10x+15y\le 450 \Rightarrow 2x+3y\le 90\\ x\ge y\\ x \text{ and } y \text{ are non-negative integers.}\end{cases}$

Let the profit be $P=300x+400y$. By sliding the line $P=0$, the maximum is attained at $(x,y)=(36,6)$.



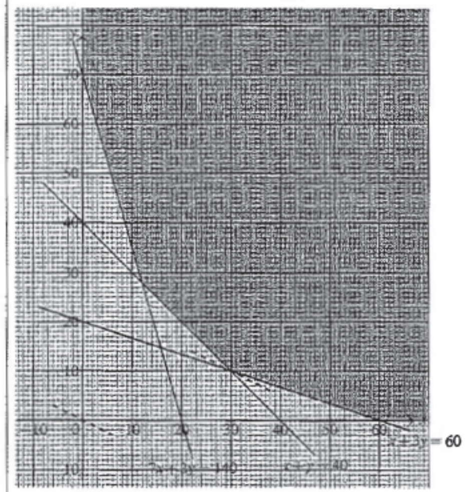
10E.3 HKCEE MA 1983(A/B) – I – 12



(c) (i) Max of $P=3(60)+2(0)=180$
Min of $P=3(10)+2(20)=70$

(ii) Max of $P=3(45)+2(10)=155$
Min of $P=3(10)+2(20)=70$ (unchanged)

10E.4 HKCEE MA 1986(A/B) – I – 11



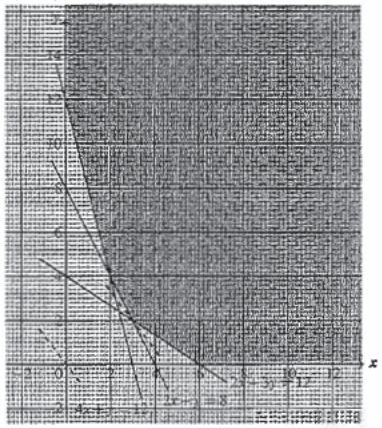
(b) Constraints: $\begin{cases}x+y\ge 40\\ x+3y\ge 60\\ 7x+2y\ge 140\\ x \text{ and } y \text{ are non-negative integers.}\end{cases}$

Let the cost be $C=1000x+2000y$. By sliding the line $C=0$, the minimum is attained at $(x,y)=(30,10)$.
$\therefore$ 30 days for A and 10 days for B

10E.5 HKCEE MA 1987(A/B) – I – 12

(a) $\begin{cases} 20000x+5000y\geq 60000 \\ 6000x+3000y\geq 24000 \end{cases}$

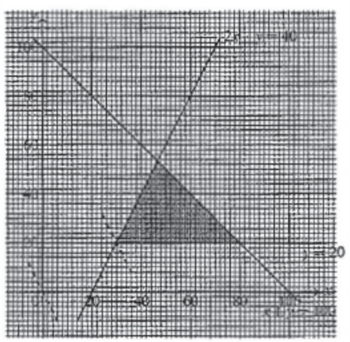
(b) $\begin{cases} x\geq 0, \quad y\geq 0 \\ 2x+3y\geq 12 \\ 4x+y\geq 12 \\ 2x+y\geq 8 \end{cases}$



(c) Let the cost be $C=4000x+3000y$. By sliding the line $C=0$, the minimum is attained at $(3,2)$.
$\therefore$ Least cost $=4000(3)+3000(2)=(\$)18000$

10E.6 HKCEE MA 1989 – I – 14

(a)



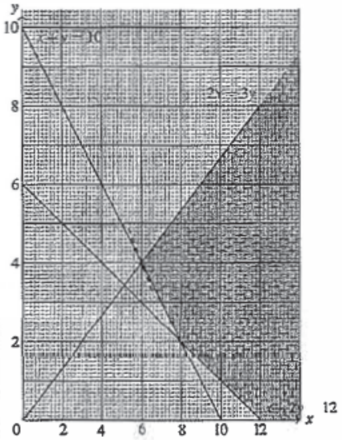
(b) (i) $z=100-x-y$

(ii) $\text{Cost}=6x+5y+4z \quad 6x+5y+4(100-x-y)$
$=(\$)2x+y+400$

(iii) $400x+600y+400z\geq 44000$
$\Rightarrow\ 2x+3y+2(100-x-y)\geq 220\ \Rightarrow\ y\geq 20$
$800x+200y+400z\geq 48000$
$\Rightarrow\ 4x+y+2(100\quad x\quad y)\geq 240\ \Rightarrow\ 2x-y\geq 40$
$z\geq 0$
$\Rightarrow\ 100\quad x\quad y\geq 0\ \Rightarrow\ x+y\leq 100$

(iv) By sliding the line Cost $=0$, the minimum is attained at $(30,20)$.
i.e. $x=30$, $y=20$, $z=100\quad 30\quad 20=50$

10E.7 HKCEE MA 1994 – I – 11

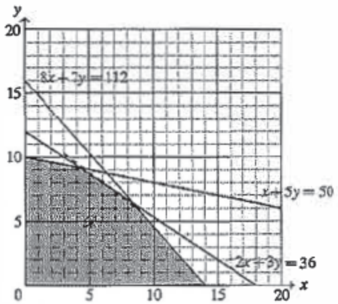


(b) (i) $\begin{cases} 2x+2y\geq 20\ \Rightarrow\ x+y\geq 10 \\ 2x\geq 3y \\ x+2y\geq 12 \\ x\geq 0, \quad y\geq 0 \end{cases}$

(ii) The feasible region is shaded above.
Let the payment be $P=500x+300(x+2y)$
$=800x+600y$
By sliding the line $P=0$, the minimum is attained at $(x,y)=(6,4)$.
$\therefore$ Length $=6$ m, Width $=4$ m,
Payment $=800(6)+600(4)=(\$)7200$

10E.8 HKCEE MA 1998 – I – 18

(a) $\begin{cases} 0.32x+0.28y\leq 4.48\ \Rightarrow\ 8x+7y\leq 112 \\ 0.24x+0.36y\leq 4.32\ \Rightarrow\ 2x+3y\leq 36 \\ 2x+10y\leq 100\ \Rightarrow\ x+5y\leq 50 \\ x\geq 0, \quad y\geq 0 \end{cases}$

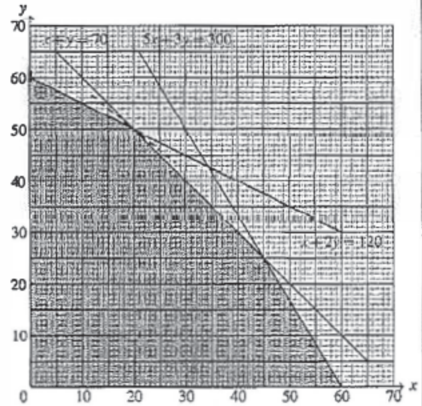


(b) Let the profit be $P=90x+120y$. By sliding the line $P=0$ among the *lattice points* in $\mathscr{R}$, the maximum is attained at $(6,8)$.
$\therefore$ Max profit $=90(6)+120(8)=(\$)1500$

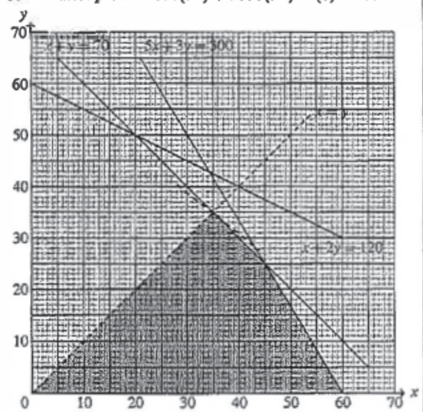
10E.9 HKCEE MA 2000 – I – 15

(a) The constraints are
$\begin{cases} 1000(0.04x)+800(0.03y)\leq 2400\ \Rightarrow\ 5x+3y\leq 300 \\ 1000(0.01x)+800(0.25y)\leq 1200\ \Rightarrow\ x+2y\leq 120 \\ x+y\leq 70 \\ x \text{ and } y \text{ are non-negative integers.} \end{cases}$

The feasible region consists of the *lattice points* in the shaded region below.
Let the profit be $P=800x+1000y$. By sliding the line $P=0$, the maximum is attained at $(x,y)=(20,50)$.

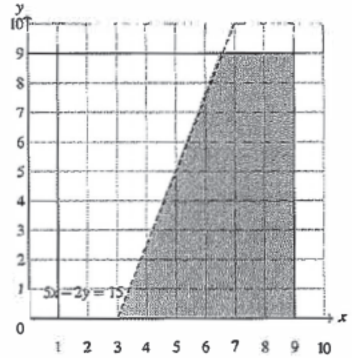


(b) Extra constraint: $x>y$.
The new feasible region consists of the lattice points in the (darker) shaded region below.
P now attains its maximum at $(36,34)$. (Note that $(35,35)$ is not in the feasible region.)
$\therefore$ Greatest profit $=800(36)+1000(34)=(\$)62800$

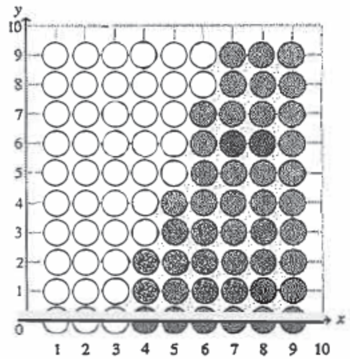


10E.10 HKCEE MA 2001 – I – 15

(a)



(b) (i)



(ii) (1) Number of tables in the smoking area $=46$
$\therefore$ Prob $=\frac{46}{90}=\frac{23}{45}$

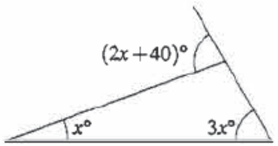
(2) Number of tables in the non-smoking area & multiple of 3 14
$\therefore$ Prob $\quad \frac{46\times 14\times 2!}{90\times 89}=\frac{644}{4005}$

11 Geometry of Rectilinear Figures

11A Angles in intersecting lines and polygons

11A.1 HKCEE MA 1980(1/1*/3) – I – 1

Find the value of x in the figure.



11A.2 HKCEE MA 1980(1*) – I – 15

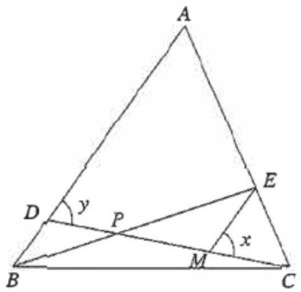
In $\triangle ABC$ (see the figure), $BD = \frac{1}{4}AB$, $CE = \frac{1}{3}AC$, BE intersects CD at P. $x = y$. Prove that

(a) $\triangle EMC$ and $\triangle ADC$ are similar and $EM = \frac{1}{4}AB$,

(b) $\triangle BDP$ and $\triangle EMP$ are congruent,

(c) $PM = CM$,

(d) area of triangle BDP is half the area of triangle PEC.



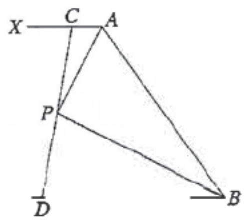
11A.3 HKCEE MA 1981(2) – I – 14

In the figure, $AX//BY$. AP and BP bisect $\angle XAB$ and $\angle YBA$ respectively, and they meet at P. A straight line passing through P meets AX and BY at C and D respectively. Prove that

(a) $\angle APB = 90°$,

(b) $CP = DP$,

(c) $AC + BD = AB$.



11A.4 HKCEE MA 1988 – I – 8(a)

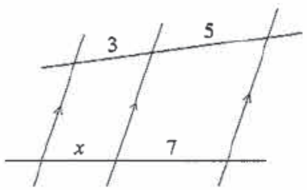
P is a point inside a square $ABCD$ such that PBC is an equilateral triangle. AP is produced to meet CD at Q.

(i) Draw a diagram to represent the above information.

(ii) Calculate $\angle PAB$ and $\angle PQC$.

11A.5 HKCEE MA 1993(I) – I – 1(c)

In the figure, find x.

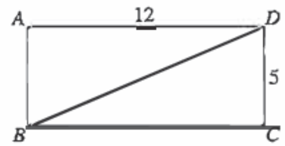


11A.6 HKCEE MA 1995 – I – 1(c)

Find the size of an interior angle of a regular octagon (8-sided polygon).

11A.7 HKCEE MA 1995 – I – 1(d)

In the figure, $ABCD$ is a rectangle. Find BD.



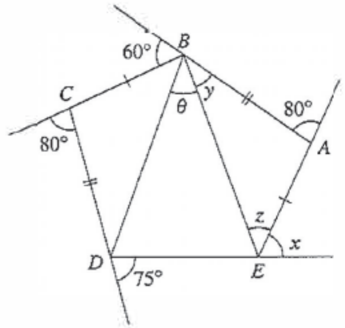
11A.8 HKCEE MA 1996 – I – 10

In the figure, $AB = CD$ and $AE = BC$.

(a) Find x.

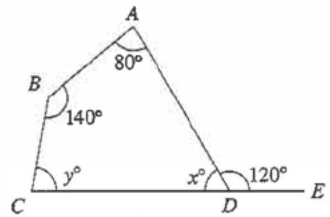
(b) Which two triangles in the figure are congruent?

(c) Find θ, y and z.



11A.9 HKCEE MA 1998 – I – 2

In the figure, CDE is a straight line. Find x and y.

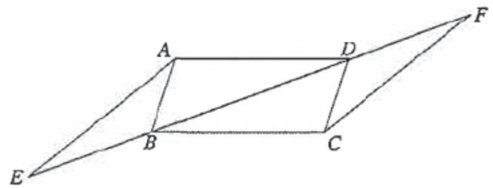


11A.10 HKCEE MA 1999 – I – 14

In the figure, $ABCD$ is a parallelogram. $EBDF$ is a straight line and $EB = DF$.

(a) Prove that $\angle ABE = \angle CDF$.

(b) Prove that $EA//CF$.

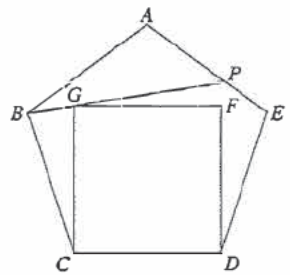


11A.11 HKCEE MA 2000 – I – 13

(To continue as **14A.6**.)

In the figure, $ABCDE$ is a regular pentagon and $CDFG$ is a square. BG produced meets AE at P.

(a) Find $\angle BCG$, $\angle ABP$ and $\angle APB$.

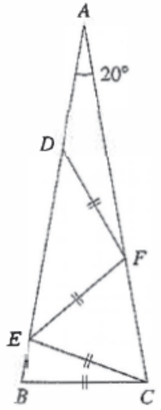


11A.12 HKCEE MA 2002 – I – 10

In the figure, ABC is a triangle in which $\angle BAC = 20°$ and $AB = AC$. D, E are points on AB and F is a point on AC such that $BC = CE = EF = FD$.

(a) Find $\angle CEF$.

(b) Prove that $AD = DF$.

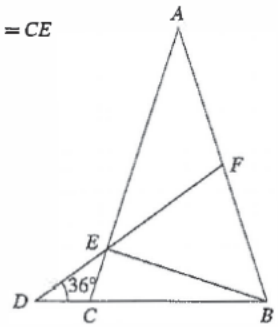


11A.13 HKCEE MA 2004 I – 12

In the figure, AEC, AFB, BCD and DEF are straight lines. $AB = AC$, $CD = CE$ and $\angle CDE = 36°$.

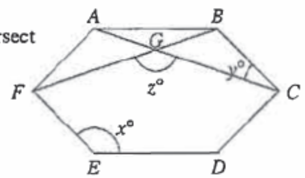
(a) Find
 (i) $\angle AEF$,
 (ii) $\angle BAC$.

(b) Suppose $AF = FB$.
 (i) Prove that $\angle AEB$ is a right angle.
 (ii) If $AE = 10$ cm, find the area of $\triangle ABC$.



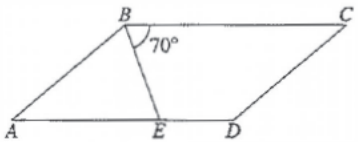
11A.14 HKCEE MA 2005 – I – 8

In the figure, $ABCDEF$ is a regular six-sided polygon. AC and BF intersect at G. Find x, y and z.



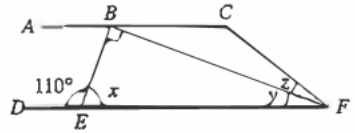
11A.15 HKCEE MA 2006 I 5

In the figure, $ABCD$ is a parallelogram. E is a point lying on AD such that $AE = AB$. It is given that $\angle EBC = 70°$. Find $\angle ABE$ and $\angle BCD$.



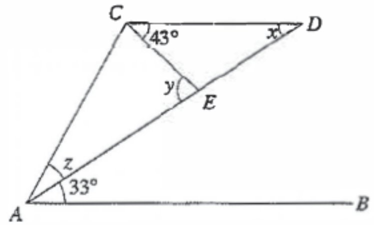
11A.16 HKCEE MA 2007 – I – 8

In the figure, ABC and DEF are straight lines. It is given that $AC // DF$, $BC = CF$, $\angle EBF = 90°$ and $\angle BED = 110°$. Find x, y and z.



11A.17 HKCEE MA 2008 I – 9

In the figure, $AB // CD$. E is a point lying on AD such that $AE = AC$. Find x, y and z.



11A.18 HKDSE MA 2020 – I – 8

In Figure 1, B and D are points lying on AC and AE respectively. BE and CD intersect at the point F. It is given that $AB = BE$, $BD // CE$, $\angle CAE = 30°$ and $\angle ADB = 42°$.

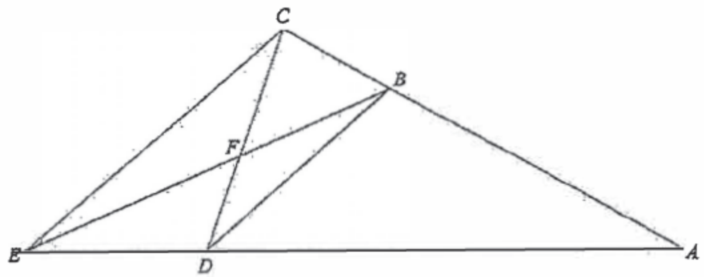


Figure 1

(a) Find $\angle BEC$.

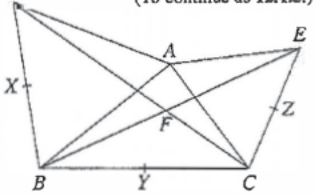
(b) Let $\angle BDC = \theta$. Express $\angle CFE$ in terms of θ.

(5 marks)

11B Congruent and similar triangles

11B.1 HKCEE MA 1982(2) I – 13

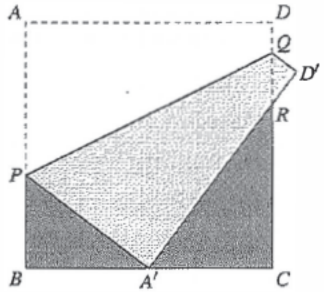
(To continue as **12A.5**.)



In the figure, $\triangle ADB$ and $\triangle ACE$ are equilateral triangles. DC and BE intersect at F.

(a) Prove that $DC = BE$. [Hint: Consider $\triangle ADC$ and $\triangle ABE$.]

11B.2 HKCEE MA 2001 – I – 11



As shown in the figure, a piece of square paper $ABCD$ of side 12 cm is folded along a line segment PQ so that the vertex A coincides with the mid-point of the side BC. Let the new positions of A and D be A' and D' respectively, and denote by R the intersection of $A'D'$ and CD.

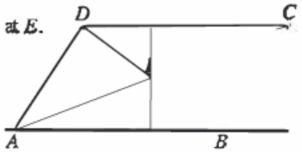
(a) Let the length of AP be x cm. By considering the triangle PBA', find x.

(b) Prove that the triangles PBA' and $A'CR$ are similar.

(c) Find the length of $A'R$.

11B.3 HKCEE MA 2003 – I – 8

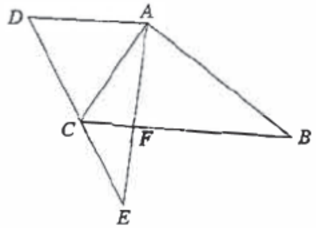
The figure shows a parallelogram $ABCD$. The diagonals AC and BD cut at E.



(a) Prove that the triangles ABC and CDA are congruent.

(b) Write down all other pairs of congruent triangles.

11B.4 HKCEE MA 2009 – I – 11



In the figure, C is a point lying on DE. AE and BC intersect at F. It is given that $AC = AD$, $BC = DE$ and $\angle BCE = \angle CAD$.

(a) Prove that $\triangle ABC \cong \triangle AED$.

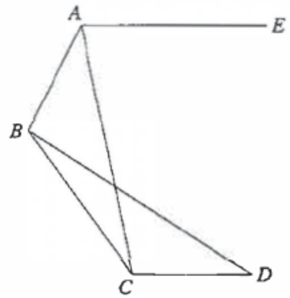
(b) If $AD // BC$,

(i) prove that $\triangle ABF \sim \triangle DEA$;

(ii) write down two other triangles which are similar to $\triangle ABF$.

11B.5 HKCEE MA 2010 – I – 9

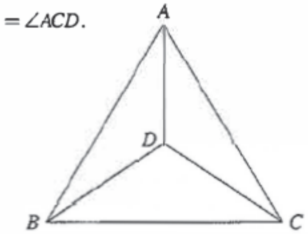
In the figure, $AB = CD$, $AE // CD$, $\angle BAE = 108°$ and $\angle BCD = 126°$.



(a) Find $\angle ABC$.

(b) Prove that $\triangle ABC \cong \triangle DCB$.

11B.6 HKCEE MA 2011 – I – 9

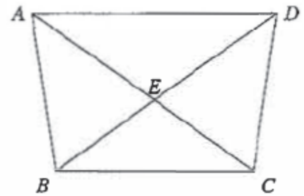


In the figure, AD is the angle bisector of $\angle BAC$. It is given that $\angle ABD = \angle ACD$.

(a) Prove that $\triangle ABD \cong \triangle ACD$.

(b) If $\angle BAD = 31°$ and $\angle ACD = 17°$, find $\angle CBD$.

11B.7 HKDSE MA 2013 – I – 7



In the figure, $ABCD$ is a quadrilateral. The diagonals AC and BD intersect at E. It is given that $BE = CE$ and $\angle BAC = \angle BDC$.

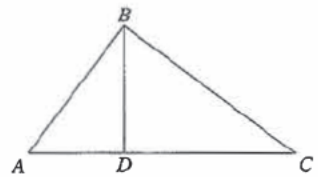
(a) Prove that $\triangle ABC \cong \triangle DCB$.

(b) Consider the triangles in the figure.

(i) How many pairs of congruent triangles are there?

(ii) How many pairs of similar triangles are there?

11B.8 HKDSE MA 2014 – I – 9



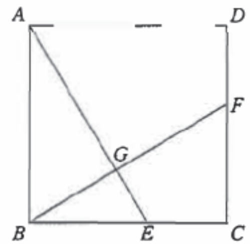
In the figure, D is a point lying on AC such that $\angle BAC = \angle CBD$.

(a) Prove that $\triangle ABC \sim \triangle BDC$.

(b) Suppose that $AC = 25$ cm, $BC = 20$ cm and $BD = 12$ cm. Is $\triangle BCD$ a right angled triangle? Explain your answer.

11B.9 HKDSE MA 2015 – I – 13

In the figure, $ABCD$ is a square. E and F are points lying on BC and CD respectively such that $AE = BF$. AE and BF intersect at G.



(a) Prove that $\triangle ABE \cong \triangle BCF$.

(b) Is $\triangle BGE$ a right-angled triangle? Explain your answer.

(c) If $CF = 15$ cm and $EG = 9$ cm, find BG.

11B.10 HKDSE MA 2016 I 13

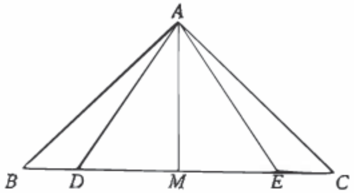
In the figure, ABC is a triangle. D, E and M are points lying on BC such that $BD = CE$, $\angle ADC = \angle AEB$ and $DM = EM$.

(a) Prove that $\triangle ACD \cong \triangle ABE$.

(b) Suppose that $AD = 15$ cm, $BD = 7$ cm and $DE = 18$ cm.

(i) Find AM.

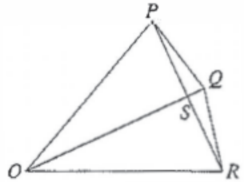
(ii) Is $\triangle ABE$ a right-angled triangle? Explain your answer.



11B.11 HKDSE MA 2017 – I – 10 (To continue as **12A.31**.)

In the figure, $OPQR$ is a quadrilateral such that $OP = OQ = OR$. OQ and PR intersect at the point S. S is the mid-point of PR.

(a) Prove that $\triangle OPS \cong \triangle ORS$.



11B.12 HKDSE MA 2018 I 13

In the figure, $ABCD$ is a trapezium with $\angle ABC = 90°$ and $AB // DC$. E is a point lying on BC such that $\angle AED = 90°$.

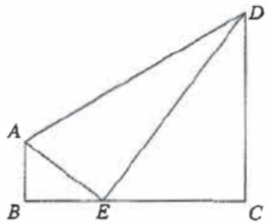
(a) Prove that $\triangle ABE \sim \triangle ECD$.

(b) It is given that $AB = 15$ cm, $AE = 25$ cm and $CE = 36$ cm.

(i) Find the length of CD.

(ii) Find the area of $\triangle ADE$.

(iii) Is there a point F lying on AD such that the distance between E and F is less than 23 cm? Explain your answer.



11B.13 HKDSE MA 2019 I 14

In the figure, $ABCD$ is a square. It is given that E is a point lying on AD. BD and CE intersect at the point F. Let G be a point such that $BG // EC$ and $CG // DB$.

(a) Prove that

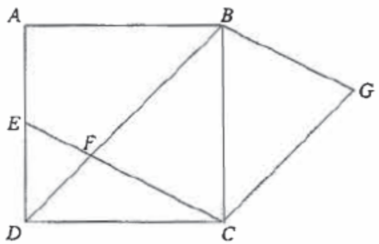
(i) $\triangle BCG \cong \triangle CBF$,

(ii) $\triangle BCF \sim \triangle DEF$.

(b) Suppose that $\angle BCF = \angle BGC$.

(i) Let $BC = \ell$. Express DF in terms of ℓ.

(ii) Someone claims that $AE > DF$. Do you agree? Explain your answer.



11B.14 HKDSE MA 2020 I 18

In Figure 2, U, V and W are points lying on a circle. Denote the circle by C. TU is the tangent to C at U such that TVW is a straight line.

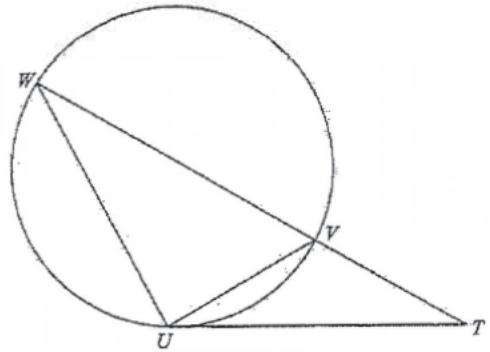


Figure 2

(a) Prove that $\Delta UTV \sim \Delta WTU$. (2 marks)

(b) It is given that VW is a diameter of C. Suppose that $TU = 780$ cm and $TV = 325$ cm.

(i) Express the circumference of C in terms of π.

(ii) Someone claims that the perimeter of ΔUVW exceeds 35 m. Do you agree? Explain your answer.

(5 marks)

11 Geometry of Rectilinear Figures

11.1 HKCEE MA 1980(1/1*/3) – I – 1

$x° + 3x° = (2x + 40)°$ (ext. $\angle$ of $\triangle$)
$x = 20$

11.2 HKCEE MA 1980(1*) – I – 15

(a) In $\triangle EMC$ and $\triangle ADC$,
$x = y$ (given)
$\angle ECM = \angle ACD$ (common)
$\angle MEC = \angle DAC$ ($\angle$ sum of $\triangle$)
$\therefore \triangle EMC \sim \triangle ADC$ (AAA)

Hence, $\dfrac{EM}{AD} = \dfrac{EC}{AC} = \dfrac{1}{3}$ (corr. sides, $\sim \triangle$s)

$EM = \frac{1}{3}AD$
$= \frac{1}{3}\left(\frac{3}{4}AB\right) = \frac{1}{4}AB$

(b) $\because x = y$ (given)
$\therefore AB // EM$ (corr. $\angle$s equal)
In $\triangle BDP$ and $\triangle EMP$,
$\angle BPD = \angle EPM$ (vert. opp. $\angle$s)
$\angle PBD = \angle PEM$ (alt. $\angle$s, $AB // EM$)
$BD = EM = \frac{1}{4}AB$ (proved)
$\therefore \triangle BDP \cong \triangle EMP$ (AAS)

(c) $PD = PM$ (corr. sides, $\cong \triangle$s)
$\dfrac{CM}{CD} = \dfrac{EC}{AC} = \dfrac{1}{3}$ (corr. sides, $\sim \triangle$s)
$\Rightarrow DM = \frac{2}{3}CD = 2CM$
$\therefore PM = CM$ ($= PD$)

(d) $\because PM = CM$ (proved)
$\therefore$ Area of $\triangle EMP$ = Area of $\triangle EMC$
$\because \triangle BDP \cong \triangle EMP$ (proved)
$\therefore$ Area of $\triangle BDP$ = Area of $\triangle EMP$
Hence, Area of $\triangle BDP = \frac{1}{2}$Area of $\triangle PEC$

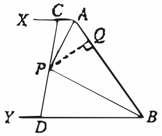
11.3 HKCEE MA 1981(2) – I – 14

(a) $\angle XAB + \angle YBA = 180°$ (int. $\angle$s, $XA // YB$)
$2\angle PAB + 2\angle PBA = 180°$ (given)
$\angle PAB + \angle PBA = 90°$
$\therefore$ In $\triangle ABP$,
$\angle APB = 180° - (\angle PAB + \angle PBA)$ ($\angle$ sum of $\triangle$)
$= 180° - 90°$ (proved)
$= 90°$

(b) Let Q be on AB such that $\angle APQ = \angle APC$.
In $\triangle APC$ and $\triangle APQ$,
$AP = AP$ (common)
$\angle CAP = \angle QAP$ (given)
$\angle APC = \angle APQ$ (by construction)
$\therefore \triangle APC \cong \triangle APQ$ (AAS)
$\therefore CP = PQ$ (corr. sides, $\cong \triangle$s)
Besides,
$\angle QPB = 90° - \angle APQ = 90° - \angle APC$ (corr. $\angle$s, $\cong \triangle$s)
$\Rightarrow \angle DPB = 180° - 90° - \angle APC$ (adj. $\angle$s on st. line)
$= 90° - \angle APC$
$= \angle QPB$

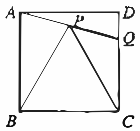
$\therefore$ In $\triangle BPD$ and $\triangle BPQ$,
$PB = PB$ (common)
$\angle PBD = \angle QBP$ (given)
$\angle DPB = \angle QPB$ (proved)
$\therefore \triangle BPD \cong \triangle BPQ$ (AAS)
$PD = PQ$ (corr. sides, $\cong \triangle$s)
$\therefore CP = DP$ ($= PQ$)

(c) $\because AC = AQ$ (corr. sides, $\cong \triangle$s)
$BD = BQ$ (corr. sides, $\cong \triangle$s)
$\therefore AC + BD = AQ + BQ = AB$



11.4 HKCEE MA 1988 – I – 8(a)

(i)



(ii) $\angle ABC = 90°$ (property of square)
$\angle PBC = 60°$ (property of equil $\triangle$)
$\Rightarrow \angle ABP = 90° - 60° = 30°$
$AB = BC$ (property of square)
$= BP$ (property of equi $\triangle$)
$\Rightarrow \angle PAB = \angle APB$ (base $\angle$s, isos. $\triangle$)
$= (180° - 30°) \div 2 = 75°$ ($\angle$ sum of $\triangle$)
$\angle PQC = 180° - \angle PAB = 105°$ (int. $\angle$s, $AB // DC$)

11.5 HKCEE MA 1993(I) – I – 1(c)

$\dfrac{x}{7} = \dfrac{3}{5}$ (intercept thm) $\Rightarrow x = \dfrac{21}{5}$

11.6 HKCEE MA 1995 – I – 1(c)

Required $\angle = (8-2)180° \div 8 = 135°$ ($\angle$ sum of polygon)

11.7 HKCEE MA 1995 – I – 1(d)

$AB = DC = 5$ and $\angle A = 90°$ (property of rectangle)
$\therefore BD = \sqrt{AB^2 + AD^2} = 13$ (Pyth. thm)

11.8 HKCEE MA 1996 – I – 10

(a) $x = 360° - 80° - 60° - 80° - 75° = 65°$
(sum of ext. $\angle$s of polygon)

(b) $\triangle ABE$ and $\triangle CDB$ (SAS)

(c) In $\triangle ABE$, $y + z = 80°$ (ext. $\angle$ of $\triangle$)
$\because \triangle ABE \cong \triangle CDB$
$\therefore \angle CDB = y$ (corr. $\angle$s, $\cong \triangle$s)
$BD = BE$ (corr. sides, $\cong \triangle$s)
$\angle BDE = \angle BED$ (base $\angle$s, isos. $\triangle$)
$= 180° - z - (65°)$ (adj. $\angle$s on st. line)
$= 115° - z$
$\therefore \angle CDB + \angle BDE + 75° = 180°$ (adj. $\angle$s on st. line)
$y + (115° - z) + 75° = 180°$
$z - y = 10°$

Hence, $\begin{cases} z - y = 10° \\ y + z = 80° \end{cases} \Rightarrow \begin{cases} y = 35° \\ z = 45° \end{cases}$

$\therefore$ In $\triangle BDE$, $\theta = 180° - 2\angle BED$ ($\angle$ sum of $\triangle$)
$= 180° - 2(115° - z) = 40°$

11.9 HKCEE MA 1998 – I – 2

$x = 180 - 120 = 60$ (adj. $\angle$s on st. line)
$y = (4-2)180 - 80 - 140 - x$ ($\angle$ sum of polygon)
$= 80$

11.10 HKCEE MA 1999 – I – 14

(a) $\angle ABE = 180° - \angle ABD$ (adj. $\angle$s on st. line)
$= 180° - \angle CDB$ (alt. $\angle$s, $AB // DC$)
$= \angle CDF$ (adj. $\angle$s on st. line)

(b) In $\triangle ABE$ and $\triangle CDF$,
$AB = CD$ (property of // gram)
$EB = FC$ (given)
$\angle ABE = \angle CDF$ (proved)
$\therefore \triangle ABE \cong \triangle CDF$ (SAS)
$\Rightarrow \angle E = \angle F$ (corr. $\angle$s, $\cong \triangle$s)
$\Rightarrow EA // CF$ (alt. $\angle$s equal)

11.11 HKCEE MA 2000 – I – 13

(a) $\angle A = \angle ABC = \angle BCD$ (given)
$= (5 - 2)180° \div 5$ ($\angle$ sum of polygon)
$= 108°$
$\angle GCD = 90°$ (property of square)
$\Rightarrow \angle BCG = 108° - 90° = 18°$
$BC = CD = CG$ (given)
$\angle GBC = \angle BGC$ (base $\angle$s, isos. $\triangle$)
In $\triangle BCG$, $\angle GBC = (180° - \angle BCG) \div 2$ ($\angle$ sum of $\triangle$)
$= 81°$
$\angle ABP = 108° - 81° = 27°$
$\angle APB = 180° - \angle A - \angle ABP = 45°$ ($\angle$ sum of $\triangle$)

11.12 HKCEE MA 2002 – I – 10

(a) In $\triangle ABC$, $\angle B = \angle C$ (base $\angle$s, isos. $\triangle$)
$= (180° - 20°) \div 2$ ($\angle$ sum of $\triangle$)
$= 80°$
In $\triangle CBE$, $\angle E = \angle B = 80°$ (base $\angle$s, isos. $\triangle$)
$\therefore \angle ECB = 180° - 2(80°)$ ($\angle$ sum of $\triangle$)
$= 20°$
$\therefore \angle ECF = 80° - 20° = 60°$
Thus, $\triangle CEF$ is equilateral. $\Rightarrow \angle CEF = 60°$

(b) $\angle EDF = \angle DEF$ (base $\angle$s, isos. $\triangle$)
$= 180° - \angle CEF - \angle BEC$ (adj. $\angle$s on st. line)
$= 40°$
$\angle DFA = 40° - \angle A = 20°$ (ext. $\angle$ of $\triangle$)
$\because \angle DFA = \angle DAF = 20°$ (proved)
$\therefore AD = DF$ (sides opp. equal $\angle$s)

11.13 HKCEE MA 2004 – I – 12

(a) (i) $\angle AEF = \angle CED$ (vert. opp. $\angle$s)
$= \angle CDE$ (base $\angle$s, isos. $\triangle$)
$= 36°$

(ii) $\angle ABC = \angle ACB$ (base $\angle$s, isos. $\triangle$)
$= \angle CDE + \angle CED$ (ext. $\angle$ of $\triangle$)
$= 72°$
$\therefore \angle BAC = 180° - 2(72°) = 36°$ ($\angle$ sum of $\triangle$)

(b) (i) $\because \angle FAE = \angle AEF = 36°$ (proved)
$\therefore AF = FE$ (sides opp. equal $\angle$s)
$\because AF = FB$, $FE = FB$ (given)
$\therefore \angle EFB = \angle A + \angle AEF = 72°$ (ext. $\angle$ of $\triangle$)
$\angle FEB = \angle FBE$ (base $\angle$s, isos. $\triangle$)
$= (180° - \angle EFB) \div 2 = 54°$
Hence, $\angle AEB = \angle AEF + \angle FEB = 36° + 54° = 90°$

(ii) $AC = AB = \dfrac{AE}{\cos\angle A} = \dfrac{10}{\cos 36°}$
$BE = AE \tan\angle A = 10 \tan 36°$
$\therefore$ Area of $\triangle ABC = \frac{1}{2}AC \cdot BE = 44.9$ (cm^2, 3.s.f.)

11.14 HKCEE MA 2005 – I – 8

$x = (6-2)180 \div 6 = 120$ ($\angle$ sum of polygon)
In $\triangle ABC$, $\angle B = 120°$
$\because AB = BC$ (given)
$\therefore y° = \angle BAC$ (base $\angle$s, isos. $\triangle$)
$y = (180 - \angle B) \div 2$ ($\angle$ sum of $\triangle$)
$= 30$
$\angle ABG = \angle BAG = 30°$
$z° = \angle AGB$ (vert. opp. $\angle$s)
$z = 180 - 30 - 30 = 120$ ($\angle$ sum of $\triangle$)

11.15 HKCEE MA 2006 – I – 5

$\angle ABE = \angle AEB$ (base $\angle$s, isos. $\triangle$)
$= \angle CBE = 70°$ (alt. $\angle$s, $BC // AD$)
$\angle BCD = 180° - \angle ABC$ (int. $\angle$s, $AB // DC$)
$= 180° - (70° + 70°) = 40°$

11.16 HKCEE MA 2007 – I – 8

$x = 180° - 110° = 70°$ (adj. $\angle$s on st. line)
$\angle CBF = z$ (base $\angle$s, isos. $\triangle$)
$\angle EBC = 110°$ (alt. $\angle$s, $AC // DF$)
$z = 110° - 90° = 20°$
$y = 180° - 90° - x = 20°$ ($\angle$ sum of $\triangle$)

11.17 HKCEE MA 2008 – I – 9

$x = 33°$ (alt. $\angle$s, $CD // AB$)
$y = 43° + x = 76°$ (ext. $\angle$ of $\triangle$)
$\angle ACE = y = 76°$ (base $\angle$s, isos. $\triangle$)
$z = 180° - \angle ACE - y = 28°$ ($\angle$ sum of $\triangle$)

11.18 HKDSE MA 2020 – I – 8

8a
$AB = BE$ (given)
$\angle AEB = \angle BAE$ (base $\angle$s, isos. $\triangle$)
$\angle AEB = 30°$
$\angle ADB = \angle BED + \angle DBE$ (ext. $\angle$ of $\triangle$)
$42° = 30° + \angle DBE$
$\angle DBE = 12°$
$\angle BEC = \angle DBE$ (alt. $\angle$s, $BD // CE$)
$= 12°$

b
$\angle DCE = \angle BDC$ (alt. $\angle$s, $BD // CE$)
$= \theta$
$\angle CEF + \angle CFE + \angle ECF = 180°$ ($\angle$ sum of $\triangle$)
$12° + \angle CFE + \theta = 180°$
$\theta = 168° - \theta$

11B Congruent and similar triangles

11B.1 HKCEE MA 1982(2) – I – 13

(a) $\angle DAB = \angle EAC = 60°$ (property of equil. $\triangle$)
$\angle DAB + \angle BAC = \angle EAC + \angle BAC$
$\angle DAC = \angle BAE$
In $\triangle ADC$ and $\triangle ABE$,
$DA = BA$ (property of equil. $\triangle$)
$\angle DAC = \angle BAE$ (proved)
$AC = AE$ (property of equil. $\triangle$)
$\therefore \triangle ADC \cong \triangle ABE$ (SAS)
$\therefore DC = BE$ (corr. sides, $\cong \triangle$s)

11B.2 HKCEE MA 2001 – I – 11

(a) $PA' = PA = x$ cm
In $\triangle PBA'$, $x^2 = PB^2 + BA'^2$ (Pyth. thm)
$x^2 = (12-x)^2 + (12 \div 2)^2$
$\cancel{x^2} = 144 - 24x + \cancel{x^2} + 36 \Rightarrow x = 7.5$

(b) In $\triangle PBA'$ and $\triangle A'CR$,
$\angle B = \angle C = 90°$ (given)
$\angle BPA' = 180° \quad \angle B - \angle PA'B$ ($\angle$ sum of $\blacktriangle$)
$= 90° - \angle PA'B$
$\angle CA'R = 180° - \angle PA'R - \angle PA'B$ (adj. $\angle$s on st. line)
$= 90° - \angle PA'B$
$\Rightarrow \angle BA'P = \angle CA'R$
$\angle BA'P = \angle CRA'$ ($\angle$ sum of $\triangle$)
$\therefore \triangle PBA' \sim \triangle A'CR$ (AAA)

(c) $\dfrac{PA'}{PB} = \dfrac{A'R}{A'C}$ (corr. sides, $\sim \triangle$s)
$\dfrac{7.5}{12-7.5} = \dfrac{A'R}{6} \Rightarrow A'R = 10$ (cm)

11B.3 HKCEE MA 2003 – I – 8

(a) In $\triangle ABC$ and $\triangle CDA$,
$AB = CD$ (property of //gram)
$BC = DA$ (property of //gram)
$AC = CA$ (common)
$\therefore \triangle ABC \cong \triangle CDA$ (SSS)

(b) $\triangle ABD \cong \triangle CDB$, $\triangle ABE \cong \triangle CDE$, $\triangle ADE \cong \triangle CBE$

11B.4 HKCEE MA 2009 – I – 11

(a) $\angle ADC = \angle ACE - \angle CAD$ (ext. $\angle$ of $\triangle$)
$= \angle ACE - \angle BCE$ (given)
$= \angle ACB$
In $\triangle ABC$ and $\triangle AED$,
$AC = AD$ (given)
$BC = ED$ (given)
$\angle ACB = \angle ADE$ (proved)
$\therefore \triangle ABC \cong \triangle AED$ (SAS)

(b) (i) In $\triangle ABF$ and $\triangle DEA$,
$\angle AFB = \angle DAE$ (alt. $\angle$s, $AD//BC$)
$\angle ABF = \angle DEA$ (corr. $\angle$s, $\cong \triangle$s)
$\angle BAF = \angle EDA$ ($\angle$ sum of $\triangle$)
$\therefore \triangle ABF \sim \triangle DEA$ (AAA)

(ii) $\triangle CEF$, $\triangle CBA$

11B.5 HKCEE MA 2010 – I – 9

(a) $\angle EAC + \angle ACD = 180°$ (int. $\angle$s, $AE//CD$)
In $\triangle ABC$, $\angle ABC + \angle BAC + \angle BCA = 180°$
($\angle$ sum of $\triangle$)
$\angle ABC + (108° - \angle EAC) + (126° - \angle ACD) = 180°$
$\angle ABC + 234° - (180°) = 180°$
(proved)
$\angle ABC = 126°$

(b) In $\triangle ABC$ and $\triangle DCB$,
$AB = DC$ (given)
$\angle ABC = \angle DCB = 126°$ (proved)
$BC = CB$ (common)
$\therefore \triangle ABC \cong \triangle DCB$ (SAS)

11B.6 HKCEE MA 2011 – I – 9

(a) In $\triangle ABD$ and $\triangle ACD$,
$\angle BAD \quad \angle CAD$ (given)
$AD = AD$ (common)
$\angle ABD = \angle ACD$ (given)
$\therefore \triangle ABD \cong \triangle ACD$ (ASA)

(b) $\angle CAD = \angle BAD = 31°$ (given)
In $\triangle ACD$,
$\angle ADC = 180° - 31° - 17° = 132°$ ($\angle$ sum of $\triangle$)
$\angle ADB = \angle ADC = 132°$ (corr. $\angle$s, $\cong \triangle$s)
$DB = DC$ (corr. sides, $\cong \triangle$s)
$\angle BDC = 360° - 132° - 132° = 96°$ ($\angle$s at a pt)
$\angle CBD = \angle BCD$ (base $\angle$s, isos. $\triangle$)
$= (180° - 96°) \div 2 = 42°$ ($\angle$ sum of $\triangle$)

11B.7 HKDSE MA 2013 – I – 7

(a) $\because BE = CE$ (given)
$\therefore \angle BCE = \angle CBE$ (base $\angle$s, isos. $\triangle$)
In $\triangle ABC$ and $\triangle DCB$,
$\angle BAC = \angle BDC$ (given)
$\angle ACB = \angle DCB$ (proved)
$BC = CB$ (common)
$\therefore \triangle ABC \cong \triangle DCB$ (AAS)

(b) (i) 3 ($\triangle ABC \cong \triangle DCB$, $\triangle ABE \cong \triangle DCE$, $\triangle ABD \cong \triangle DCA$)
(ii) 4 (the 3 in (i) and $\triangle ADE \sim \triangle CBE$)

11B.8 HKDSE MA 2014 – I – 9

(a) In $\triangle ABC$ and $\triangle BDC$,
$\angle C = \angle C$ (common)
$\angle BAC = \angle DBC$ (given)
$\angle ABC = \angle BDC$ ($\angle$ sum of $\triangle$)
$\therefore \triangle ABC \sim \triangle BDC$ (AAA)

(b) $\dfrac{AC}{BC} = \dfrac{BC}{DC}$ (corr. sides, $\sim \triangle$s)
$\dfrac{25}{20} = \dfrac{20}{DC}$
$DC = 16$
$BC^2 = 20^2 = 400$
$BD^2 + CD^2 = 12^2 + 16^2 = 400 = BC^2$
$\therefore \triangle BCD$ is a right-$\angle$ed $\triangle$. (converse of Pyth. thm)

11B.9 HKDSE MA 2015 – I – 13

(a) In $\triangle ABE$ and $\triangle BCF$,
$AB = BC$ (property of square)
$\angle B = \angle C = 90°$ (property of square)
$AE = BF$ (given)
$\therefore \triangle ABE \cong \triangle BCF$ (RHS)

(b) $\angle AEB = \angle BFC$ (corr. sides, $\cong \triangle$s)
In $\blacktriangle BEG$,
$\angle BGE = 180° \quad \angle GBE \quad \angle GEB$ ($\angle$ sum of $\triangle$)
$= 180° - \angle GBE - \angle BFC$ (proved)
$= \angle BCF = 90°$ ($\angle$ sum of $\triangle$)
$\therefore$ YES.

(c) $BE = CF = 15$ cm (corr. sides, $\cong \triangle$s)
$BG = \sqrt{BE^2 - EG^2} = 12$ cm (Pyth. thm)

11B.10 HKDSE MA 2016 – I – 13

(a) $DE = ED$ (common)
$BD + DE = CE + ED$ (given)
$BE = CD$
In $\triangle ACD$ and $\triangle ABE$,
$BE = CD$ (proved)
$\angle AEB = \angle ADC$ (given)
$AE = AD$ (sides opp. equal $\angle$s)
$\therefore \triangle ACD \cong \triangle ABE$ (SAS)

(b) (i) $\because DM = EM$ (given)
$\therefore AM \perp DE$ (property of isos. $\triangle$)
$AM = \sqrt{AD^2 - (DE \div 2)^2} = 12$ (cm) (Pyth. thm)
(ii) $AB = \sqrt{AM^2 + BM^2} = 20$ (cm) (Pyth. thm)
$BE^2 = 25^2 = 625$
$AB^2 + AE^2 = AB^2 + AD^2$ (corr. sides, $\cong \triangle$s)
$= 20^2 + 15^2 = 625 = BE^2$
$\therefore$ YES. (converse of Pyth. thm)

11B.11 HKDSE MA 2017 – I – 10

(a) $\because OP = OR$ and $PS = RS$ (given)
$\therefore OS \perp PR$ (property of isos. $\triangle$)
In $\triangle OPS$ and $\triangle ORS$,
$OP = OR$ (given)
$OS = OS$ (common)
$\angle OSP = \angle OSR$ (proved)
$\therefore \triangle OPS \cong \triangle ORS$ (RHS)

11B.12 HKDSE MA 2018 – I – 13

(a) $\angle C = 180° \quad \angle B = 90°$ (int. $\angle$s, $AB//DC$)
$\angle BAE = 180° - \angle ABE - \angle AEB$quad ($\angle$ sum of $\triangle$)
$= 90° \quad \angle AEB$
$\angle CED = 180° - \angle AED - \angle AEB$ (adj. $\angle$s on st. line)
$= 90° \quad \angle AEB$
$\therefore BAE = \angle CED$
In $\triangle ABE$ and $\triangle ECD$,
$\angle B = \angle C = 90°$ (proved)
$\angle BAE = \angle CED$ (proved)
$\angle BEA = \angle CDE$ ($\angle$ sum of $\triangle$)
$\therefore \triangle ABE \sim \triangle ECD$ (AAA)

(b) (i) $BE = \sqrt{AE^2 - AB^2} = 20$ cm (Pyth. thm)
$\dfrac{AB}{BE} = \dfrac{EC}{CD}$ (corr. sides, $\sim \triangle$s)
$\dfrac{15}{20} = \dfrac{36}{CD}$
$CD = 48$ cm

(ii) $DE = \sqrt{CD^2 + CE^2} = 60$ cm (Pyth. thm)
Area of $\triangle ADE = \dfrac{1}{2}(25)(60) = 750$ (cm²)

(iii) $AD = \sqrt{25^2 + 60^2} = 65$ (cm) (Pyth. thm)
Let ℓ cm be the shortest distance from E to AD.
$\dfrac{AD \cdot \ell}{2} =$ Area of $\triangle ADE$
$\ell = 2 \times 750 \div 65$
$= 23.077 > 23$
$\therefore$ NO.

11B.13 HKDSE MA 2020 – I – 18

18a
$\angle TUV = \angle TWU$ ($\angle$ in alt. segment)
$\angle UTV = \angle WTU$ (common $\angle$)
$\angle UVT = \angle WUT$ (3rd $\angle$ of $\triangle$)
$\triangle UTV \sim \triangle WTU$ (A.A.A.)

bi
$\triangle UTV \sim \triangle WTU$ (from (a))
$\dfrac{TU}{TW} = \dfrac{TV}{TU}$ (corr. sides, ~△s)
$\dfrac{TU}{TV + VW} = \dfrac{TV}{TU}$
$\dfrac{780}{325 + VW} = \dfrac{325}{780}$
$VW = 1547$ cm
The circumference of $C = \pi(1547)$
$= 1547\pi$ cm

ii
$\triangle UTV \sim \triangle WTU$ (from (a))
$\dfrac{UV}{WU} \quad \dfrac{TV}{TU}$ (corr. sides, ~△s)
$\dfrac{UV}{UW} = \dfrac{325}{780}$
$UV = \dfrac{5}{12}UW$
$\angle VUW = 90°$ ($\angle$ in semi-circle)
$UV^2 + UW^2 = VW^2$ (Pyth. Thm.)
$\left(\dfrac{5}{12}UW\right)^2 + UW^2 \quad 1547^2$
$UW = 1428$ cm
The perimeter of $\triangle UVW = UV + UW + VW$
$= \dfrac{5}{12}UW + UW + VW$
$= \dfrac{5}{12}(1428) + 1428 + 1547$
$= 3570$ cm
$= 35.7$ m
> 35 m
Therefore, the perimeter of $\triangle UVW$ exceeds 35 m.
The claim is agreed with.

12 Geometry of Circles

12A Angles and chords in circles

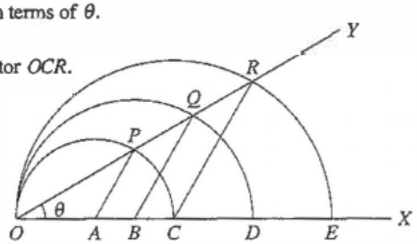
12A.1 HKCEE MA 1980(1/1*/3) – I 10 (Continued from 15A.1.)

A, B and C are three points on the line OX such that $OA = 2$, $OB = 3$ and $OC = 4$. With A, B, C as centres and OA, OB, OC as radii, three semi-circles are drawn as shown in the figure. A line OY cuts the three semi circles at P, Q, R respectively.

(a) If $\angle YOX = \theta$, express $\angle PAX$, $\angle QBX$ and $\angle RCX$ in terms of θ.

(b) Find the following ratios:
area of sector OAP : area of sector OBQ : area of sector OCR.

(c) If $RD \perp OX$, calculate the angle θ.



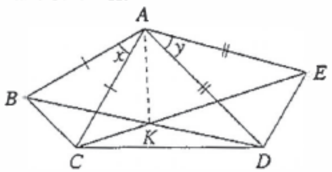
12A.2 HKCEE MA 1980(1*) – I 14

In the figure, $AB = AC$, $AD = AE$, $x = y$. Straight lines BD and CE intersect at K.

(a) Prove that $\triangle ABD$ and $\triangle ACE$ are congruent.

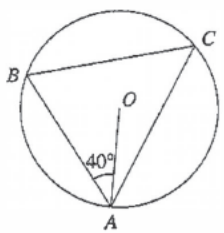
(b) Prove that $ABCK$ is a cyclic quadrilateral.

(c) Besides the quadrilateral $ABCK$, there is another cyclic quadrilateral in the figure. Write it down (proof is not required).



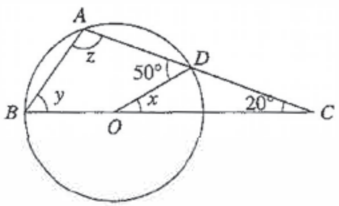
12A.3 HKCEE MA 1981(2) I 7

In the figure, O is the centre of circle ABC. $\angle OAB = 40°$. Calculate $\angle BCA$.



12A.4 HKCEE MA 1982(2) – I 6

In the figure, O is the centre of the circle BAD. BOC and ADC are straight lines. If $\angle ADO = 50°$ and $\angle ACB = 20°$, find x, y and z.



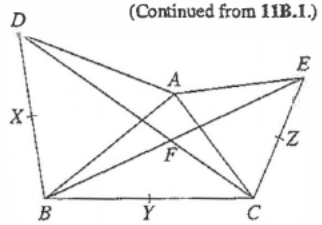
12A.5 HKCEE MA 1982(2) I 13 (Continued from 11B.1.)

In the figure, $\triangle ADB$ and $\triangle ACE$ are equilateral triangles. DC and BE intersect at F.

(a) Prove that $DC = BE$. [Hint: Consider $\triangle ADC$ and $\triangle ABE$.]

(b) (i) Prove that A, D, B and F are concyclic.
(ii) Find $\angle BFD$.

(c) Let the mid points of DB, BC and CE be X, Y and Z respectively. Find the angles of $\triangle XYZ$.



12A.6 HKCEE MA 1989 – I – 4

AB is a diameter of a circle and M is a point on the circumference. C is a point on BM produced such that $BM = MC$.

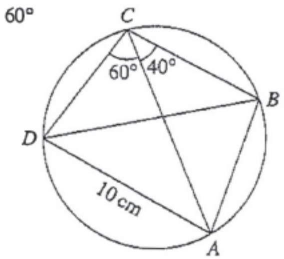
(a) Draw a diagram to represent the above information.

(b) Show that AM bisects $\angle BAC$.

12A.7 HKCEE MA 1989 I – 6 (To continue as 14A.4.)

In the figure, $ABCD$ is a cyclic quadrilateral with $AD = 10$ cm, $\angle ACD = 60°$ and $\angle ACB = 40°$.

(a) Find $\angle ABD$ and $\angle BAD$.



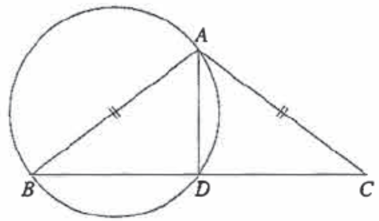
12A.8 HKCEE MA 1990 I – 9

In the figure, AB is a diameter of the circle ADB and ABC is an isosceles triangle with $AB = AC$.

(a) Prove that $\triangle ABD$ and $\triangle ACD$ are congruent.

(b) The tangent to the circle at D cuts AC at the point E. Prove that $\triangle ABD$ and $\triangle ADE$ are similar.

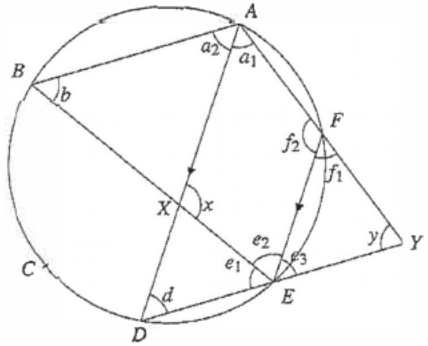
(c) In (b), let $AB = 5$ and $BD = 4$.
(i) Find DE.
(ii) CA is produced to meet the circle at the point F. Find AF.



12A.9 HKCEE MA 1992 I – 11

In the figure, A, B, C, D, E and F are points on a circle such that $AD//FE$ and $\widehat{BCD} = \widehat{AFE}$. AD intersects BE at X. AF and DE are produced to meet at Y.

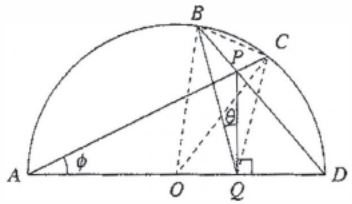
(a) Prove that $\triangle EFY$ is isosceles.
(b) Prove that $BA//DE$.
(c) Prove that A, X, E, Y are concyclic.
(d) If $b = 47°$, find f_1, y and x.



12A.10 HKCEE MA 1993 – I – 11

The figure shows a semicircle with diameter AD and centre O. The chords AC and BD meet at P. Q is the foot of the perpendicular from P to AD.

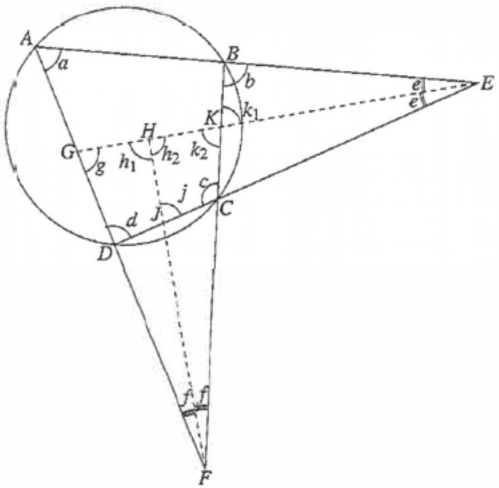
(a) Show that A, Q, P, B are concyclic.
(b) Let $\angle BQP = \theta$. Find, in terms of θ,
  (i) $\angle BQC$,
  (ii) $\angle BOC$.
(c) Let $\angle CAD = \phi$. Find $\angle CBQ$ in terms of ϕ.



12A.11 HKCEE MA 1994 I – 13

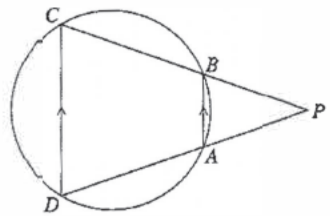
In the figure, A, B, C, D are points on a circle and ABE, $GHKE$, $DJCE$, $AGDF$, HJF, $BKCF$ are straight lines. FH bisects $\angle AFB$ and GE bisects $\angle AED$.

(a) Prove that $\angle FGH = \angle FKH$.
(b) Prove that $FH \perp GK$.
(c) (i) If $\angle AED = \angle AFB$, prove that D, J, H, G are concyclic.
  (ii) If $\angle AED = 28°$ and $\angle AFB = 46°$, find $\angle BCD$.



12A.12 HKCEE MA 1996 – I – 6

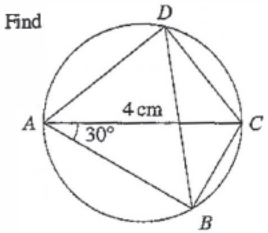
In the figure, A, B, C, D are points on a circle. CB and DA are produced to meet at P. If $AB//DC$, prove that $AP = BP$.



12A.13 HKCEE MA 1997 – I – 9

In the figure, AC is a diameter of the circle. $AC = 4$ cm and $\angle BAC = 30°$. Find

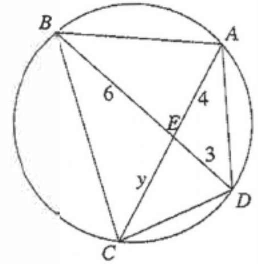
(a) $\angle BDC$ and $\angle ADB$,
(b) $\widehat{AB} : \widehat{BC}$,
(c) $AB : BC$.



12A.14 HKCEE MA 1998 – I – 6

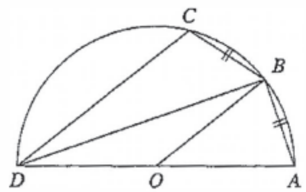
In the figure, A, B, C, D are points on a circle. AC and BD meet at E.

(a) Which triangle is similar to $\triangle ECD$?
(b) Find y.



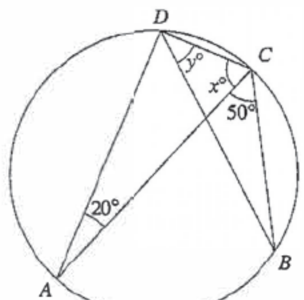
12A.15 HKCEE MA 1998 – I – 14

In the figure, O is the centre of the semicircle $ABCD$ and $AB = BC$. Show that $BO//CD$.



12A.16 HKCEE MA 1999 – I – 5

In the figure, A, B, C, D are points on a circle and AC is a diameter. Find x and y.

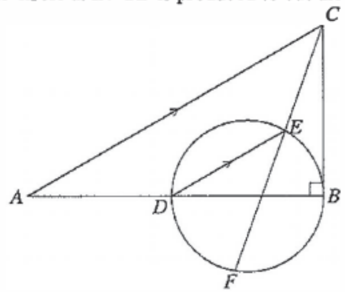


12A.17 HKCEE MA 1999 – I – 16 (To continue as 16C.20.)

(a) In the figure, ABC is a triangle right angled at B. D is a point on AB. A circle is drawn with DB as a diameter. The line through D and parallel to AC cuts the circle at E. CE is produced to cut the circle at F.

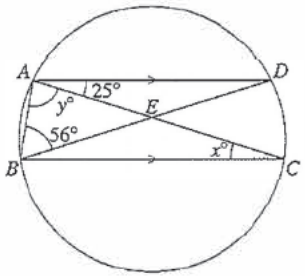
(i) Prove that A, F, B and C are concyclic.

(ii) If M is the mid point of AC, explain why $MB = MF$.



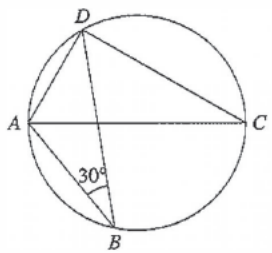
12A.18 HKCEE MA 2000 – I – 7

In the figure, AD and BC are two parallel chords of the circle. AC and BD intersect at E. Find x and y.



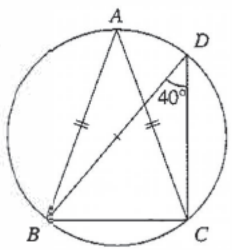
12A.19 HKCEE MA 2001 – I – 5

In the figure, AC is a diameter of the circle. Find $\angle DAC$.



12A.20 HKCEE MA 2002 – I – 9

In the figure, BD is a diameter of the circle $ABCD$. $AB = AC$ and $\angle BDC = 40°$. Find $\angle ABD$.



12A.21 HKCEE MA 2002 – I – 16 (To continue as 16C.23.)

In the figure, AB is a diameter of the circle $ABEG$ with centre C. The perpendicular from G to AB cuts AB at O. AE cuts OG at D. BE and OG are produced to meet at F.

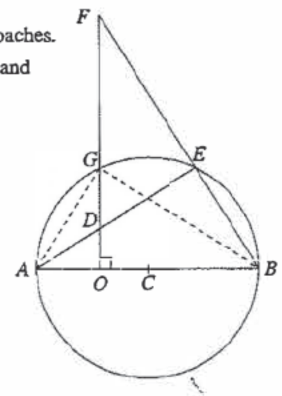
Mary and John try to prove $OD \cdot OF = OG^2$ by using two different approaches.

(a) Mary tackles the problem by first proving that $\triangle AOD \sim \triangle FOB$ and $\triangle AOG \sim \triangle GOB$. Complete the following tasks for Mary.

(i) Prove that $\triangle AOD \sim \triangle FOB$.

(ii) Prove that $\triangle AOG \sim \triangle GOB$.

(iii) Using (a)(i) and (a)(ii), prove that $OD \cdot OF = OG^2$.

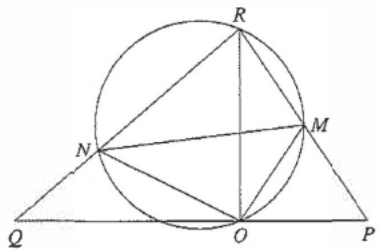


12A.22 HKCEE MA 2005 – I – 17 (To continue as 16C.26.)

(a) In the figure, MN is a diameter of the circle $MONR$. The chord RO is perpendicular to the straight line POQ. RNQ and RMP are straight lines.

(i) By considering triangles OQR and ORP, prove that $OR^2 = OP \cdot OQ$.

(ii) Prove that $\triangle MON \sim \triangle POR$.



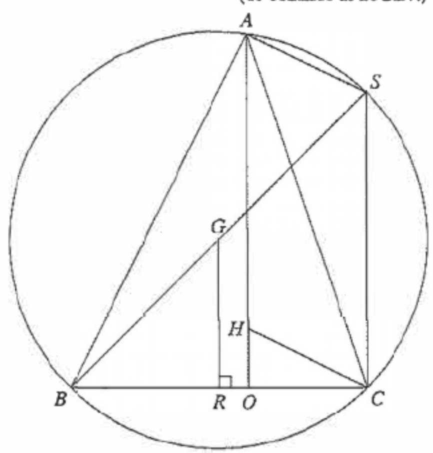
12A.23 HKCEE MA 2006 – I – 16 (To continue as 16C.27.)

In the figure, G and H are the circumcentre and the orthocentre of $\triangle ABC$ respectively. AH produced meets BC at O. The perpendicular from G to BC meets BC at R. BS is a diameter of the circle which passes through A, B and C.

(a) Prove that

(i) $AHCS$ is a parallelogram,

(ii) $AH = 2GR$.

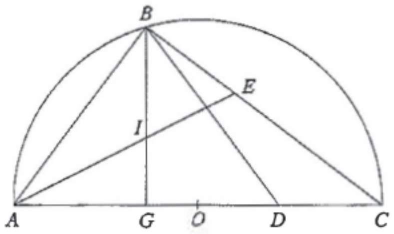


12A.24 HKCEE MA 2007 – I – 17 (To continue as **16C.28.**)

(a) In the figure, AC is the diameter of the semi circle ABC with centre O. D is a point lying on AC such that $AB = BD$. I is the in-centre of $\triangle ABD$. AI is produced to meet BC at E. BI is produced to meet AC at G.

(i) Prove that $\triangle ABG \cong \triangle DBG$.

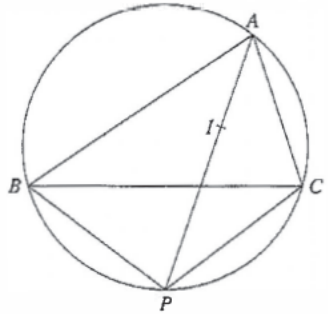
(ii) By considering the triangles AGI and ABE, prove that $\dfrac{GI}{AG} = \dfrac{BE}{AB}$.



12A.25 HKCEE MA 2008 – I – 17 (To continue as **16C.29.**)

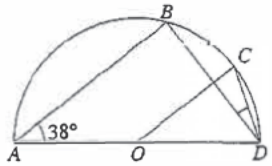
The figure shows a circle passing through A, B and C. I is the in centre of $\triangle ABC$ and AI produced meets the circle at P.

(a) Prove that $BP = CP = IP$.



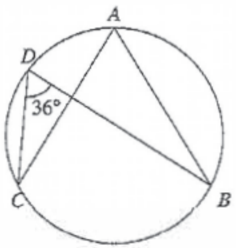
12A.26 HKDSE MA SP – I – 7

In the figure, O is the centre of the semicircle $ABCD$. If $AB // OC$ and $\angle BAD = 38°$, find $\angle BDC$.



12A.27 HKDSE MA PP – I – 7

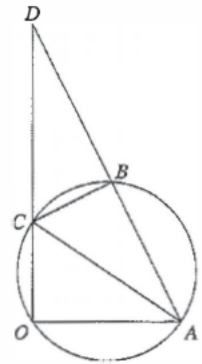
In the figure, BD is a diameter of the circle $ABCD$. If $AB = AC$ and $\angle BDC = 36°$, find $\angle ABD$.



12A.28 HKDSE MA PP – I – 14 (To continue as **16C.51.**)

In the figure, $OABC$ is a circle. It is given that AB produced and OC produced meet at D.

(a) Write down a pair of similar triangles in the fi gure.

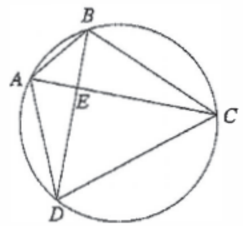


12A.29 HKDSE MA 2012 – I – 8

In the fi gure, AB, BC, CD and AD are chords of the circle. AC and BD intersect at E. It is given that $BE = 8$ cm, $CE = 20$ cm and $DE = 15$ cm.

(a) Write down a pair of similar triangles in the figure. Also find AE.

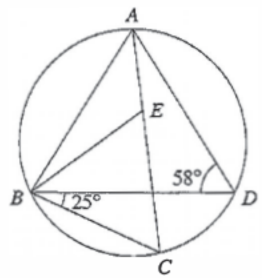
(b) Suppose that $AB = 10$ cm. Are AC and BD perpendicular to each other? Explain your answer.



12A.30 HKDSE MA 2015 – I – 8

In the figure, $ABCD$ is a circle. E is a point lying on AC such that $BC = CE$. It is given that $AB = AD$, $\angle ADB = 58°$ and $\angle CBD = 25°$.

Find $\angle BDC$ and $\angle ABE$.

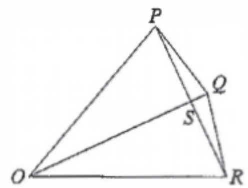


12A.31 HKDSE MA 2017 – I – 10 (Continued from **11B.11.**)

In the figure, $OPQR$ is a quadrilateral such that $OP = OQ = OR$. OQ and PR intersect at the point S. S is the mid-point of PR.

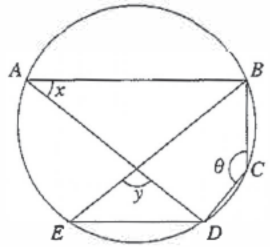
(a) Prove that $\triangle OPS \cong \triangle ORS$.

(b) It is given that O is the centre of the circle which passes through P, Q and R. If $OQ = 6$ cm and $\angle PRQ = 10°$, fi nd the area of the sector $OPQR$ in terms of π.



12A.32 HKDSE MA 2018 – I – 8

In the figure, $ABCDE$ is a circle. It is given that $AB//ED$. AD and BE intersect at the point F.
Express x and y in terms of θ.

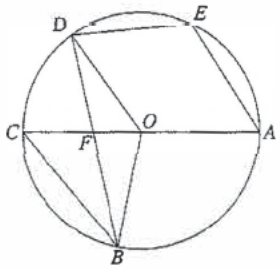


12A.33 HKDSE MA 2019 – I – 13

In the figure, O is the centre of circle $ABCDE$. AC is a diameter of the circle. BD and OC intersect at the point F. It is given that $\angle AED = 115°$.

(a) Find $\angle CBF$.

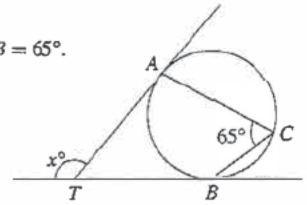
(b) Suppose that $BC//OD$ and $OB = 18$ cm. Is the perimeter of the sector OBC less than 60 cm? Explain your answer.



12B Tangents of circles

12B.1 HKCEE MA 1980(1*) – I 8

In the figure, TA and TB touch the circle at A and B respectively. $\angle ACB = 65°$.
Find the value of x.



12B.2 HKCEE MA 1981(2) I – 13

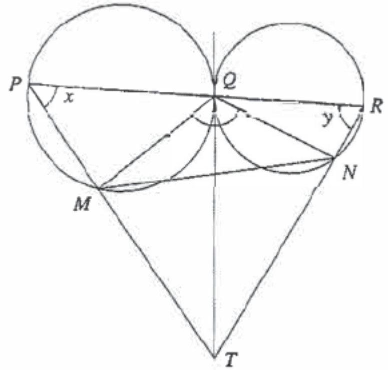
In the figure, circles PMQ and QNR touch each other at Q. QT is a common tangent. PQR is a straight line. TP and TR cut the circles at M and N respectively.

(a) If $\angle P = x$ and $\angle R = y$, express $\angle MQN$ in terms of x and y.

(b) Prove that Q, M, T and N are concyclic.

(c) Prove that P, M, N and R are concyclic.

(d) There are several pairs of similar triangles in the figure. Name any two pairs (no proof is required).



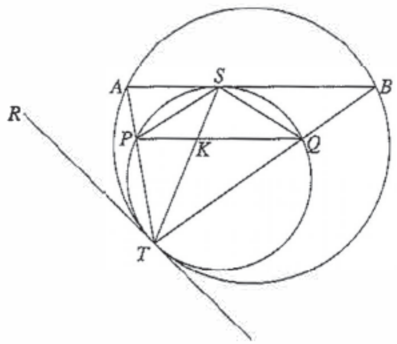
12B.3 HKCEE MA 1982(2) I – 14

In the figure, two circles touch internally at T. TR is their common tangent. AB touches the smaller circle at S. AT and BT cut the smaller circle at P and Q respectively. PQ and ST intersect at K.

(a) Prove that $PQ//AB$.

(b) Prove that ST bisects $\angle ATB$.

(c) ΔSTQ is similar to four other triangles in the figure. Write down any three of them.
(No proof is required.)

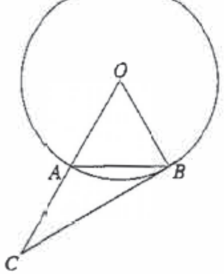


12B.4 HKCEE MA 1983(A/B) – I 2

In the figure, O is the centre of the circle. A and B are two points on the circle such that OAB is an equilateral triangle. OA is produced to C such that $OA = AC$.

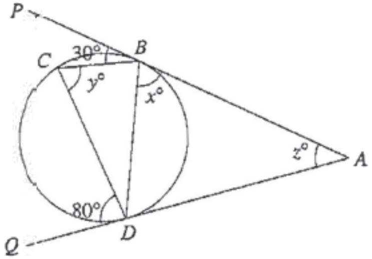
(a) Find $\angle ABC$.

(b) Is CB a tangent to the circle at B? Give a reason for your answer.



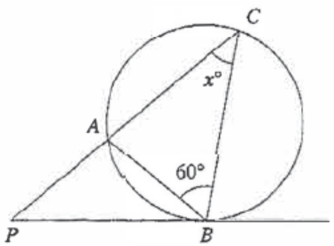
12B.5 HKCEE MA 1984(A/B) I – 5

In the figure, AP and AQ touch the circle BCD at B and D respectively. $\angle PBC = 30°$ and $\angle CDQ = 80°$. Find the values of x, y and z.



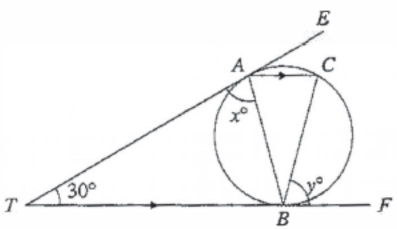
12B.6 HKCEE MA 1985(A/B) – I – 2

In the figure, PB touches the circle ABC at B. PAC is a straight line. $\angle ABC = 60°$. $AP = AB$. Find the value of x.



12B.7 HKCEE MA 1986(A/B) I – 2

In the figure, TAE and TBF are tangents to the circle ABC. If $\angle ATB = 30°$ and $AC//TF$, find x and y.

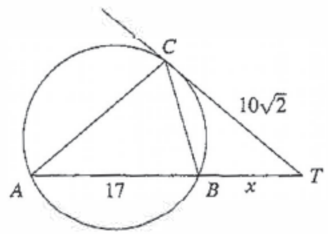


12B.8 HKCEE MA 1986(A/B) – I – 6

In the figure, A, B and C are three points on the circle. CT is a tangent and ABT is a straight line.

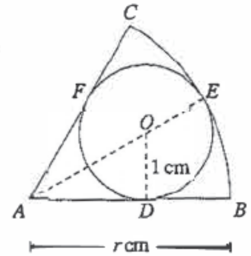
(a) Name a triangle which is similar to $\triangle BCT$.

(b) Let $BT = x$, $AB = 17$ and $CT = 10\sqrt{2}$. Find x.



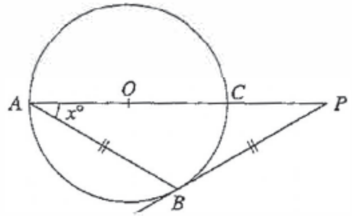
12B.9 HKCEE MA 1987(A/B) I 6

The figure shows a circle, centre O, inscribed in a sector ABC. D, E and F are points of contact. $OD = 1$ cm, $AB = r$ cm and $\angle BAC = 60°$. Find r.



12B.10 HKCEE MA 1987(A/B) I – 7

In the figure, O is the centre of the circle. $AOCP$ is a straight line, PB touches the circle at B, $BA = BP$ and $\angle PAB = x°$. Find x.

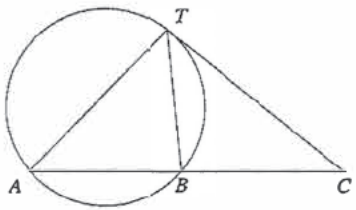


12B.11 HKCEE MA 1988 I – 8(b)

In the figure, CT is tangent to the circle ABT.

(i) Find a triangle similar to $\triangle ACT$ and give reasons.

(ii) If $CT = 6$ and $BC = 5$, find AB.



12B.12 HKCEE MA 1991 I – 13

In the figure, A, B are the centres of the circles DEC and DFC respectively. ECF is a straight line.

(a) Prove that triangles ABC and ABD are congruent.

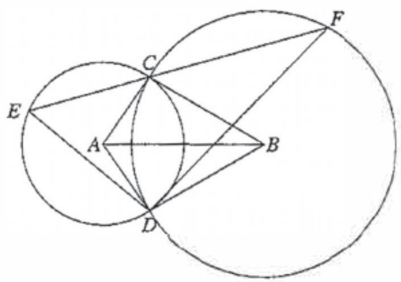
(b) Let $\angle FED = 55°$, $\angle ACB = 95°$.

  (i) Find $\angle CAB$ and $\angle EFD$.

  (ii) A circle S is drawn through D to touch the line CF at F.

    (1) Draw a labelled rough diagram to represent the above information.

    (2) Show that the diameter of the circle S is $2DF$.



12B.13 HKCEE MA 1995 – I – 14

In Figure (1), AP and AQ are tangents to the circle at P and Q. A line through A cuts the circle at B and C and a line through Q parallel to AC cuts the circle at R. PR cuts BC at M.

(a) Prove that
  (i) M, P, A and Q are concyclic;
  (ii) $MR = MQ$.

(b) If $\angle PAC = 20^\circ$ and $\angle QAC = 50^\circ$, find $\angle QPR$ and $\angle PQR$. (You are not required to give reasons.)

(c) The perpendicular from M to RQ meets RQ at H (see Figure (2)).
  (i) Explain briefly why MH bisects RQ.
  (ii) Explain briefly why the centre of the circle lies on the line through M and H.

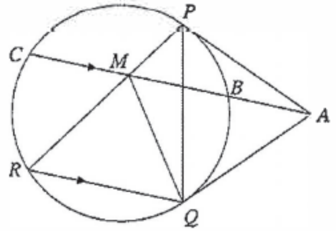


Figure (1)

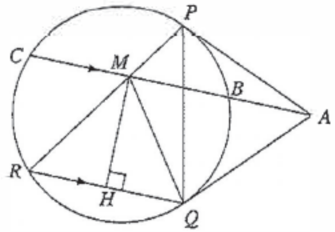
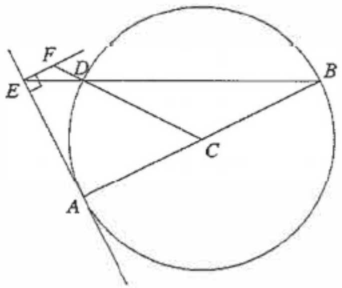


Figure (2)

12B.14 HKCEE MA 1997 – I – 16 (To continue as **16C.18.**)

(a) In the figure, D is a point on the circle with AB as diameter and C as the centre. The tangent to the circle at A meets BD produced at E. The perpendicular to this tangent through E meets CD produced at F.
  (i) Prove that $AB // EF$.
  (ii) Prove that $FD = FE$.
  (iii) Explain why F is the centre of the circle passing through D and touching AE at E.

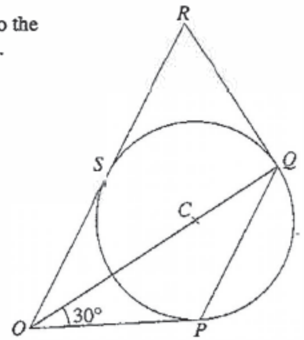


12B.15 HKCEE MA 2000 – I – 16 (To continue as **16C.21.**)

In the figure, C is the centre of the circle PQS. OR and OP are tangent to the circle at S and P respectively. OCQ is a straight line and $\angle QOP = 30^\circ$.

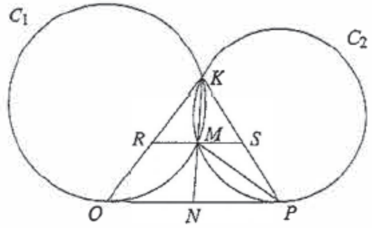
(a) Show that $\angle PQO = 30^\circ$.

(b) Suppose $OPQR$ is a cyclic quadrilateral.
  (i) Show that RQ is tangent to circle PQS at Q.



12B.16 HKCEE MA 2003 – I – 17 (To continue as **16C.24.**)

(a) In the figure, OP is a common tangent to the circles C_1 and C_2 at the points O and P respectively. The common chord KM when produced intersects OP at N. R and S are points on KO and KP respectively such that the straight line RMS is parallel to OP.
  (i) By considering triangles NPM and NKP, prove that $NP^2 = NK \cdot NM$.
  (ii) Prove that $RM = MS$.



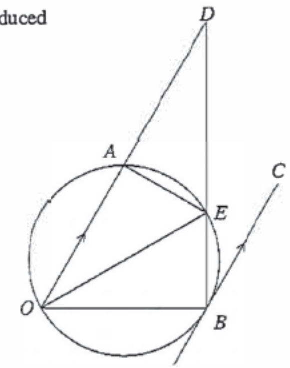
12B.17 HKCEE MA 2004 I 16(a),(b),(c)(i) (To continue as **16C.25.**)

In the figure, BC is a tangent to the circle OAB with $BC // OA$. OA is produced to D such that $AD = OB$. BD cuts the circle at E.

(a) Prove that $\triangle ADE \cong \triangle BOE$.

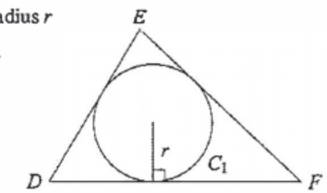
(b) Prove that $\angle BEO = 2\angle BOE$.

(c) Suppose OE is a diameter of the circle $OAEB$.
  (i) Find $\angle BOE$.



12B.18 HKCEE AM 2002 – 15 (To continue as **16C.45.**)

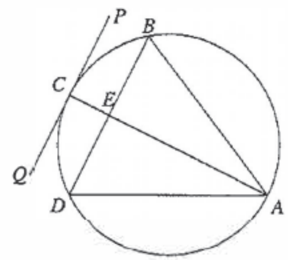
(a) DEF is a triangle with perimeter p and area A. A circle C_1 of radius r is inscribed in the triangle (see the figure). Show that $A = \frac{1}{2}pr$.



12B.19 HKDSE MA SP – I – 19 (To continue as **16C.50.**)

In the figure, the circle passes through four points A, B, C and D. PQ is the tangent to the circle at D and is parallel to BD. AC and BD intersect at E. It is given that $AB = AD$.

(a) (i) Prove that $\triangle ABE \cong \triangle ADE$.
  (ii) Are the in centre, the orthocentre, the centroid and the circum centre of $\triangle ABD$ collinear? Explain your answer.



12B.20 HKDSE MA 2016 – I – 20 (To continue as **16C.54.**)

$\triangle OPQ$ is an obtuse-angled triangle. Denote the in-centre and the circumcentre of $\triangle OPQ$ by I and J respectively. It is given that P, I and J are collinear.

(a) Prove that $OP = PQ$.

12B.21 HKDSE MA 2019 I 17 (To continue as **16D.14.**)

(a) Let a and p be the area and perimeter of $\triangle CDE$ respectively. Denote the radius of the inscribed circle of $\triangle CDE$ by r. Prove that $pr = 2a$.

12 Geometry of Circles

12A Angles and chords in circles

12A.1 HKCEE MA 1980(1/1*/3) – I – 10

(a) $\angle PAX = 2\theta$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
Similarly, $\angle QBX = \angle RCX = 2\theta$

(b) Areas of sector $OAP : OBQ : OCR = (OA : OB : OC)^2$
$= 4 : 9 : 16$

(c) $\cos\angle RCX = \dfrac{CD}{CR} = \dfrac{2}{4} = \dfrac{1}{2} \Rightarrow 2\theta = 60° \Rightarrow \theta = 30°$

12A.2 HKCEE MA 1980(1*) – I – 14

(a) $\angle CAD = \angle CAD$ (common)
$x + \angle CAD = \angle CAD + y$ (given)
$\Rightarrow \angle BAD = \angle CAE$
In $\triangle ABD$ and $\triangle ACE$,
$AB = AC$ (given)
$\angle BAD = \angle CAE$ (proved)
$AD = AE$ (given)
$\therefore \triangle ABD \cong \triangle ACE$ (SAS)

(b) $\because \angle ABK = \angle ACK$ (corr. $\angle$s, $\cong \triangle$s)
$\therefore ABCK$ is cyclic. (converse of $\angle$s in the same segment)

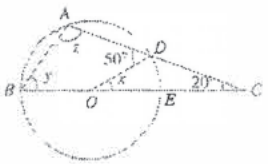
(c) $AEDK$

12A.3 HKCEE MA 1981(2) – I – 7

$\angle OBA = 40°$ (base $\angle$s, isos. $\triangle$)
$\angle BOA = 180° - 40° - 40° = 100°$ ($\angle$ sum of $\triangle$)
$\angle BCA = 100° \div 2 = 50°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

12A.4 HKCEE MA 1982(2) – I – 6

$x = 50° - 20° = 30°$ (ext. $\angle$ of $\triangle$)
Let OC meet the circle at E. Then
$\angle BOD = 180° \quad x = 150°$ (adj. $\angle$s on st. line)
$\Rightarrow \angle BED = 150° \div 2 = 75°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
$\therefore z = 180° - \angle BED = 105°$ (opp. $\angle$s, cyclic quad.)
$\Rightarrow y = 180° - 20° - z = 55°$ ($\angle$ sum of $\triangle$)



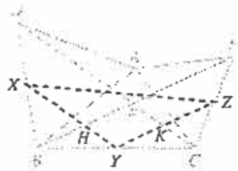
12A.5 HKCEE MA 1982(2) – I – 13

(a) $\angle DAB = \angle EAC = 60°$ (property of equil. $\triangle$)
$\angle DAB + \angle BAC = \angle EAC + \angle BAC$
$\angle DAC = \angle BAE$
In $\triangle ADC$ and $\triangle ABE$,
$DA = BA$ (property of equil. $\triangle$)
$\angle DAC = \angle BAE$ (proved)
$AC = AE$ (property of equil. $\triangle$)
$\therefore \triangle ADC \cong \triangle ABE$ (SAS)
$\therefore DC = BE$ (corr. sides, $\cong \triangle$s)

(b) (i) $\because \angle ADC = \angle ABF$ (corr. $\angle$s, $\cong \triangle$s)
$\therefore A, D, B$ and F are concyclic.
(converse of $\angle$s in the same segment)

(ii) $\angle BFD = \angle BAD = 60°$ ($\angle$s in the same segment)

(c)



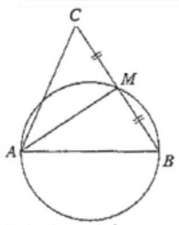
$\because BX = XD$ and $BY = YC$ (given)
$\therefore XY = \dfrac{1}{2}DC$ and $XY // DC$ (mid-pt thm)
Similarly, $YZ = \dfrac{1}{2}BE$ and $YZ // BE$ (mid-pt thm)
$\because DC = BE$ (proved) $\therefore XY = YZ$
$\because \angle BFD = 60°$ (proved)
$\therefore \angle BFC = 180° - 60° = 120°$ (adj. $\angle$s on st. line)
and $\angle CFE = 60°$ (vert. opp. $\angle$s)

Suppose XY meets BE at H and YZ meets DC at K. Then
$\angle YHF = \angle CFE = 60°$ (corr. $\angle$s, $XY // DC$)
$\angle YKF = \angle BFD = 60°$ (corr. $\angle$s, $YZ // BE$)

Hence,
$\angle XYZ = 360° - \angle YHF - \angle YKF - \angle BFC = 120°$
($\angle$ sum of polygon)
$\angle XZY = \angle ZXY$ (base $\angle$s, isos. $\triangle$)
$= (180° - 120°) \div 2 = 30°$ ($\angle$ sum of $\triangle$)

12A.6 HKCEE MA 1989 – I – 4

(a)



(b) In $\triangle ABM$ and $\triangle ACM$,
$AM = AM$ (common)
$MB = MC$ (given)
$\angle AMB = \angle AMC = 90°$ ($\angle$ in semi-circle)
$\therefore \triangle ABM \cong \triangle ACM$ (SAS)
$\therefore \angle BAM = \angle CAM$ (corr. $\angle$s, $\cong \triangle$s)
i.e. AM bisects $\angle BAC$.

12A.7 HKCEE MA 1989 – I – 6

(a) $\angle ABD = \angle ACD = 60°$ ($\angle$s in the same segment)
$\angle BAD = 180° - (60° + 40°)$ (opp. $\angle$s, cyclic quad.)
$= 80°$

12A.8 HKCEE MA 1990 – I – 9

(a) In $\triangle ABD$ and $\triangle ACD$,
$\angle ADB = \angle ADC = 90°$ ($\angle$ in semi-circle)
$AB = AC$ (given)
$AD = AD$ (common)
$\therefore \triangle ABD \cong \triangle ACD$ (RHS)

(b) In $\triangle ABD$ and $\triangle ADE$,
$\angle ABD = \angle ADE$ ($\angle$ in alt. segment)
$\angle BAD = \angle DAE$ (corr. $\angle$s, $\cong \triangle$s)
$\angle ADB = \angle AED$ ($\angle$ sum of $\triangle$)
$\therefore \triangle ABD \sim \triangle ADE$ (AAA)

(c) (i) $AD = \sqrt{AB^2 - BD^2} = 3$ (Pyth. thm)
$\dfrac{AB}{BD} = \dfrac{AD}{DE}$ (corr. sides, $\cong \triangle$s)
$\dfrac{5}{4} = \dfrac{3}{DE}$
$DE = 2.4$

(ii) $\angle AED = \angle ADB = 90°$ (corr. $\angle$s, $\sim \triangle$s)
$\angle CFB = 90°$ ($\angle$ in semi-circle)
In $\triangle CFB$ and $\triangle CDA$,
$\angle CFB = \angle CDA = 90°$ (proved)
$\angle C = \angle C$ (common)
$\angle CBF = \angle CAD$ ($\angle$ sum of $\triangle$)
$\therefore \triangle CFB \sim \triangle CDA$ (AAA)
$\therefore \dfrac{CF}{CB} = \dfrac{CD}{CA}$ (corr. sides, $\cong \triangle$s)
$\dfrac{AC + AF}{CD + DB} = \dfrac{CD}{CA}$
$\dfrac{5 + AF}{4 + 4} = \dfrac{4}{5} \Rightarrow AF = 1.4$

12A.9 HKCEE MA 1992 – I – 11

(a) $e_3 = d$ (corr. $\angle$s, $FE // AD$)
$b = d$ ($\angle$s in the same segment)
$d = f_1$ (ext. $\angle$, cyclic quad.)
$\therefore e_3 = f_1$
i.e. $\triangle EFY$ is isosceles. (sides opp. equal $\angle$s)

(b) $\because \widehat{BCD} = \widehat{AFE}$ (given)
$\therefore e_1 = b$ (equal arcs, equal $\angle$s)
$\therefore BA // DE$ (alt. $\angle$s equal)

(c) $f_1 = b$ (ext. $\angle$, cyclic quad.)
$= e_1$ (proved)
$e_3 = d$ (proved)
$\therefore f_1 + e_3 + y = 180°$ ($\angle$ sum of $\triangle$)
$\Rightarrow (e_1) + (d) + y = 180°$
$x + y = 180°$ (ext. $\angle$ of $\triangle$)
$\therefore A, X, E$ and Y are concyclic. (opp. $\angle$s supp.)

(d) $f_1 = b = 47°$ (proved)
$e_3 = f_1 = 47°$ (proved)
$\therefore y = 180° \quad f_1 - e_3 = 86°$ ($\angle$ sum of $\triangle$)
$x = 180° - y = 94°$ (opp. $\angle$s, cyclic quad.)

12A.10 HKCEE MA 1993 – I – 11

(a) $\angle ABP = 90°$ ($\angle$ in semi-circle)
$\angle PQD = 90°$ (given)
$\because \angle ABP = \angle PQD$
$\therefore A, Q, P$ and B are concyclic. (ext. $\angle$ = int. opp. $\angle$)

(b) (i) $\angle BAC = \angle BQP = \theta$ ($\angle$s in the same segment)
$\Rightarrow \angle BDC = \theta$ ($\angle$s in the same segment)
Similar to (a), we get D, Q, P and C are concyclic.
$\Rightarrow \angle PQC = \angle BDC = \theta$ ($\angle$s in the same segment)
$\therefore \angle BQC = \angle BQP + \angle PQC = 2\theta$

(ii) $\angle BOC = 2\angle BAC = 2\theta$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

(c) $\because \angle BQC = \angle BOC = 2\theta$ (proved)
$\therefore BOQC$ is cyclic. (converse of $\angle$s in the same segment)
$\therefore \angle CBQ = \angle COQ$ ($\angle$s in the same segment)
$2\angle CAD = 2\phi$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

12A.11 HKCEE MA 1994 – I – 13

(a) $d = b$ (ext. $\angle$, cyclic quad.)
$\therefore g = 180° - d - \angle DEG$ ($\angle$ sum of $\triangle$)
$= 180° - d - e$
$k_2 = k_1$ (vert. opp. $\angle$s)
$= 180° - b - \angle AEG$ ($\angle$ sum of $\triangle$)
$= 180° - d - e = g$ (proved)
$\therefore \angle FGH = \angle FKH$

(b) $h_2 = g + \angle GFH = g + f$ (ext. $\angle$ of $\triangle$)
$h_1 = k_2 + \angle KFH = k_2 + f$ (ext. $\angle$ of $\triangle$)
$= g + f = h_2$ (proved)
$\therefore h_1 = h_2 = 180° \div 2 = 90°$ (adj. $\angle$s on st. line)
i.e. $FH \perp GK$

(c) (i) $d = 180° - a - 2e$ ($\angle$ sum of $\triangle$)
$= 180° - a \quad 2f$ (given)
$= \angle ABF$ ($\angle$ sum of $\triangle$)
$\because d + \angle ABF = 180°$ (opp. $\angle$s, cyclic quad.)
$\therefore d = 180° \div 2 = 90°$
Hence, $d = h_2 = 90°$ (proved)
$\Rightarrow D, J, H$ and G are concyclic. (ext. $\angle$ = int. opp. $\angle$)

(ii) $d = 180° \quad 28° - a = 152° - a$ ($\angle$ sum of $\triangle$)
$b = a + 46°$ (ext. $\angle$ of $\triangle$)
$152° \quad a = a + 46°$ (ext. $\angle$, cyclic quad.)
$a = 53°$
$\therefore \angle BCD = 180° \quad 53°$ (opp. $\angle$s, cyclic quad.)
$= 127°$

12A.12 HKCEE MA 1996 – I – 6

$\angle BAP = \angle DCP$ (ext. $\angle$, cyclic quad.)
$= \angle ABP$ (corr. $\angle$s, $AB // DC$)
$\therefore AP = BP$ (sides opp. equal $\angle$s)

12A.13 HKCEE MA 1997 – I – 9

(a) $\angle BDC = \angle BAC = 30°$ ($\angle$s in the same segment)
$\angle ADB = 90° - \angle BDC = 60°$ ($\angle$ in semi-circle)

(b) $\widehat{AB} : \widehat{BC} = \angle ADB : \angle BDC = 2 : 1$ (arcs prop. to $\angle$s at $\odot^{ce}$)

(c) $\angle ABC = 90°$ ($\angle$ in semi-circle)
$\Rightarrow AB = 4\cos 30° = 2\sqrt{3}$, $BC = 4\sin 30° = 2$
$\therefore AB : BC = \sqrt{3} : 1$

12A.14 HKCEE MA 1998 – I – 6

(a) $\triangle EBA$

(b) $\dfrac{y}{3} = \dfrac{6}{4} \Rightarrow y = \dfrac{9}{2}$ (corr. sides, $\sim \triangle$s)

12A.15 HKCEE MA 1998 – I – 14

$\because OB = OD$ (radii)
$\therefore \angle ODB = \angle OBD$ (base $\angle$s, isos. $\triangle$)
$\because CB = BA$ (given)
$\therefore \angle CDB = \angle BDA$ (equal chords, equal $\angle$s)
$= \angle OBD$
$\therefore BO // CD$ (alt. $\angle$s equal)

12A.16 HKCEE MA 1999 – I – 5

$\angle ADC = 90°$ ($\angle$ in semi-circle)
$\angle ADB = 50°$ ($\angle$s in the same segment)
$\therefore\ y = 90 - 50 = 40$
$x = 180 - 20 \quad 90 = 70$ ($\angle$ sum of $\triangle$)

12A.17 HKCEE MA 1999 – I – 16

(a) (i) $\angle BFE = \angle BDE$ ($\angle$s in the same segment)
$= \angle BAC$ (corr. $\angle$s, $AC // DE$)
$\therefore$ A, F, B and C are concyclic.
(converse of $\angle$s in the same segment)
(ii) $\because\ \angle ABC = 90°$ (given)
$\therefore$ AC is a diameter of circle $AFBC$.
(converse of $\angle$ in sem-circle)
$\Rightarrow$ M is the centre of circle $AFBC$ $\Rightarrow$ $MB = MF$

12A.18 HKCEE MA 2000 – I – 7

$x = 25$ ($\angle$ in alt. segment) AD//BC
$\angle DBC = \angle DAC = 25°$ ($\angle$s in the same segment)
$\angle DAB + \angle ABC = 180°$ (int. $\angle$s, $AD // BC$)
$\therefore\ y = 180 - 25 - 56 - 25 = 74$

12A.19 HKCEE MA 2001 – I – 5

$\angle ADC = 90°$ ($\angle$ in semi-circle)
$\angle ACD = 30°$ ($\angle$s in the same segment)
$\therefore\ \angle DAC = 180° - 90° - 30° = 60°$ ($\angle$ sum of $\triangle$)

12A.20 HKCEE MA 2002 – I – 9

$\angle BCD = 90°$ ($\angle$ in semi-circle)
$\angle DBC = 180° - 90° - 40° = 50°$ ($\angle$ sum of $\triangle$)
$\angle BAC = 40°$ ($\angle$s in the same segment)
$\angle ABC = \angle ACB$ (base $\angle$s, isos. $\triangle$)
$= (180° - 40°) \div 2 = 70°$ ($\angle$ sum of $\triangle$)
$\therefore\ \angle ABD = 70° - 50° = 20°$

12A.21 HKCEE MA 2002 – I – 16

(a) (i) $\angle AEB = 90°$ ($\angle$ in semi-circle)
$\angle DAO = 180° - \angle AEB - \angle ABE$ ($\angle$ sum of $\triangle$)
$= 90° - \angle ABE$
$\angle BFO = 180° - \angle FOB - \angle ABE$ ($\angle$ sum of $\triangle$)
$= 90° \quad \angle ABE$
$\therefore\ \angle DAO = \angle BFO$
In $\triangle AOD$ and $\triangle FOB$,
$\angle DAO = \angle BFO$ (proved)
$\angle AOD = \angle FOB = 90°$ (given)
$\angle ADO = \angle FBO$ ($\angle$ sum of $\triangle$)
$\therefore\ \triangle AOD \sim \triangle FOB$ (AAA)
(ii) $\angle AGB = 90°$ ($\angle$ in semi-circle)
$\angle GAO = 180° - \angle AGO - \angle AOG$ ($\angle$ sum of $\triangle$)
$= 90° - \angle AGO = \angle BGO$
In $\triangle AOG$ and $\triangle GOB$,
$\angle GAO = \angle BGO$ (proved)
$\angle AOG = \angle GOB = 90°$ (given)
$\angle OGA = \angle OBG$ ($\angle$ sum of $\triangle$)
$\therefore\ \triangle AOG \sim \triangle GOB$ (AAA)
(iii) From (i), $\dfrac{AO}{OD} = \dfrac{FO}{OB}$ (corr. sides, $\sim\triangle$s)
$AO \cdot OB = OD \cdot OF$
From (ii), $\dfrac{AO}{OG} = \dfrac{GO}{OB}$ (corr. sides, $\sim\triangle$s)
$AO \cdot OB = OG^2$
$\therefore\ OD \cdot OF = OG^2$

12A.22 HKCEE MA 2005 – I – 17

(a) (i) $\because$ MN is a diameter (given)
$\therefore\ \angle NOM = \angle QRP = 90°$ ($\angle$ in semi-circle)
In $\triangle OQR$ and $\triangle ORP$,
$\angle ROQ = \angle POR = 90°$ (given)
$\angle QRO = \angle QRP - \angle PRO$
$= 90° - \angle PRO$
$\angle POR = 180° - \angle ROP - \angle PRO$
($\angle$ sum of $\triangle$)
$= 90° - \angle PRO$
$\Rightarrow\ \angle QPO = \angle PRO$
$\angle RQO = \angle PRO$ ($\angle$ sum of $\triangle$)
$\therefore\ \triangle OQR \sim \triangle ORP$ (AAA)
$\Rightarrow\ \dfrac{OR}{OQ} = \dfrac{OP}{OR}$ (corr. sides, $\sim\triangle$s)
$OR^2 = OP \cdot OQ$
(ii) In $\triangle MON$ and $\triangle POR$,
$\angle NMO = \angle QRO$ ($\angle$s in the same segment)
$= \angle RPO$ (proved)
$\angle MON = \angle POR$ (proved)
$\angle MNO = \angle RQO$ ($\angle$ sum of $\triangle$)
$\therefore\ \triangle MON \sim \triangle RQO$ (AAA)

12A.23 HKCEE MA 2006 – I – 16

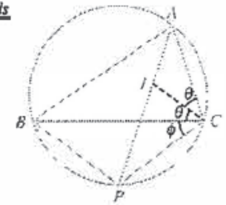
(a) (i) $\because$ G is the circumcentre (given)
$\therefore\ SC \perp BC$ and $SA \perp AB$ ($\angle$ in semi-circle)
$\because$ H is the orthocentre (given)
$\therefore\ AH \perp BC$ and $CH \perp AB$
Thus, $SC // AH$ and $SA // CH$ $\Rightarrow$ $AHCS$ is a //gram.
(ii) *Method 1*
$\because\ \angle GRB = \angle SCB = 90°$ (proved)
$\therefore\ GR // SC$ (corr. $\angle$s equal)
$\because\ BG = GS =$ radius
$\therefore\ BR = RC$ (intercept thm)
$\Rightarrow\ SC = 2GR$ (mid-pt thm)
Hence, $AH = SC = 2GR$ (property of //gram)
Method 2
$\because\ BG = GS =$ radius
and $BR = RC$ ($\perp$ from centre to chord bisects chord)
$\Rightarrow\ SC = 2GR$ (mid-pt thm)
Hence, $AH = SC = 2GR$ (property of //gram)

12A.24 HKCEE MA 2007 – I – 17

(a) (i) $\because$ I is the incentre of $\triangle ABD$ (given)
$\therefore\ \angle ABG = \angle DBG$ and $\angle BAE = \angle CAE$
In $\triangle ABG$ and $\triangle DBG$,
$\angle ABG = \angle DBG$ (proved)
$AB = DB$ (given)
$BG = BG$ (common)
$\therefore\ \triangle ABG \cong \triangle DBG$ (SAS)
(ii) $\because\ \triangle ABD$ is isosceles and $\angle ABG = \angle DBG$
$\therefore\ \angle BGA = 90°$ (property of isos. $\triangle$)
In $\triangle AGI$ and $\triangle ABE$,
$\angle AGI = 90° = \angle ABE$ ($\angle$ in semi-circle)
$\angle IAG = \angle EAB$ (proved)
$\angle AIG = \angle AEB$ ($\angle$ sum of $\triangle$)
$\therefore\ \triangle AGI \sim \triangle ABE$ (AAA)
$\Rightarrow\ \dfrac{GI}{AG} = \dfrac{BE}{AB}$ (corr. sides, $\sim\triangle$s)

12A.25 HKCEE MA 2008 – I – 17

(a) *Method 1*
$\because$ I is the incentre of $\triangle ABC$ (given)
$\therefore\ \angle BAP = \angle CAP$
$\therefore\ BP = CP$ (equal $\angle$s, equal chords)
Method 2
$\because$ I is the incentre of $\triangle ABC$ (given)
$\therefore\ \angle BAP = \angle CAP$
$\angle BCP = \angle BAP$ ($\angle$s in the same segment)
$= \angle CAP$ (proved)
$= \angle CBP$ ($\angle$s in the same segment)
$\Rightarrow\ BP = CP$ (sides opp. equal $\angle$s)
Both methods

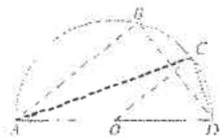


Join CI. Let $\angle ACI = \angle BCI = \theta$ and $\angle BCP = \phi$.
$\angle PAC = \phi$ (equal chords, equal $\angle$s)
$\Rightarrow\ \angle PIC = \angle PAC + \angle ACI = \theta + \phi$ (ext. $\angle$ of $\triangle$)
$= \angle PCI$
$\therefore\ IP = CP$ (sides opp. equal $\angle$s)
i.e. $BP = CP = IP$

12A.26 HKDSE MA SP – I – 7

Method 1
$\angle ABD = 90°$ ($\angle$ in semi-circle)
$\angle BDA = 180° - 90° - 38° = 52°$ ($\angle$ sum of $\triangle$)
$\angle COD = 38°$ (corr. $\angle$s, $AB // OC$)
$\because\ OC = OD$ (radii)
$\therefore\ \angle ODC = \angle OCD$ (base $\angle$s, isos. $\triangle$)
$= (180° - 38°) \div 2 = 71°$ ($\angle$ sum of $\triangle$)
Hence, $\angle BDC = 71° - 52° = 19°$
Method 2

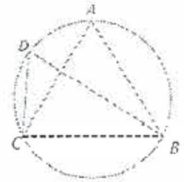
$\angle BOD = 2(38°) = 76°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
$\angle COD = 38°$ (corr. $\angle$s, $AB // OC$)
$\Rightarrow\ \angle BOC = 76° - 38° = 38°$
$\therefore\ \angle BDC = 38° \div 2 = 19°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
Method 3



$\angle COD = 38°$ (corr. $\angle$s, $AB // OC$)
$OA = OC$ (radii)
$\Rightarrow\ \angle OAC = \angle OCA$ (base $\angle$s, isos. $\triangle$)
$= \angle COD \div 2 = 19°$ (ext. $\angle$ of $\triangle$)
$\therefore\ \angle BAC = 38° - 19° = 19°$
$\Rightarrow\ \angle BDC \quad \angle BAC \quad 19°$ ($\angle$s in the same segment)

12A.27 HKDSE MA PP – I – 7

$\angle DCB = 90°$ ($\angle$ in semi-circle)
$\Rightarrow\ \angle DBC = 180° \quad 90° \quad 36° = 54°$ ($\angle$ sum of $\triangle$)
$\angle CAB = 36°$ ($\angle$s in the same segment)
$\angle ABC = \angle ACB$ (base $\angle$s, isos. $\triangle$) / (equal chords, equal $\angle$s)
$= (180° - \angle CAB) \div 2 = 72°$ ($\angle$ sum of $\triangle$)
$\therefore\ \angle ABD = 72° - 54° = 18°$



12A.28 HKDSE MA PP – I – 14

(a) $\triangle AOD \sim \triangle CBD$

12A.29 HKDSE MA 2012 – I – 8

(a) $\triangle AED \sim \triangle BEC$
$\therefore\ \dfrac{AE}{DE} = \dfrac{BE}{CE}$ (corr. sides, $\sim\triangle$s)
$\Rightarrow\ AE = \dfrac{8}{20} \times 15 = 6$ (cm)
(b) $AB^2 = 10^2 = 100$
$AE^2 + EB^2 = 6^2 + 8^2 = 100 = AB^2$
$\therefore\ AC \perp BD$ (converse of Pyth. thm)

12A.30 HKDSE MA 2015 – I – 8

Method 1
$\angle ACB = \angle ADB = 58°$ ($\angle$s in the same segment)
$\angle ABD = \angle ADB$ (base $\angle$s, isos. $\triangle$) / (equal chords, equal $\angle$s)
$= 58°$
$\angle BDC = \angle BAC$ ($\angle$s in the same segment)
$= 180° \quad \angle ABC - \angle ACB$ ($\angle$ sum of $\triangle$)
$= 180° - (58° + 25°) - 58° = 39°$
Method 2
$\angle ABD = \angle ADB$ (base $\angle$s, isos. $\triangle$) / (equal chords, equal $\angle$s)
$= 58°$
$\angle ADC + \angle ABC = 180°$ (opp. $\angle$s, cyclic quad.)
$58° + \angle BDC + (58° + 25°) = 180°$
$\angle BDC = 39°$
Both methods
$\angle BAC = \angle BDC = 39°$ ($\angle$s in the same segment)
In $\triangle BCE$, $\angle BEC = \angle EBC$ (base $\angle$s, isos. $\triangle$)
$= (180° - \angle BCA) \div 2$ ($\angle$ sum of $\triangle$)
$= 61°$
$\therefore\ \angle ABE = \angle BEC - \angle BAC = 22°$ (ext. $\angle$ of $\triangle$)

12A.31 HKDSE MA 2017 – I – 10

(a) In $\triangle OPS$ and $\triangle ORS$,
$OP = OR$ (given)
$OS = OS$ (common)
$PS = RS$ (given)
$\therefore\ \triangle OPS \cong \triangle ORS$ (SSS)
(b) $\angle ROQ = \angle POQ$ (corr. $\angle$s, $\cong\triangle$s)
$= 2\angle PRQ = 20°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
$\therefore$ Area of sector $= \dfrac{2(20°)}{360°} \times \pi(6)^2 = 4\pi$ (cm²)

12A.32 HKDSE MA 2018 – I – 8

$x = 180° - \theta$ (opp. $\angle$s, cyclic quad.)

$\angle BED = \angle BAD = x$ ($\angle$s in the same segment)

$= \angle ADE$ (alt. $\angle$s, $AB//ED$)

$\therefore$ $y = 180° - \angle BED - \angle ADE$ ($\angle$ sum of $\triangle$)

$= 180° - 2(180° - \theta) = 2\theta - 180°$

12A.33 HKDSE MA 2019 – I – 13

(a) *Method 1*

Reflex $\angle DOA = 2\angle DEA$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$= 230°$

$\Rightarrow$ $\angle DOC = 230° - 180° = 50°$

$\therefore$ $\angle CBF = \angle DOC \div 2 = 25°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

Method 2

$\angle ABD = 180° - \angle AED = 65°$ (opp. $\angle$s, cyclic quad.)

$\angle ABC = 90°$ ($\angle$ in semi-circle)

$\therefore$ $\angle CBF = 90° - 65° = 25°$

(b) $\angle OCB = \angle DOC = 50°$ (alt. $\angle$s, $BC//OD$)

$\Rightarrow$ $\angle BOC = 180° - 2\angle OCB = 80°$

$\therefore$ Perimeter of sector $OBC = 2 \times 18 + \widehat{BC}$

$= 36 + \dfrac{80°}{360°} \times 2\pi(18)$

$= 61.13 > 60$ (cm)

$\therefore$ NO

12B Tangents of circles

12B.1 HKCEE MA 1980(1*) – I – 8

$\angle TAB = \angle TBA = 65°$ ($\angle$ in alt. segment)

$\therefore$ $x = \angle TAB + \angle TBA = 130°$ (ext. $\angle$ of $\triangle$)

12B.2 HKCEE MA 1981(2) – I – 13

(a) $\angle MQT = x$ ($\angle$ in alt. segment)

$\angle NQT = y$ ($\angle$ in alt. segment)

$\therefore$ $\angle MQN = x + y$

(b) $\angle PTR = 180° - \angle TPR - \angle PRT$ ($\angle$ sum of ▲)

$= 180° - x - y$

$\because$ $\angle MQN + \angle MTN = (x+y) + (180° - x - y) = 180°$

$\therefore$ Q, M, T and N are concyclic. (opp. $\angle$s supp.)

(c) $\because$ $QMTN$ is cyclic, (proved)

$\therefore$ $\angle NMT = \angle NQT = y$ ($\angle$s in the same segment)

$\because$ $\angle NMT = \angle PRN = y$ (proved)

$\therefore$ P, M, N and R are concyclic. (ext. $\angle$ = int. opp. $\angle$)

(d) $\triangle MNT \sim \triangle RPT$, $\triangle MQT \sim \triangle QPT$, $\triangle NQT \sim \triangle QRT$

12B.3 HKCEE MA 1982(2) – I – 14

(a) $\angle ABT = \angle ATR$ ($\angle$ in alt. segment)(large circle)

$= \angle PQT$ ($\angle$ in alt. segment)(small circle)

$\therefore$ $AB//PQ$ (corr. $\angle$s equal)

(b) Consider the small circle.

$\angle QTS = \angle BSQ$ ($\angle$ in alt. segment)

$= \angle SQP$ (alt. $\angle$s, $AB//PQ$)

$= \angle STP$ ($\angle$s in the same segment)

i.e. ST bisects $\angle ATB$.

(c) ▲PTK, $\triangle ATS$, $\triangle ASP$, $\triangle SQK$

12B.4 HKCEE MA 1983(A/B) – I – 2

(a) $\angle OAB = \angle OBA = 60°$ (property of equil $\triangle$)

$AC = OA = AB$ (given)

$\therefore$ $\angle ABC = \angle ACB$ (base $\angle$s, isos. $\triangle$)

$= \angle OAB \div 2 = 30°$ (ext. $\angle$ of $\triangle$)

(b) $\because$ $\angle OBC = 60° + 30° = 90°$

$\therefore$ CB is tangent to the circle at B.

(converse of tangent $\perp$ radius)

12B.5 HKCEE MA 1984(A/B) – I – 5

$\angle CBD = 80°$ ($\angle$ in alt. segment)

$x = 180 - 30 - 80 = 70$ (adj. $\angle$s on st. line)

$y = x = 70$ ($\angle$ in alt. segment)

$AB = AD$ (tangent properties)

$\Rightarrow$ $\angle BDA = x°$ (base $\angle$s, isos. $\triangle$)

$\therefore$ $z = 180 - x - x = 40$ ($\angle$ sum of $\triangle$)

12B.6 HKCEE MA 1985(A/B) – I – 2

$\angle APB = \angle ABP$ (base $\angle$s, isos. ▲)

$= x°$ ($\angle$ in alt. segment)

$\therefore$ In $\triangle BCP$, $x° + x° + (x° + 60°) = 180°$ ($\angle$ sum of $\triangle$)

$x = 40$

12B.7 HKCEE MA 1986(A/B) – I – 2

$TA = TB$ (tangent properties)

$\angle ABT = x°$ (base $\angle$s, isos. $\triangle$)

$= (180° - 30°) \div 2$ ($\angle$ sum of $\triangle$) $\Rightarrow$ $x = 75$

$y° = \angle ACB$ (alt. $\angle$s, $AC//TF$)

$= \angle ABT = x°$ ($\angle$ in alt. segment) $\Rightarrow$ $y = 75$

12B.8 HKCEE MA 1986(A/B) – I – 6

(a) $\triangle CAT$

(b) $\because$ $\triangle BCT \sim \triangle CAT$

$\therefore$ $\dfrac{BT}{CT} = \dfrac{CT}{AT}$ (corr. sides, $\sim$ $\triangle$s)

$\dfrac{x}{10\sqrt{2}} = \dfrac{10\sqrt{2}}{17+x}$

$17x + x^2 = 200$ $\Rightarrow$ $x = 8$ or -25 (rejected)

12B.9 HKCEE MA 1987(A/B) – I – 6

$\angle ODA = 90°$ (tangent $\perp$ radius)

$\angle OAD = 60° \div 2 = 30°$ (tangent properties)

$\therefore$ $AO = \dfrac{1}{\sin 30°} = 2$ (cm)

$\therefore$ $r = AE = 2 + 1 = 3$

12B.10 HKCEE MA 1987(A/B) – I – 7

$\angle ABC = 90°$ ($\angle$ in semi-circle)

$\angle APB = \angle PAB = x°$ (base $\angle$s, isos. $\triangle$)

$= \angle CBP$ ($\angle$ in alt. segment)

$\therefore$ In $\triangle ABP$, $x° + x° + (90° + x°) = 180°$ ($\angle$ sum of $\triangle$)

$x = 30$

12B.11 HKCEE MA 1988 – I – 8(b)

(i) In $\triangle ACT$ and $\triangle TCB$,

$\angle TCA = \angle BCT$ (common)

$\angle TAC = \angle BTC$ ($\angle$ in alt. segment)

$\angle CTA = \angle CBT$ ($\angle$ sum of $\triangle$)

$\therefore$ $\triangle ACT \sim \triangle TCB$ (AAA)

(ii) $\dfrac{AC}{CT} = \dfrac{TC}{CB}$ (corr. sides, $\sim$ $\triangle$s)

$\dfrac{AB+5}{6} = \dfrac{6}{5}$ $\Rightarrow$ $AB = \dfrac{11}{5}$

12B.12 HKCEE MA 1991 – I – 13

(a) In $\triangle ABC$ and $\triangle ABD$,

$AC = AD$ (radii)

$BC = BD$ (radii)

$AB = AB$ (common)

$\therefore$ $\triangle ABC \cong \triangle ABD$ (SSS)

(b) (i) $\because$ $\angle CAD = 2(55°)$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$= 110°$

and $\angle CAB = \angle DAB$ (corr. $\angle$s, $\cong$ $\triangle$s)

$\therefore$ $\angle CAB = 110 \div 2 = 55°$

$\angle DBA = \angle CBA$ (corr. $\angle$s, $\cong$ $\triangle$s)

$= 180° - \angle ACB - \angle CAB$ ($\angle$ sum of $\triangle$)

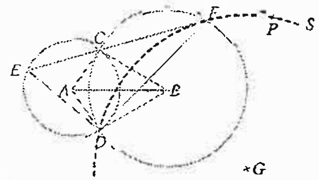
$= 30°$

$\Rightarrow$ $\angle CBD = 30° + 30° = 60°$

$\therefore$ $\angle EFD = \dfrac{1}{2}\angle CBD$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$= \dfrac{1}{2}(60°) = 30°$

(ii) (1)



(The centre of S lies on the intersection of the perpendicular bisector of DF and the line at F perpendicular to CF.)

(2) Let P be a point on major $\widehat{DF}$ and G be the centre of S.

$\angle CFD = \angle FPD = 30°$ ($\angle$ in alt. segment)

$\angle FGD = 2 \times 30°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$= 60°$

Hence, $\triangle FGD$ is equilateral.

$\Rightarrow$ Diameter $= 2GF = 2DF$

12B.13 HKCEE MA 1995 – I – 14

(a) (i) $\because$ $\angle PQA = \angle PRQ$ ($\angle$ in alt. segment)

$= \angle PMA$ (corr. $\angle$s, $AC//QR$)

$\therefore$ M, P, A and Q are concyclic.

(converse of $\angle$s in the same segment)

(ii) $\angle MQR = \angle AMQ$ (alt. $\angle$s, $AC//QR$)

$= \angle APQ$ ($\angle$s in the same segment)

$= \angle MRQ$ ($\angle$ in alt. segment)

$\therefore$ $MR = MQ$ (sides opp. equal $\angle$s)

(b) $\angle QPR = \angle QAC = 50°$ ($\angle$s in the same segment)

$\angle RMQ = \angle PAQ = 70°$ (opp. $\angle$s, cyclic quad.)

$\angle MQR = (180° - 70°) \div 2 = 55°$ ($\angle$ sum of $\triangle$)

$\angle MQP = \angle PAC = 20°$ ($\angle$s in the same segment)

$\therefore$ $\angle PQR = \angle MQR + \angle MQP = 75°$

(c) (i) Property of isos. $\triangle$

(ii) $\perp$ bisector of chord passes through centre

12B.14 HKCEE MA 1997 – I – 16

(a) (i) $\angle EAB = 90°$ (tangent $\perp$ radius)

$\because$ $\angle FEA + \angle EAB = 90° + 90° = 180°$

$\therefore$ $AB//EF$ (int. $\angle$s supp.)

(ii) $\angle FDE = \angle BDC$ (vert. opp. $\angle$s)

$= \angle DBC$ (base $\angle$s, isos. $\triangle$)

$= \angle FED$ (alt. $\angle$s, $AB//EF$)

$\therefore$ $FD = FE$ (sides opp. equal $\angle$s)

(iii) If the circle touches AE at E, its centre lies on EF.

If ED is a chord, the centre lies on the $\perp$ bisector of ED.

$\therefore$ The intersection of these two lines, F, is the centre of the circle described.

12B.15 HKCEE MA 2000 – I – 16

(a) In $\triangle OCP$, $\angle CPO = 90°$ (tangent $\perp$ radius)

$\angle PCO = 180° - 30° - 90°$ ($\angle$ sum of $\triangle$)

$\therefore$ $\angle PQO = 60° \div 2 = 30°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

(b) (i) $\angle SOC = \angle POC = 30°$ (tangent properties)

$\angle PQR = 180° - \angle POS$ (opp. $\angle$s, cyclic quad.)

$= 120°$

$\Rightarrow$ $\angle RQO = 120° - 30° = 90°$

$\therefore$ RQ is tangent to the circle at Q.

(converse of tangent $\perp$ radius)

12B.16 HKCEE MA 2003 – I – 17

(a) (i) In $\triangle NPM$ and $\triangle NKP$,

$\angle PNM = \angle KNP$ (common)

$\angle NPM = \angle NKP$ ($\angle$ in alt. segment)

$\angle PMN = \angle KPN$ ($\angle$ sum of $\triangle$)

$\therefore \triangle NPM \sim \triangle NKP$ (AAA)

$\Rightarrow \dfrac{NP}{NM} = \dfrac{NK}{NP}$ (corr. sides, $\sim\triangle$s)

$NP^2 = NK \cdot NM$

(ii) $\because RS // OP$ (given)

$\therefore \triangle KRM \sim \triangle KON$ and $\triangle KSM \sim \triangle KPN$

$\Rightarrow \dfrac{RM}{ON} = \dfrac{KM}{KN}$ and $\dfrac{SM}{PN} = \dfrac{KM}{KN}$

$\Rightarrow \dfrac{RM}{ON} = \dfrac{SM}{PN}$

Similar to (a), $NO^2 = NK \cdot NM \Rightarrow NP = NO$

Hence, $RM = MS$.

12B.17 HKCEE MA 2004 – I – 16

(a) In $\triangle ADE$ and $\triangle BOE$,

$\angle ADE = \angle EBC$ (alt. $\angle$s, $OD // BC$)

$= \angle BOE$ ($\angle$ in alt. segment)

$\angle DAE = \angle OBE$ (ext. $\angle$, cyclic quad.)

$AD = BO$ (given)

$\therefore \triangle ADE \cong \triangle BOE$ (ASA)

(b) $DE = OE$ (corr. sides, $\cong\triangle$s)

$\angle BOE = \angle ADE$ (proved)

$= \angle AOE$ (base $\angle$s, isos. $\triangle$)

i.e. $\angle AOB = 2\angle BOE$

$\therefore \angle BEO = \angle AED$ (corr. $\angle$s, $\cong\triangle$s)

$= \angle AOB$ (ext. $\angle$, cyclic quad.)

$= 2\angle BOE$ (proved)

(c) Suppose OE is a diameter of the circle $OAEB$.

(i) $\angle OBE = 90°$ ($\angle$ in semi-circle)

In $\triangle OBE$, $\angle BOE = 180° - 90° - (2\angle BOE)$ ($\angle$ sum of $\triangle$)

$3\angle BOE = 90° \Rightarrow \angle BOE = 30°$

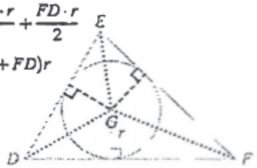
12B.18 HKCEE AM 2002 – 15

(a) Cut the triangle into $\triangle ODE$, $\triangle OEF$ and $\triangle OFD$. Then the radii are the heights of the triangles. (tangent $\perp$ radius)

$\therefore A = \dfrac{DE \cdot r}{2} + \dfrac{EF \cdot r}{2} + \dfrac{FD \cdot r}{2}$

$= \dfrac{1}{2}(DE + EF + FD)r$

$= \dfrac{1}{2}pr$



12B.19 HKDSE MA SP – I – 19

(a) (i) In $\triangle ABE$ and $\triangle ADE$,

$AB = AD$ (given)

$AE = DE$ (common)

$\angle BAE = \angle BCP$ ($\angle$ in alt. segment)

$= \angle EBC$ (alt. $\angle$s, $BD // PQ$)

$= \angle DAE$ ($\angle$s in the same segment)

$\therefore \triangle ABE \cong \triangle ADE$ (SAS)

(ii) $\because AB = AD$ (given)

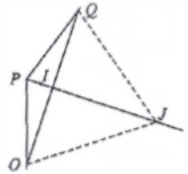
and AE is an $\angle$ bisector of $\triangle ADE$ (proved)

$\therefore AE$ is an altitude, a median and $\perp$ bisector of $\triangle ADE$. (property of isos. $\triangle$)

i.e. The in-centre, orthocentre, centroid and circumcentre of $\triangle ABD$ all lie on AE, and are thus collinear.

12B.20 HKDSE MA 2016 – I – 20

(a) *Method 1*



Let $\angle OPJ = \angle QPJ = \theta$. (in-centre)

$OJ = PJ = QJ$ (radii)

In $\triangle POJ$, $\angle POJ = \angle OPJ = \theta$ (base $\angle$s, isos. $\triangle$)

In $\triangle PQJ$, $\angle PQJ = \angle QPJ = \theta$ (base $\angle$s, isos. $\triangle$)

In $\triangle POJ$ and $\triangle PQJ$,

$\angle OPJ = \angle QPJ = \theta$ (in-centre)

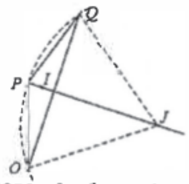
$\angle POJ = \angle PQJ = \theta$ (proved)

$PJ = PJ$ (common)

$\therefore \triangle POJ \cong \triangle PQJ$ (AAS)

$\therefore PO = PQ$ (corr. sides, $\cong\triangle$s)

Method 2



Let $\angle OPJ = \angle QPJ = \theta$. (in-centre)

$OJ = PJ = QJ$ (radii)

In $\triangle POJ$, $\angle POJ = \angle OPJ = \theta$ (base $\angle$s, isos. $\triangle$)

$\Rightarrow \angle PJO = 180° - 2\theta$ ($\angle$ sum of $\triangle$)

$\Rightarrow \angle PQO = (180° - 2\theta) \div 2 = 90° - \theta$

($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

In $\triangle PQJ$, $\angle PQJ = \angle QPJ = \theta$ (base $\angle$s, isos. $\triangle$)

$\Rightarrow \angle PJQ = 180° - 2\theta$ ($\angle$ sum of $\triangle$)

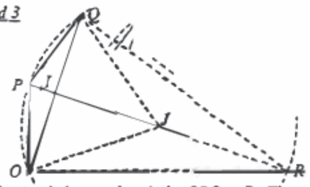
$\Rightarrow \angle POQ = (180° - 2\theta) \div 2 = 90° - \theta$

($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$\because \angle PQO = \angle POQ = 90° - \theta$ (proved)

$\therefore PO = PQ$ (sides opp. equal $\angle$s)

Method 3



Let PJ extended meet the circle OPQ at R. Then PR is a diameter of the circle.

$\therefore \angle POR = \angle PQR = 90°$ ($\angle$ in semi-circle)

Let $\angle OPR = \angle QPR = \theta$. (in-centre)

In $\triangle OPR$, $PO = PR\cos\theta$

In $\triangle QPR$, $PQ = PR\cos\theta$

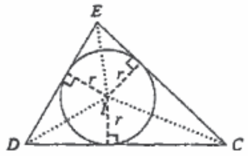
$\therefore PO = PQ$

12B.21 HKDSE MA 2019 – I – 17

(a) Let I be the in-centre of $\triangle CDE$. Then the perpendiculars from I to CD, DE and EC are all r.

$a = \dfrac{r \cdot CD}{2} + \dfrac{r \cdot DE}{2} + \dfrac{r \cdot EC}{2}$

$= \dfrac{r(CD + DE + EC)}{2} = \dfrac{r(p)}{2} \Rightarrow pr = 2a$



13 Basic Trigonometry

13A Trigonometric functions

13A.1 HKCEE MA 1980(1/1*/3) – I – 4

If $0° < \theta < 360°$ and $\sin\theta = \cos 120°$, find θ.

13A.2 HKCEE MA 1981(1/2/3) – I – 4

Solve $\cos(200° + \theta) = \sin 120°$ where $0° \le \theta \le 180°$.

13A.3 HKCEE MA 1982(1/2/3) I 5

Solve $2\sin^2\theta + 5\sin\theta - 3 = 0$ for θ, where $0° \le \theta < 360°$.

13A.4 HKCEE MA 1983(A/B) – I – 7

Find all the values of θ, where $0° \le \theta \le 360°$, such that $2\cos^2\theta + 5\sin\theta + 1 \quad 0$.

13A.5 HKCEE MA 1984(A/B) I 7

Given $\tan\theta = \dfrac{1+\cos\theta}{\sin\theta}$ $(0° < \theta < 90°)$,

(a) rewrite the above equation in the form $a\cos^2\theta + b\cos\theta + c = 0$ where a, b and c are integers;

(b) hence, solve the given equation.

13A.6 HKCEE MA 1985(A/B) I 6

Solve $2\tan^2\theta = 1 - \tan\theta$, where $0° \le \theta < 360°$. (Give your answers correct to the nearest degree.)

13A.7 HKCEE MA 1986(A/B) I – 4

Solve $\sin^2\theta + 7\sin\theta = 5\cos^2\theta$ for $0° \le \theta < 360°$.

13A.8 HKCEE MA 1987(A/B) – I – 4

Solve the equation $\sin^2\theta = \dfrac{3}{2}\cos\theta$, where $0° \le \theta < 360°$.

13A.9 (HKCEE MA 1988 – I – 2)

Simplify

(a) $\dfrac{\sin(180° - \theta)}{\sin(90° + \theta)}$,

(b) $\sin^2(180° - \phi) + \sin^2(270° + \phi)$.

13A.10 HKCEE MA 1989 – I – 7

Rewrite the equation $3\tan\theta = 2\cos\theta$ in the form $a\sin^2\theta + b\sin\theta + c = 0$, where a, b and c are integers. Hence solve the equation for $0° \le \theta < 360°$.

13A.11 HKCEE MA 1990 – I – 3

Rewrite $\sin^2\theta : \cos\theta = -3 : 2$ in the form $a\cos^2\theta + b\cos\theta + c = 0$, where a, b and c are integers. Hence solve for θ, where $0° \le \theta < 360°$.

13A.12 HKCEE MA 1991 – I – 5

Solve $\sin^2\theta - 3\cos\theta - 1 = 0$ for $0° \le \theta < 360°$.

13A.13 HKCEE MA 1992 I 1(b)

Find x if $\sin x = \dfrac{1}{2}$ and $90° < x < 180°$.

13A.14 HKCEE MA 1992 I 1(c)

Simplify $\dfrac{1-\sin^2 A}{\cos A}$.

13A.15 HKCEE MA 1993 – I – 3

Solve $\dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta} = \dfrac{3}{2}$ for $0° \le \theta < 360°$.

13A.16 HKCEE MA 1994 – I – 2(b)

If $\sin x° = \sin 36°$ and $90 < x < 270$, find the value of x.

13A.17 HKCEE MA 1994 I 2(c)

If $\cos y° = -\cos 36°$ and $180 < y < 360$, find the value of y.

13A.18 HKCEE MA 1995 – I – 6

Solve the trigonometric equation $2\sin^2\theta + 5\sin\theta - 3 = 0$ for $0° \le \theta < 360°$.

13A.19 HKCEE MA 2010 – I – 4

For each positive integer n, the nth term of a sequence is $\tan\dfrac{180°}{n+2}$.

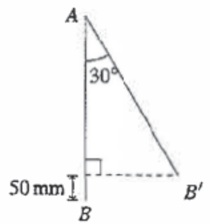
(a) Find the 2nd term of the sequence.

(b) Write down, in surd form, two different terms of the sequence such that the product of these two terms is equal to the 2nd term of the sequence.

13B Trigonometric ratios in right-angled triangles

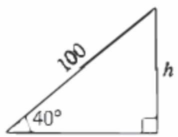
13B.1 HKCEE MA 1980(1/1*/3) I – 5

In the figure, AB is a vertical thin rod. It is rotated about A to position AB' such that $\angle BAB' = 30°$. If B' is 50mm higher than B, find the length of the rod.



13B.2 HKCEE MA 1993 I 1(b)

In the figure, find h.

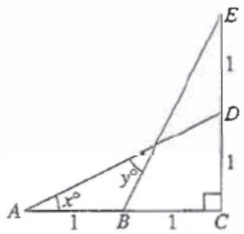


13B.3 HKCEE MA 1994 – I – 5

In the figure, calculate

(a) the length of BE,

(b) the values of x and y.



13B.4 HKCEE MA 1995 I 1(e)

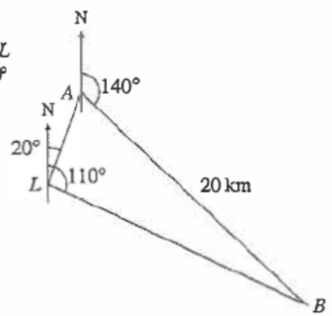
In the figure, ABC is a right-angled triangle. If $\cos A = \frac{1}{3}$, find AC.

13B.5 HKCEE MA 1997 I – 6

In the figure, the bearings of two ships A and B from a lighthouse L are 020° and 110° respectively. B is 20km and at a bearing of 140° from A. Find

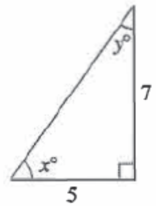
(a) the distance of L from B,

(b) the bearing of L from B.



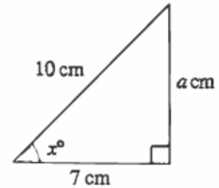
13B.6 HKCEE MA 1998 – I – 3

In the figure, find x and y.



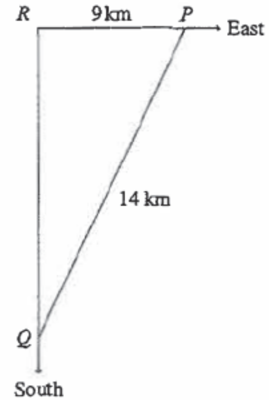
13B.7 HKCEE MA 2000 – I – 4

In the figure, find a and x.



13B.8 HKCEE MA 2008 I 4

In the figure, P, Q and R are three posting boxes on the horizontal ground. P is 9km due east of R and Q is due south of R. The distance between P and Q is 14km. Find the bearing of Q from P.



13 Basic Trigonometry

13A Trigonometric functions

13A.1 HKCEE MA 1980(1/1*/3) – I – 4

$\sin\theta = \cos 120° = -\frac{1}{2} \Rightarrow \theta = 210°$ or $330°$

13A.2 HKCEE MA 1981(1/2/3) – I – 4

$0° \le \theta \le 180° \Rightarrow 200° \le 200° + \theta \le 380°$

$\therefore \cos(200° + \theta) = \sin 120° = \frac{\sqrt{3}}{2} \Rightarrow 200° + \theta = 330°$

$\theta = 130°$

13A.3 HKCEE MA 1982(1/2/3) – I – 5

$$\begin{aligned}2\sin^2\theta + 5\sin\theta - 3 &= 0\\(2\sin\theta - 1)(\sin\theta + 3) &= 0\\\sin\theta &= \frac{1}{2} \text{ or } -3 \text{ (rej.)} \Rightarrow \theta = 30° \text{ or } 150°\end{aligned}$$

13A.4 HKCEE MA 1983(A/B) – I – 7

$$\begin{aligned}2\cos^2\theta + 5\sin\theta + 1 &= 0\\2(1 - \sin^2\theta) + 5\sin\theta + 1 &= 0\\2\sin^2\theta - 5\sin\theta - 3 &= 0\\(2\sin\theta + 1)(\sin\theta - 3) &= 0\\\sin\theta &= -\frac{1}{2} \text{ or } 3 \text{ (rej.)} \Rightarrow \theta = 210° \text{ or } 330°\end{aligned}$$

13A.5 HKCEE MA 1984(A/B) – I – 7

(a)
$$\begin{aligned}\frac{\sin\theta}{\cos\theta} &= \frac{1 + \cos\theta}{\sin\theta}\\\sin^2\theta &= \cos\theta + \cos^2\theta\\0 &= \cos\theta + \cos^2\theta - (1 - \cos^2\theta)\\2\cos^2\theta + \cos\theta - 1 &= 0\end{aligned}$$

(b)
$$\begin{aligned}(2\cos\theta - 1)(\cos\theta + 1) &= 0\\\cos\theta &= \frac{1}{2} \text{ or } -1 \text{ (rej.)} \Rightarrow \theta = 60°\end{aligned}$$

13A.6 HKCEE MA 1985(A/B) – I – 6

$$\begin{aligned}2\tan^2\theta &= 1 - \tan\theta\\2\tan^2\theta + \tan\theta - 1 &= 0\\(2\tan\theta - 1)(\tan\theta + 1) &= 0\\\tan\theta &= \frac{1}{2} \text{ or } -1\\\theta &= 27°, 180° + 27° \text{ or } 135°, 180° + 135°\\&= 27°, 207° \text{ (nearest deg)}, 135° \text{ or } 315°\end{aligned}$$

13A.7 HKCEE MA 1986(A/B) – I – 4

$$\begin{aligned}\sin^2\theta + 7\sin\theta &= 5\cos^2\theta = 5(1 - \sin^2\theta)\\6\sin^2\theta + 7\sin\theta - 5 &= 0\\(2\sin\theta - 1)(3\sin\theta + 5) &= 0\\\sin\theta &= \frac{1}{2} \text{ or } -\frac{5}{3} \text{ (rejected)}\\\theta &= 30° \text{ or } 180° - 30° = 150°\end{aligned}$$

13A.8 HKCEE MA 1987(A/B) – I – 4

$$\begin{aligned}2\sin^2\theta &= 3\cos\theta\\2(1 - \cos^2\theta) &= 3\cos\theta\\2\cos^2\theta + 3\cos\theta - 2 &= 0\\(2\cos\theta - 1)(\cos\theta + 2) &= 0\\\cos\theta &= \frac{1}{2} \text{ or } -2 \text{ (rejected)}\\\theta &= 60° \text{ or } 360° - 60° = 300°\end{aligned}$$

13A.9 (HKCEE MA 1988 – I – 2)

(a) $\frac{\sin(180° - \theta)}{\sin(90° + \theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

(b) $\sin^2(180° - \phi) + \sin^2(270° + \phi) = \sin^2\phi + (-\cos\phi)^2 = 1$

13A.10 HKCEE MA 1989 – I – 7

$$\begin{aligned}\frac{3\sin\theta}{\cos\theta} &= 2\cos\theta\\3\sin\theta &= 2\cos^2\theta = 2(1 - \sin^2\theta)\\2\sin^2\theta + 3\sin\theta - 2 &= 0\\(2\sin\theta - 1)(\sin\theta + 2) &= 0\\\sin\theta &= \frac{1}{2} \text{ or } -2 \text{ (rejected)}\\\theta &= 30° \text{ or } 180° - 30° = 150°\end{aligned}$$

13A.11 HKCEE MA 1990 – I – 3

$$\begin{aligned}\frac{1 - \cos^2\theta}{\cos\theta} &= -\frac{3}{2}\\2 - 2\cos^2\theta &= -3\cos\theta\\2\cos^2\theta - 3\cos\theta - 2 &= 0\\(2\cos\theta + 1)(\cos\theta - 2) &= 0\\\cos\theta &= \frac{-1}{2} \text{ or } 2 \text{ (rejected)}\\\theta &= 120° \text{ or } 360 - 120° = 240°\end{aligned}$$

13A.12 HKCEE MA 1991 – I – 5

$$\begin{aligned}\sin^2\theta - 3\cos\theta - 1 &= 0\\(1 - \cos^2\theta) - 3\cos\theta - 1 &= 0\\\cos^2\theta + 3\cos\theta &= 0\\\cos\theta(\cos\theta + 3) &= 0\\\cos\theta &= 0 \text{ or } -3 \text{ (rejected)}\\\theta &= 90° \text{ or } 270°\end{aligned}$$

13A.13 HKCEE MA 1992 – I – 1(b)

$\sin x = \frac{1}{2} \Rightarrow x = 180° - 30° = 150°$

13A.14 HKCEE MA 1992 – I – 1(c)

$\frac{1 - \sin^2 A}{\cos A} = \frac{\cos^2 A}{\cos A} = \cos A$

13A.15 HKCEE MA 1993 – I – 3

$$\begin{aligned}\frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta} &= \frac{3}{2}\\2\sin\theta + 2\cos\theta &= 3\sin\theta - 3\cos\theta\\-\sin\theta &= 5\cos\theta\\\tan\theta &= -5\\\theta &= 78.7° \text{ or } 180° + 78.7° = 259° \text{ (3 s.f.)}\end{aligned}$$

13A.16 HKCEE MA 1994 – I – 2(b)

$\sin x° = \sin 36° \Rightarrow x = 180 - 36 = 144$

13A.17 HKCEE MA 1994 – I – 2(c)

$\cos y° = -\cos 36° = \cos(180° + 36°) \Rightarrow y = 216$

13A.18 HKCEE MA 1995 – I – 6

$$\begin{aligned}2\sin^2\theta + 5\sin\theta - 3 &= 0\\(2\sin\theta - 1)(\sin\theta + 3) &= 0\\\sin\theta &= \frac{1}{2} \text{ or } -3 \text{ (rejected)}\\\theta &= 30° \text{ or } 180° - 30° = 150°\end{aligned}$$

13A.19 HKCEE MA 2010 – I – 4

(a) 2nd term $= \tan\frac{180°}{(2)+2} = \tan 45° = 1$

(b) (Note that if the product of two different numbers is 1, one of them is > 1 and the other < 1. Besides, the sequence is decreasing when n increases. Hence, the larger term must come before the 2nd term.)

$\tan\frac{180°}{(1)+2} = \tan 60° = \sqrt{3}$

$\Rightarrow \frac{1}{\sqrt{3}} = \tan 30° = \tan\frac{180°}{6} = \frac{180°}{(5)+1}$

$\therefore$ Required terms are the 1st one, $\sqrt{3}$, and 5th one, $\frac{1}{\sqrt{3}}$.

13B Trigonometric ratios in right angled triangles

13B.1 HKCEE MA 1980(1/1*/3) – I – 5

Let ℓ mm be the length of rod. Then

$$\begin{aligned}\frac{\sqrt{3}}{2} = \cos 30° &= \frac{\ell - 50}{\ell}\\\sqrt{3}\ell &= 2(\ell - 50)\\100 &= (2 - \sqrt{3})\ell \Rightarrow \ell = 373 \text{ (3 s.f.)}\end{aligned}$$

Hence, the rod is 373 mm long.

13B.2 HKCEE MA 1993 – I – 1(b)

$h = 100\cos 40° = 76.6$ (3 s.f.)

13B.3 HKCEE MA 1994 – I – 5

(a) $BE = \sqrt{1^2 + 2^2} = \sqrt{5}$ $(= 2.24)$

(b) $\tan x° = \frac{1}{2} \Rightarrow x = 26.5651 = 26.6$ (3 s.f.)

$\tan\angle EBC = 2 \Rightarrow \angle EBC = 63.4349$

$\Rightarrow y = 63.4349 - x = 36.9$ (3 s.f.)

13B.4 HKCEE MA 1995 – I – 1(e)

$\frac{1}{3} = \cos A = \frac{2}{AC} \Rightarrow AC = 6$

13B.5 HKCEE MA 1997 – I – 6

(a) $\angle LAB = 20° + (180° - 140°) = 60°$

$\angle ALB = 110° - 20° = 90°$

$\therefore$ Distance $= LB = 20\sin 60° = 10\sqrt{3} = 17.3$ (km, 3 s.f.)

(b) $\angle ABL = 180° - 90° - 60° = 30°$

$\therefore$ Bearing $= 180° + 140° - 30° = 290°$

13B.6 HKCEE MA 1998 – I – 3

$\tan x° = \frac{7}{5} \Rightarrow x = 54.5$

$\Rightarrow y = 180 - 90 - 54.5 = 35.5$

13B.7 HKCEE MA 2000 – I – 4

$a = \sqrt{10^2 - 7^2} = \sqrt{51} = 7.14$

$\cos x° = \frac{7}{10} \Rightarrow x = 45.6$

13B.8 HKCEE MA 2008 – I – 4

$\sin\angle RQP = \frac{9}{14} \Rightarrow \angle RQP = 40.01°$

$\therefore$ Bearing $=$ S40.0°W or $(180° + 40.0°) = 220°$

14 Applications of Trigonometry

14A Two-dimensional applications

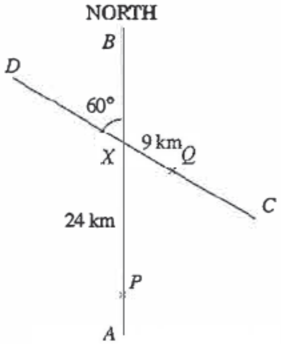
14A.1 HKCEE MA 1981(2/3) I 11

AB and CD are two straight roads intersecting at X. AB runs North and makes an angle of 60° with CD. At noon, two people P and Q are respectively 24 km and 9 km from X as shown in the figure. P walks at a speed of 4.5 km/h towards B and Q walks at a speed of 6 km/h towards D.

(a) Calculate the distance between P and Q at noon.

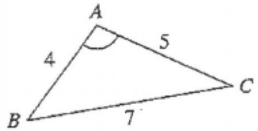
(b) What are the distances of P and Q from X at 4 p.m.?

(c) Calculate the bearing of Q from P at 4 p.m. to the nearest degree.



14A.2 HKCEE MA 1982(3) – I – 2

In the figure, $AB = 4$, $AC = 5$ and $BC = 7$.
Calculate $\angle A$ to the nearest degree.



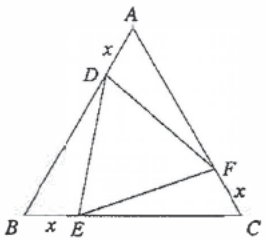
14A.3 HKCEE MA 1985(A/B) I 13

(To continue as 7C.1.)

In the figure, ABC is an equilateral triangle. $AB = 2$. D, E, F are points on AB, BC, CA respectively such that $AD = BE = CF = x$.

(a) By using the cosine formula or otherwise, express DE^2 in terms of x.

(b) Show that the area of $\triangle DEF = \dfrac{\sqrt{3}}{4}(3x^2 - 6x + 4)$.



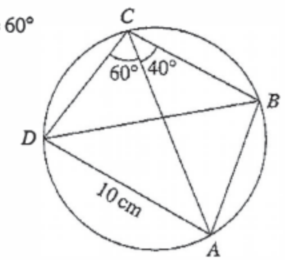
14A.4 HKCEE MA 1989 – I – 6

(Continued from 12A.7.)

In the figure, $ABCD$ is a cyclic quadrilateral with $AD = 10$ cm, $\angle ACD = 60°$ and $\angle ACB = 40°$.

(a) Find $\angle ABD$ and $\angle BAD$.

(b) Find the length of BD in cm, correct to 2 decimal places.



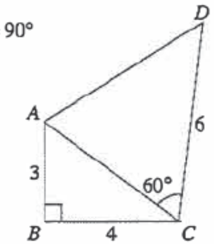
14A.5 HKCEE MA 1997 I 5

In the figure, ABC is a right angled triangle. $AB = 3$, $BC = 4$, $CD = 6$, $\angle ABC = 90°$ and $\angle ACD = 60°$. Find

(a) AC,

(b) AD,

(c) the area of $\triangle ACD$.



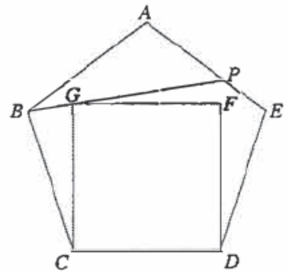
14A.6 HKCEE MA 2000 I 13

(Continued from 11A.11.)

In the figure, $ABCDE$ is a regular pentagon and $CDFG$ is a square. BG produced meets AE at P.

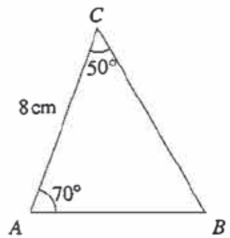
(a) Find $\angle BCG$, $\angle ABP$ and $\angle APB$.

(b) Using the fact that $\dfrac{AP}{\sin\angle ABP} = \dfrac{AB}{\sin\angle APB}$, or otherwise, determine which line segment, AP or PE, is longer.



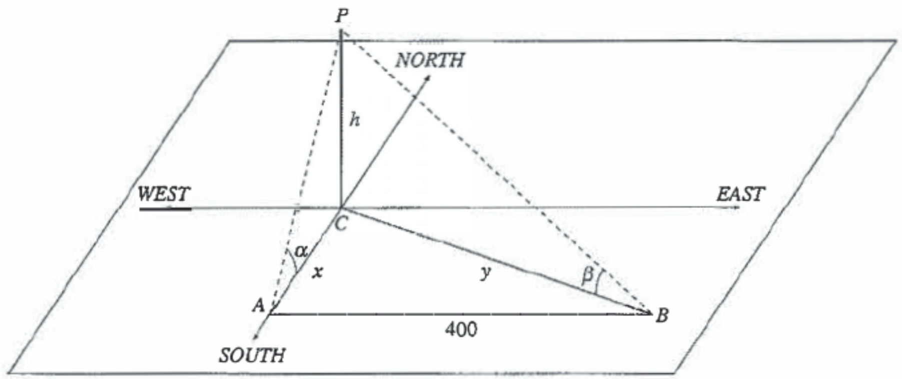
14A.7 HKCEE MA 2001 I 9

In the figure, find AB and the area of $\triangle ABC$.



14B Three-dimensional applications

14B.1 HKCEE MA 1980(1/1*/3) – I – 9



In the figure, PC represents a vertical object of height h metres. From a point A, south of C, the angle of elevation of P is α. From a point B, 400 metres east of A, the angle of elevation of P is β. AC and BC are x metres and y metres respectively.

(a) (i) Express x in terms of h and α.

(ii) Express y in terms of h and β.

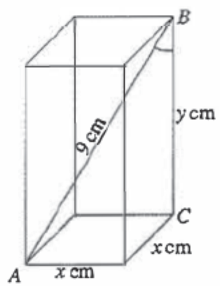
(b) If $\alpha = 60°$ and $\beta = 30°$, find the value of h correct to 3 significant figures.

14B.2 HKCEE MA 1982(1/2/3) I 8

The figure represents the framework of a cuboid made of iron wire. It has a square base of side x cm and a height of y cm. The length of the diagonal AB is 9 cm. The total length of wire used for the framework (including the diagonal AB) is 69 cm.

(a) Find all the values of x and y.

(b) Hence calculate $\angle ABC$ to the nearest degree for the case in which $y > x$.



14B.3 HKCEE MA 1983(A/B) I – 13

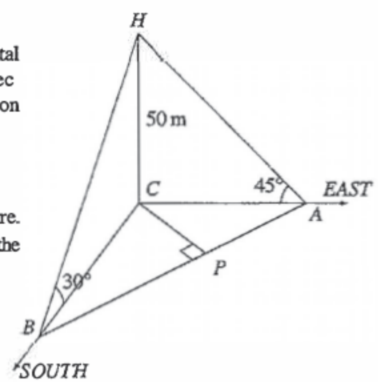
In the figure, A, B and C are three points on the same horizontal ground. HC is a vertical tower 50 m high. A and B are respectively due east and due south of the tower. The angles of elevation of H observed from A and B are respectively 45° and 30°.

(a) Find the distance between A and B.

(b) P is a point on AB such that $CP \perp AB$.

(i) Find the distance between C and P to the nearest metre.

(ii) Find the angle of elevation of H observed from P to the nearest degree.



14B.4 HKCEE MA 1984(A/B) – I 13

In the figure, A, B and C lie in a horizontal plane. $AC = 20$ m. HA is a vertical pole. The angles of elevation of H from B and C are 30° and 15° respectively.

(In this question, give your answers correct to 2 decimal places.)

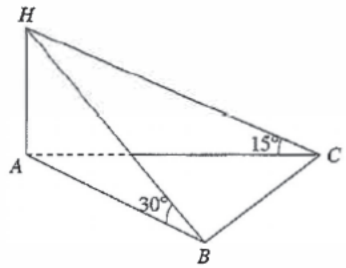
(a) (i) Find, in m, the length of the pole HA.

(ii) Find, in m, the length of AB.

(b) If A, B and C lie on a circle with AC as diameter,

(i) find, in m, the distance between B and C;

(ii) find, in m^2, the area of $\triangle ABC$.



14B.5 HKCEE MA 1985(A/B) – I – 8

In the figure, A, B and C are three points in a horizontal plane. $AB = 100$ m, $\angle CAB = 30°$, $\angle ABC = 45°$.

(a) Find BC and AC, in metres, correct to 1 decimal place.

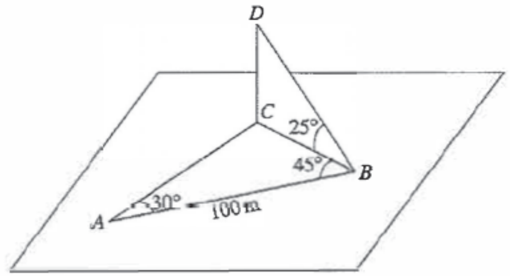
(b) D is a point vertically above C. From B, the angle of elevation of D is 25°.

(i) Find CD, in metres, correct to 1 decimal place.

(ii) X is a point on AB such that $CX \perp AB$.

(1) Find CX, in metres, correct to 1 decimal place.

(2) Find the angle of elevation of D from X, correct to the nearest degree.

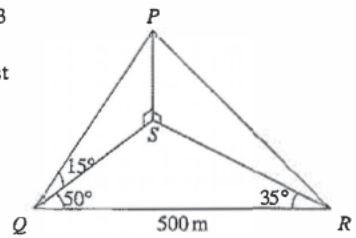


14B.6 HKCEE MA 1986(A/B) I 10

In the figure, Q, R and S are three points on the same horizontal plane. $QR = 500$ m, $\angle SQR = 50°$ and $\angle QRS = 35°$. P is a point vertically above S. The angle of elevation of P from Q is 15°.

(a) Find the distance, in metres, from P to the plane, correct to 3 significant figures.

(b) Find the angle of elevation of P from R, correct to the nearest degree.



14B.7 HKCEE MA 1987(A/B) I – 11

In this question, you should give your answers in cm or degrees, correct to 3 decimal places.

The figure shows a solid in which $ABCD$, $DCFE$ and $ABFE$ are rectangles. DG is the perpendicular from D to AE. $AB = 3$ cm, $AD = 3$ cm and $DE = 2$ cm. $\angle ADE = 80°$.

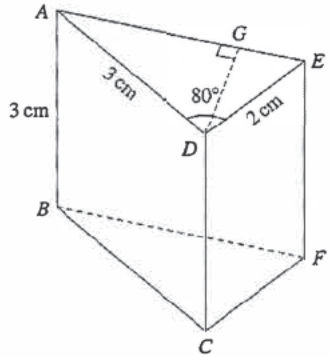
(a) Find AE.

(b) Find $\angle DAE$.

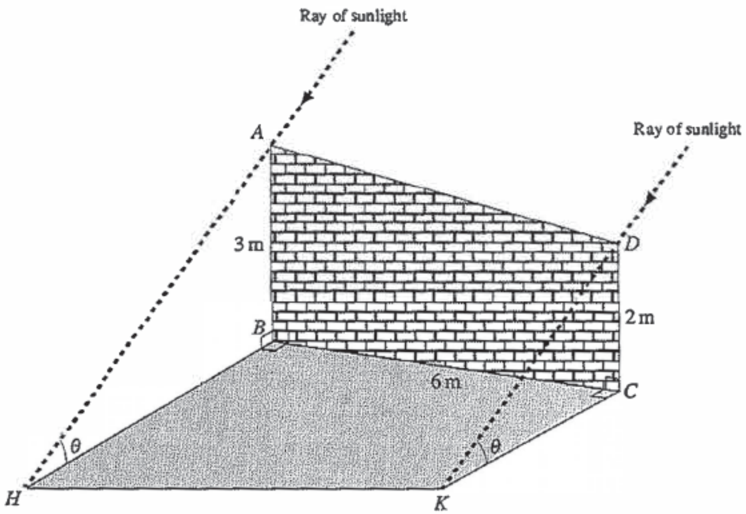
(c) Find DG.

(d) Find BD.

(e) Find the angle between the line BD and the face $ABFE$.



14B.8 HKCEE MA 1988 – I – 13



In the figure, $ABCD$ is a wall in the shape of a trapezium with AB and DC vertical. Rays of sunlight coming from the back of the wall cast a shadow $HBCK$ on the horizontal ground such that the edges HB and KC of the shadow are perpendicular to BC. Suppose the angle of elevation of the sun is θ, $AB = 3$ m, $CD = 2$ m and $BC = 6$ m.

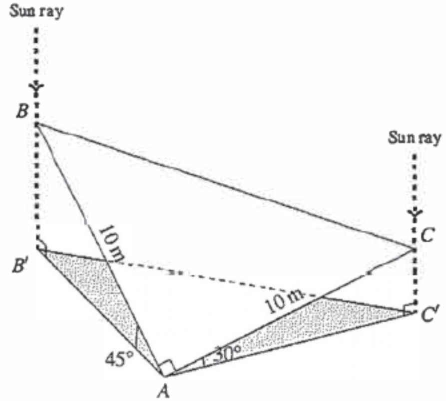
(a) Express HB and KC in terms of θ.

(b) (i) Find the area S_1 of the wall.

(ii) Find, in terms of θ, the area S_2 of the shadow. Hence show that $\dfrac{S_1}{S_2} = \tan\theta$.

(c) If $\theta = 30°$, find the length of the edge HK, leaving your answer in surd form.

14B.9 HKCEE MA 1989 – I – 10



Answers in this question should be given correct to at least 3 significant figures or in surd form.

In the figure, a triangular board ABC, right angled at A with $AB = AC = 10$ m, is placed with the vertex A on the horizontal ground. AB and AC make angles of 45° and 30° with the horizontal respectively. The sun casts a shadow $AB'C'$ of the board on the ground such that B' and C' are vertically below B and C respectively.

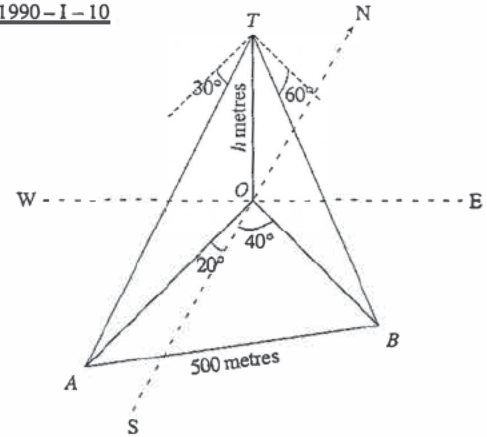
(a) Find the lengths of AB' and AC'.

(b) Find the lengths of BC, BB' and CC'.

(c) Using the results of (b), or otherwise, find the length of $B'C'$.

(d) Find $\angle B'AC'$. Hence find the area of the shadow.

14B.10 HKCEE MA 1990 – I – 10



In the figure, OT represents a vertical tower of height h metres. From the top T of the tower, two landmarks A and B, 500 metres apart on the same horizontal ground, are observed to have angles of depression 30° and 60° respectively. The bearings of A and B from the tower OT are S20°W and S40°E respectively.

(a) Find the lengths of OA and OB in terms of h.

(b) Express the length of AB in terms of h. Hence, or otherwise, find the value of h.

(c) Find $\angle OAB$, correct to the nearest degree. Hence write down

(i) the bearing of B from A,

(ii) the bearing of A from B.

14B.11 HKCEE MA 1992 – I – 15

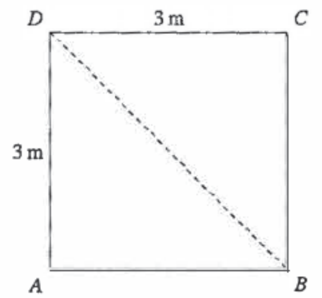


Figure (1)

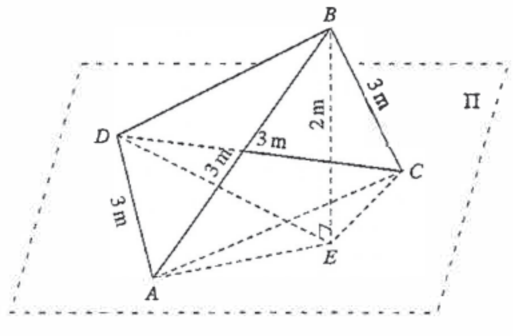


Figure (2)

In Figure (1), $ABCD$ is a thin square metal sheet of side three metres. The metal sheet is folded along BD and the edges AD and CD of the folded metal sheet are placed on a horizontal plane Π with B two metres vertically above the plane Π. E is the foot of the perpendicular from B to the plane Π. (See Figure (2).)

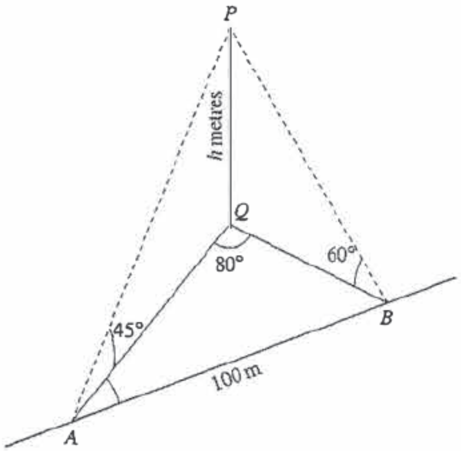
(a) Find the lengths of BD, ED and AE, leaving your answers in surd form.

(b) Find $\angle ADE$.

(c) Find the angle between BD and the plane Π.

(d) Find the angle between the planes ABD and CBD.

14B.12 HKCEE MA 1993 – I – 12



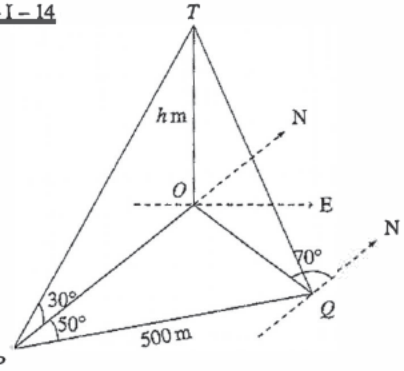
In the figure, PQ is a vertical television tower h metres high. A and B are two points 100 m apart on a straight road in front of the tower with A, B and Q on the same horizontal ground and $\angle AQB = 80°$. The angles of elevation of P from A and B are $45°$ and $60°$ respectively.

(a) (i) Express the lengths of AQ and BQ in terms of h.

(ii) Find h and $\angle QAB$.

(b) A person walks from A along the road towards B. At a certain point R between A and B, the person finds that the angle of elevation of P is $50°$. How far away is R from A?

14B.13 HKCEE MA 1994 – I – 14



In the figure, OT is a vertical tower of height h metres and O, P and Q are points on the same horizontal plane. When a man is at P, he finds that the tower is due north and that the angle of elevation of the top T of the tower is $30°$. When he walks a distance of 500 metres in the direction N50°E to Q, he finds that the bearing of the tower is N70°W.

(a) Find OQ and OP.

(b) Find h.

(c) Find the angle of elevation of T from Q, giving your answer correct to the nearest degree.

(d) (i) If he walks a further distance of 400 metres from Q in a direction Nθ°E to a point R (not shown in the figure) on the same horizontal plane, he finds that the angle of elevation of T is $20°$. Find $\angle OQR$ and hence write down the value of θ to the nearest integer.

(ii) If he starts from Q again and walks the same distance of 400 metres in another direction to a point S on the same horizontal plane, he finds that the angle of elevation of T is again $20°$. Find the bearing of S from Q, giving your answer correct to the nearest degree.

14B.14 HKCEE MA 1995 – I – 15

The figure shows a triangular road sign ABC attached to a vertical pole OAB standing on the horizontal ground. The plane ABC is vertical with $OA = 2$ m, $AB = 0.6$ m, $AC = 0.7$ m and $BC = 0.8$ m. D is a point on the horizontal ground vertically below C and is due north of the foot O of the pole.

The sun is due west. When its angle of elevation is $30°$, the shadow of the road sign on the horizontal ground is $A'B'C'$.

(a) Find the lengths of OA' and $A'B'$.

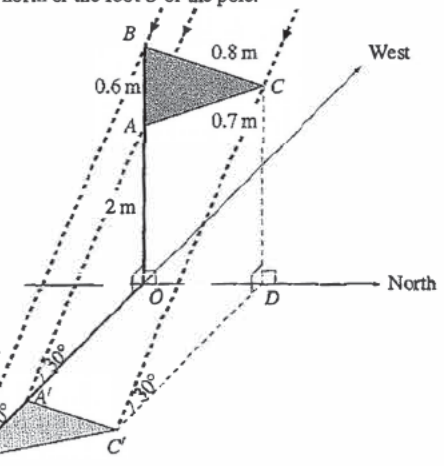
(b) Calculate $\angle BAC$ and hence find the length of OD.

(c) Find the area of the shadow $A'B'C'$.

(d) If the angle of elevation of the sun is less than $30°$,

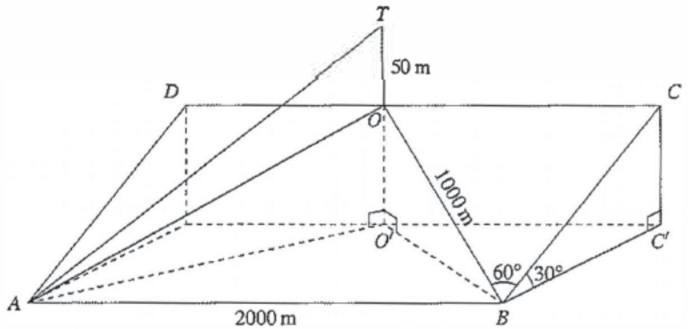
(i) state whether the shadow of AB is longer than, shorter than, or equal to $A'B'$ in (a); and hence

(ii) state with reasons whether the area of the shadow of the road sign ABC is larger than, smaller than, or equal to that of $A'B'C'$ in (c).



14B.15 <u>HKCEE MA 1996 – I – 15</u>

In the figure, the rectangular plane $ABCD$ is a hillside with inclination 30°. C' and O' are vertically below C and O respectively so that A, B, C', O' are on the same horizontal plane. BO is a straight path on the hillside which makes an angle 60° with BC, and OT is a vertical tower. $AB = 2000$ m, $BO = 1000$ m and $OT = 50$ m.



(a) Find BC and CC'.

(b) Find the inclination of BO with the horizontal.

(c) Find AT.

(d) There are cable cars going directly from A to T. A man wants to go to T from B and he can do this by taking either one of the following two routes:

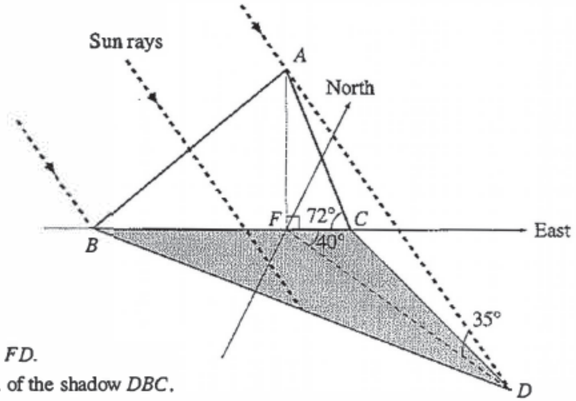
Route I: Walking uphill along BO at an average speed of 0.3 m/s and taking a lift in the tower for 1 minute from O to T.

Route II: Walking along BA at an average speed of 0.8 m/s and taking a cable car from A to T at an average speed of 3.2 m/s.

Determine which route takes a shorter time.

14B.16 <u>HKCEE MA 1998 – I – 17</u>

In the figure, triangular sign post ABC stands vertically on the horizontal ground along the east west direction. $AC = 4$ m, $BC = 6$ m, $\angle ACB = 72°$ and F is the foot of the perpendicular from A to BC. When the sun shines from N50°W with an angle of elevation 35°, the shadow of the sign post on the horizontal ground is DBC.



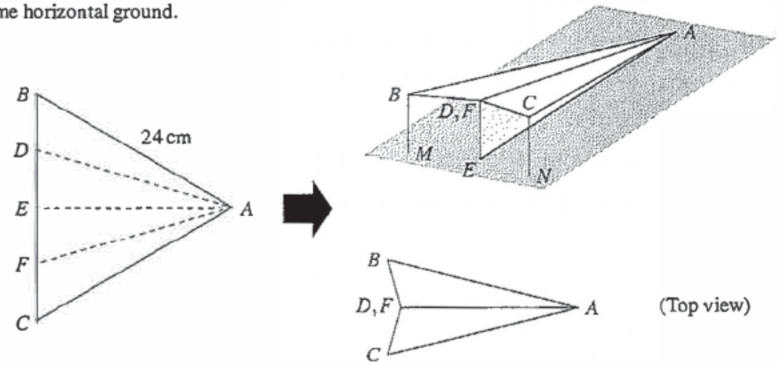
(a) Find AF and FD.

(b) Find the area of the shadow DBC.

(c) Suppose the sun shines from Nx°W, where $50 < x < 90$, but its angle of elevation is still 35°. State **with reasons** whether the area of the shadow of the sign post on the horizontal ground is greater than, smaller than or equal to the area obtained in (b).

14B.17 <u>HKCEE MA 1999 – I – 18</u>

In the figure, a paper card ABC in the shape of an equilateral triangle of side 24 cm is folded to form a paper aeroplane. D, E and F are points on edge BC so that $BD = DE = EF = FC$. The aeroplane is formed by folding the paper card along the lines AD, AE and AF so that AD and AF coincide. It is supported by two vertical sticks BM and CN of equal length so that A, B, D, F, C lie on the same plane and A, E, M, N lie on the same horizontal ground.



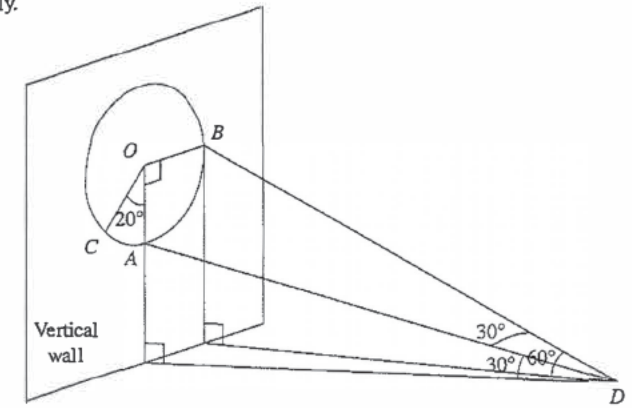
(a) Find the distance between the tips, B and C, of the wings of the aeroplane.

(b) Find the inclination of the wings of the aeroplane to the horizontal ground.

(c) Find the length of the stick CN.

14B.18 <u>HKCEE MA 2000 – I – 17</u>

The figure shows a circle with centre O and radius 10 m on a vertical wall which stands on the horizontal ground. A, B and C are three points on the circumference of the circle such that A is vertically below O, $\angle AOB = 90°$ and $\angle AOC = 20°$. A laser emitter D on the ground shoots a laser beam at B. The laser beam then sweeps through an angle of 30° to shoot at A. The angles of elevation of B and A from D are 60° and 30° respectively.



(a) Let A be h m above the ground.

(i) Express AD and BD in terms of h.

(ii) Find h.

(b) Another laser emitter E on the ground shoots a laser beam at A with angle of elevation 25°. The laser beam then sweeps through an angle of 5° to shoot at C. Find $\angle ACE$.

14B.19 HKCEE MA 2001 – I – 16

Figure (1) shows a piece of pentagonal cardboard $ABCDE$. It is formed by cutting off two equilateral triangular parts, each of side x cm, from an equilateral triangular cardboard AFG. AB is 6 cm long and the area of $BCDE$ is $5\sqrt{3}\,\text{cm}^2$.

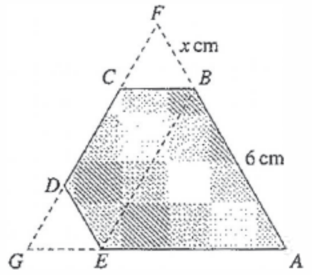


Figure (1)

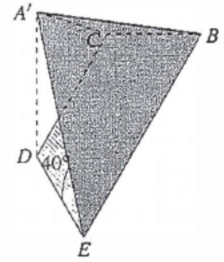


Figure (2)

(a) Show that x^2 $12x+20=0$. Hence find x.

(b) The triangular part ABE is folded up along the line BE until the vertex A comes to the position A' (as shown in Figure (2)) such that $\angle A'ED=40°$.

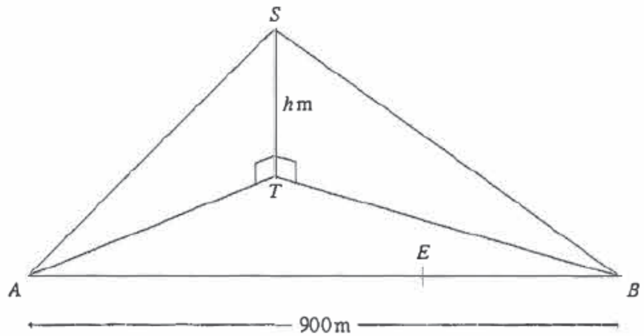
(i) Find the length of $A'D$.

(ii) Find the angle between the planes $BCDE$ and $A'BE$.

(iii) If A', B, C, D, E are the vertices of a pyramid with base $BCDE$, find the volume of the pyramid.

14B.20 HKCEE MA 2002 – I – 14

In the figure, AB is a straight track 900 m long on the horizontal ground. E is a small object moving along AB. ST is a vertical tower of height h m standing on the horizontal ground. The angles of elevation of S from A and B are 20° and 15° respectively. $\angle TAB=30°$.



(a) Express AT and BT in terms of h. Hence find h.

(b) (i) Find the shortest distance between E and S.

(ii) Let θ be the angle of elevation of S from E. Find the range of values of θ as E moves along AB.

14B.21 HKCEE MA 2003 – I 14

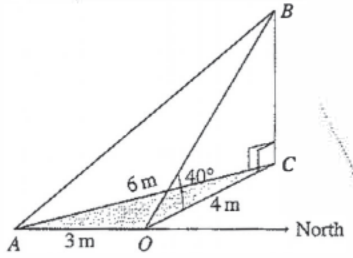


Figure (1)

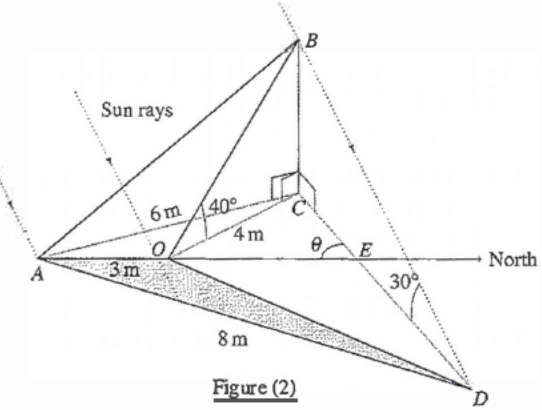


Figure (2)

Figure (1) shows a triangular metal plate OAB standing on the horizontal ground. The side OA lies along the north south direction on the ground. OB is inclined at an angle of 40° to the horizontal. The overhead sun casts a shadow of the plate, OAC, on the ground. $OA=3$ m, $OC=4$ m and $AC=6$ m.

(a) Find $\angle OAC$.

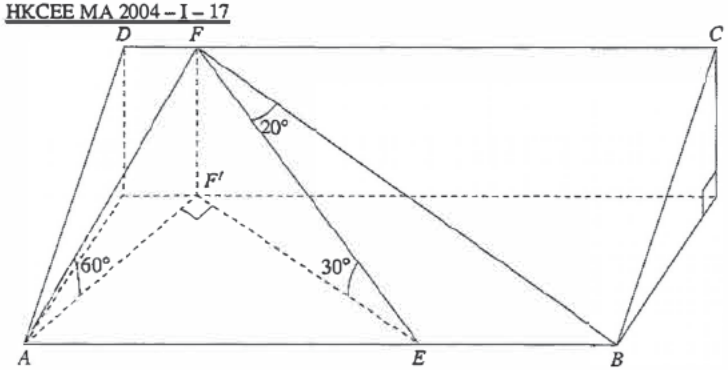
(b) In Figure (2), OAD is the shadow of the plate cast on the horizontal ground when the sun shines from SθW with an angle of elevation 30°. AO is produced to cut CD at E. $AD=8$ m.

(i) Find CD.

(ii) Find $\angle CAD$.

(iii) Using $CE+ED=CD$, or otherwise, find θ.

14B.22 HKCEE MA 2004 – I – 17



In the figure, $ABCD$ is a rectangular inclined plane. E and F are points on the straight lines AB and CD respectively. F' is vertically below F. A, E, B and F' are on the same horizontal ground. $\angle AF'E=90°$, $\angle FAF'=60°$, $\angle FEF'=30°$, $\angle EFB=20°$ and $EF=20$ m.

(a) Find

(i) FF' and AE,

(ii) $\angle AEF$.

(b) A small red toy car goes straight from E to B at an average speed of 2 m/s while a small yellow toy car goes straight from F to B at an average speed of 3 m/s. The two toy cars start going at the same time. Will the yellow toy car reach B before the red one? Explain your answer.

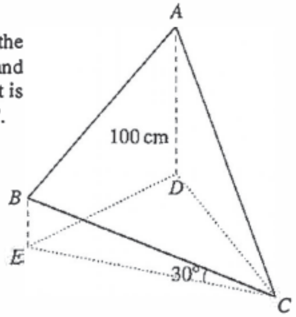
14B.23 HKCEE MA 2005 – I – 14

In the figure, a thin triangular board ABC is held with the vertex C on the horizontal ground. D and E are points on the ground vertically below A and B respectively. BC is inclined at an angle of $30°$ with the horizontal. It is known that $AD = 100\text{ cm}$, $BC = 120\text{ cm}$, $\angle CAB = 60°$ and $\angle ABC = 80°$.

(a) Find BE and CE.

(b) Find AB and AC.

(c) Find $\angle CDE$ and the shortest distance from C to DE.



14B.24 HKCEE MA 2006 – I – 17

In Figure (1), ABC is a triangular paper card. D is a point lying on AC such that BD is perpendicular to AC. It is known that $AB = 40\text{ cm}$, $BC = 60\text{ cm}$ and $AC = 90\text{ cm}$.

(a) Find AD.

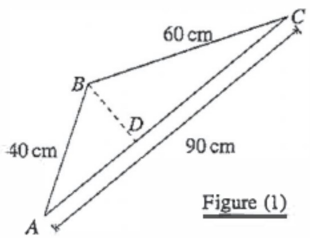


Figure (1)

(b) The triangular paper card in Figure (1) is folded along BD such that AB and BC lie on a horizontal plane as shown in Figure (2).

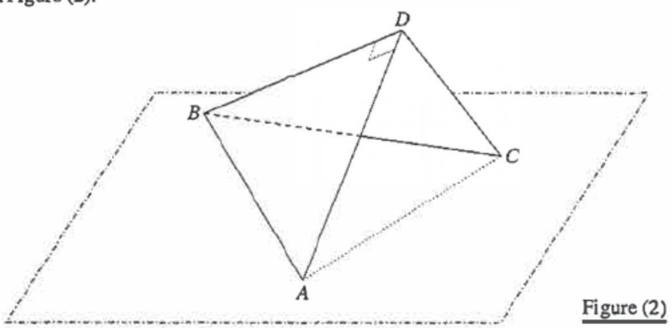


Figure (2)

(i) Suppose $\angle DAC = 62°$.

(1) Find the distance between A and C on the horizontal plane.

(2) Using Heron's formula, or otherwise, find the area of $\triangle ABC$ on the horizontal plane.

(3) Find the height of the tetrahedron $ABCD$ from the vertex D to the base $\triangle ABC$.

(ii) Describe how the volume of the tetrahedron $ABCD$ varies when $\angle ADC$ increases from $30°$ to $150°$. Explain your answer.

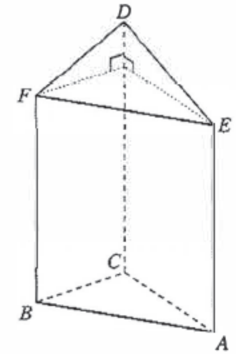
14B.25 HKCEE MA 2007 – I – 16

The figure shows a solid wooden souvenir $ABCDEF$ with the triangular base ABC lying on the horizontal ground. A, B and C are vertically below E, F and D respectively. DEF is an inclined triangular plane. It is given that $AB = 9\text{ cm}$, $BC = 5\text{ cm}$, $AC = 6\text{ cm}$, $AE = BF = 20\text{ cm}$ and $CD = 23\text{ cm}$.

(a) Find the area of the triangular base ABC and the volume of the souvenir $ABCDEF$.

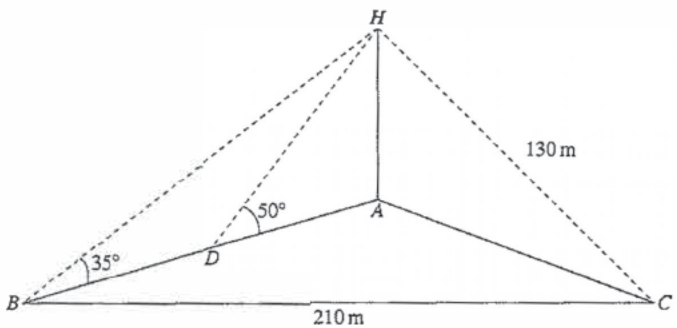
(b) Find $\angle DFE$ and the shortest distance from D to EF.

(c) Can a piece of thin rectangular metal plate of dimensions $5\text{ cm} \times 4\text{ cm}$ be fixed onto the triangular surface DEF so that the thin metal plate completely lies in the triangle DEF? Explain your answer.



14B.26 HKCEE MA 2008 – I – 15

In the figure, H is the top of a tower and A is vertically below H. AB, BC and CA are straight paths on the horizontal ground and D is a point on AB. Christine walks from A to D along AD and finds that the angle of elevation of H from D is $50°$. She then walks 50 m to B along DB and finds that the angle of elevation of H from B is $35°$.



(a) Find the distance between B and H.

(b) Christine walks 210m from B to C along BC. It is given that the distance between C and H is 130m.

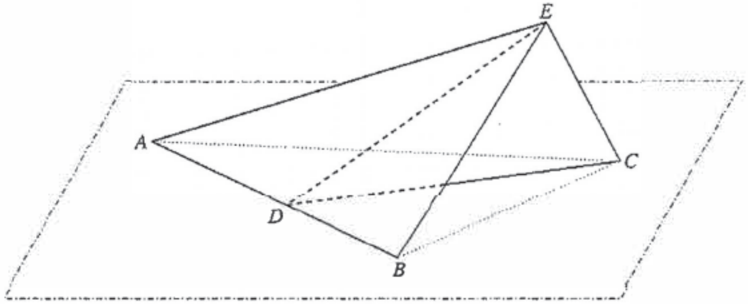
(i) Find $\angle CBH$.

(ii) Find the angle between the plane BCH and the horizontal ground.

(iii) When Christine walks from B to C along BC, is it possible for her to find a point K on BC such that the angle of elevation of H from K is $75°$? Explain your answer.

14B.27 HKCEE MA 2009 – I – 17

The figure shows a geometric model fixed on the horizontal ground. The model consists of two thin triangular metal plates ABE and CDE, where D lies on AB and CE is perpendicular to the thin metal plate ABE. It is given that A, B, C and D lie on the horizontal ground. It is found that $AC = 28$ cm, $BC = 25$ cm, $BD = 6$ cm, $BE = 24$ cm and $\angle ABC = 57°$.



(a) Find
  (i) the length of CD,
  (ii) $\angle BAC$,
  (iii) the area of $\triangle ABC$,
  (iv) the shortest distance from E to the horizontal ground.

(b) A student claims that the angle between DE and the horizontal ground is $\angle CDE$. Do you agree? Explain your answer.

14B.28 HKCEE MA 2010 – I – 15

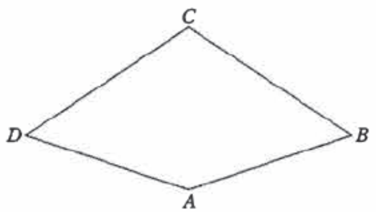


Figure (1)

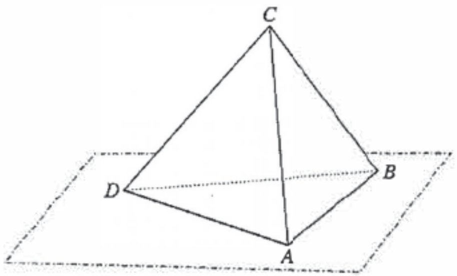


Figure (2)

(a) Figure (1) shows a piece of paper card $ABCD$ in the form of a quadrilateral with $AB = AD$ and $BC = CD$. It is given that $BC = 24$ cm, $\angle BAD = 146°$ and $\angle ABC = 59°$. Find the length of AB.

(b) The paper card described in (a) is folded along AC such that AB and AD lie on the horizontal ground as shown in Figure (2). It is given that $\angle BAD = 92°$.
  (i) Find the distance between B and D on the horizontal ground.
  (ii) Find the angle between the plane ABC and the plane ACD.
  (iii) Let P be a movable point on the slant edge AC. Describe how $\angle BPD$ varies as P moves from A to C. Explain your answer.

14B.29 HKCEE MA 2011 – I – 17

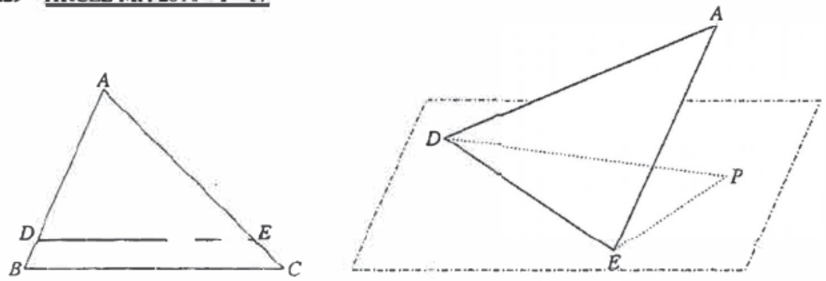


Figure (1) Figure (2)

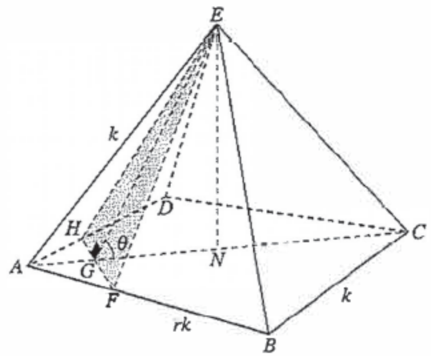
In Figure (1), ABC is a thin triangular metal sheet. D and E are points lying on AB and AC respectively such that DE is parallel to BC and the distance between DE and BC is 4 cm. It is found that $AB = 20$ cm, $AC = 30$ cm and $\angle BAC = 56°$.

(a) Find
  (i) the length of BC,
  (ii) $\angle ACB$,
  (iii) the perpendicular distance from A to DE,
  (iv) the length of DE.

(b) The thin triangular metal sheet in Figure (1) is cut along DE. The metal sheet ADE is held with DE lying on the horizontal ground as shown in Figure (2). It is given that P is the projection of A on the horizontal ground and the area of $\triangle PDE$ is 120 cm^2. Find
  (i) the angle between the metal sheet ADE and the horizontal ground,
  (ii) the shortest distance from A to the horizontal ground.

14B.30 HKCEE AM 1981 – II – 10

In the figure, $ABCDE$ is a right pyramid with a square base $ABCD$. Each of the eight edges of the pyramid is of length k. F, G and H are points on AB, AC and AD, respectively, such that FGH is a straight line and $BF = DH = rk$, where $0 \le r \le 1$. $EG \perp HF$, $\angle EGC = \theta$ and N is the foot of the perpendicular from E to the base.



(a) Express FE^2 and FG^2 in terms of k and r.

(b) Express EG and EN in terms of k and r.
Hence, or otherwise, show that $\sin\theta = \dfrac{1}{\sqrt{1+r^2}}$.

(c) Using the results of (b), find the range of the inclination of the plane EFH to the base as r varies from 0 to 1.

14B.31 HKCEE AM 1983 II – 8

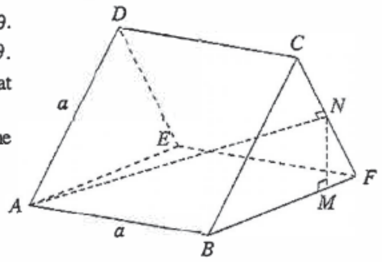
The figure shows a tent consisting of two inclined square planes $ABCD$ and $EFCD$ standing on the horizontal ground $ABFE$. The length of each side of the inclined planes is a. N is a point on CF such that $AN \perp CF$. Let $NF = x(\neq 0)$, $\angle CFB = \theta$ and M be a point on BF such that $NM \perp BF$.

(a) By considering $\triangle ABM$, express AM in terms of a, x and θ.

(b) By considering $\triangle ANF$, express AN in terms of a, x and θ.

(c) Using the results of (a) and (b), or otherwise, show that $x = 2a\cos^2\theta$.

(d) Given that $x = \frac{a}{2}$, find (correct to the nearest degree) the inclination of AN to the horizontal.

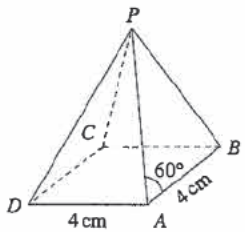


14B.32 HKCEE AM 1991 – II – 6

In the figure, $PABCD$ is a right pyramid with a square base of sides of length 4 cm. $\angle PAB = 60°$. Find, correct to the nearest 0.1 degree,

(a) the angle between the plane PAB and the base $ABCD$,

(b) the angle between the planes PAB and PAD.

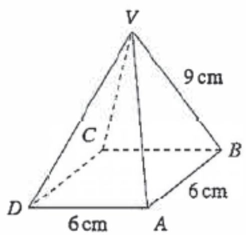


14B.33 HKCEE AM 1992 – II – 7

In the figure, $VABCD$ is a right pyramid with a square base of side 6 cm. $VB = 9$ cm. Find, correct to the nearest 0.1 degree,

(a) the angle between edge VB and the base $ABCD$,

(b) the angle between the planes VAB and VAD.

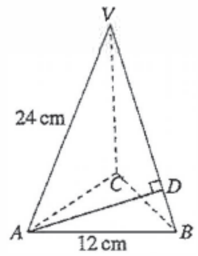


14B.34 HKCEE AM 1993 II 7

In the figure, $VABC$ is a right pyramid whose base ABC is an equilateral triangle. $AB = 12$ cm and $VA = 24$ cm. D is a point on VB such that AD is perpendicular to VB. Find, correct to 3 significant figures,

(a) $\angle VBA$ and AD,

(b) the angle between the faces VAB and VBC.



14B.35 HKCEE AM 1994 – II – 12

A, B and C are three points on the horizontal ground and $AB = 100$ km. P is a point vertically above C (see Figure (1)). Let $\angle CAB = \alpha$, $\angle CBA = \beta$, $\angle PAC = \theta$.

(a) Show that

(i) $AC = \dfrac{100\sin\beta}{\sin(\alpha+\beta)}$ km.

(ii) $PC = \dfrac{100\sin\beta\tan\theta}{\sin(\alpha+\beta)}$ km.

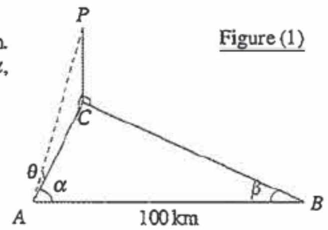


Figure (1)

(b) Suppose at P, $\alpha = 45°$, $\beta = 30°$ and $\theta = 20°$. An aeroplane climbs from P to a point P' along a straight path. The projection of P' on the ground is the point C' (see Figure (2)).
Given that $\angle C'AB = 37°$, $\angle C'BA = 43°$ and $\angle P'AC'$ 17°, find, correct to 2 decimal places,

(i) AC and AC',

(ii) the distance between C and C',

(iii) the increase in height of the aeroplane as it climbs from P to P',

(iv) the angle of inclination PP'.

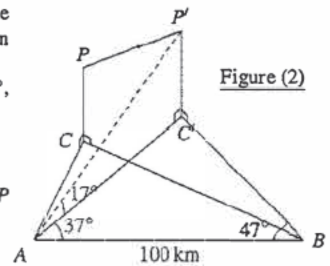


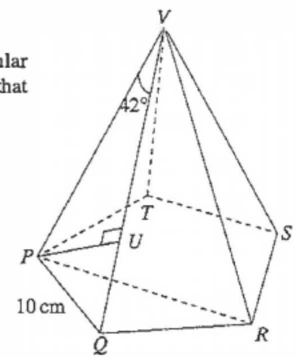
Figure (2)

14B.36 HKCEE AM 1995 – II – 7

In the figure, $VPQRST$ is a right pyramid whose base $PQRST$ is a regular pentagon. $PQ = 10$ cm and $\angle PVQ = 42°$. U is a point on VQ such that PU is perpendicular to VQ. Find, correct to 3 significant figures,

(a) PU and PR,

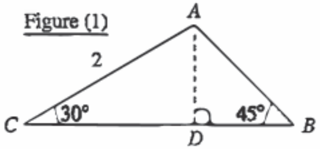
(b) the angle between the faces VPQ and VQR.



14B.37 HKCEE AM 1996 – II – 12

In Figure (1), ABC is a triangular piece of paper such that $\angle B = 45°$, $\angle C = 30°$ and $AC = 2$. D is the foot of perpendicular from A to BC.

Figure (1)



(a) Find AB, BD and DC.

(b) The paper is folded along AD. It is then placed on a horizontal table such that the edges AB and AC lie on the table and the plane DAB is vertical. (See Figure (2).) E is the foot of perpendicular from D to AB.

(i) If θ is the angle between DC and the horizontal, show that $\sin\theta = \frac{\sqrt{6}}{6}$.

(ii) Find CE.
Hence show that $\angle EAC = 45°$.

(iii) Find the angle between the two planes DAB and DAC to the nearest degree.

[Hint: You may tear off Figure (3) to help you answer part (b).]

Figure (2)

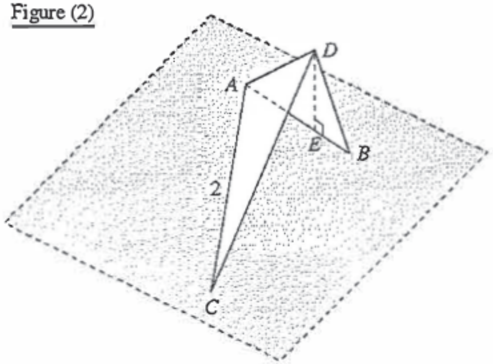
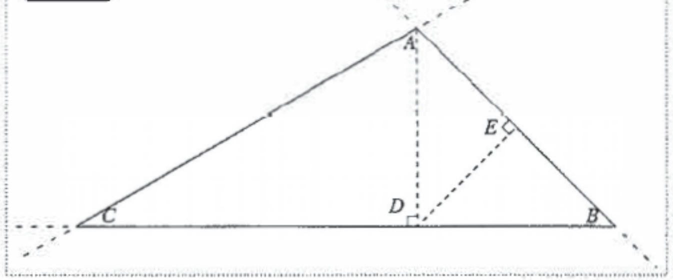


Figure (3)



14B.38 HKCEE AM 1997 – II – 12

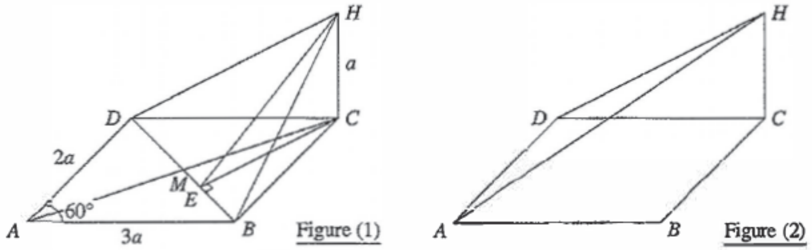


Figure (1) Figure (2)

In Figure (1), $ABCD$ is a parallelogram on a horizontal plane with $AB = 3a$, $AD = 2a$ and $\angle BAD = 60°$. H is a point vertically above C and $HC = a$.

(a) (i) Find AC in terms of a.

(ii) If M is the mid-point of AC, find the angle of elevation of H from M to the nearest degree.

(b) E is a point on BD such that CE is perpendicular to BD.

(i) Find BD and CE in terms of a.

(ii) Using Pythagoras' theorem and its converse, show that HE is perpendicular to BD.
Hence find the angle between the planes HBD and $ABCD$ to the nearest degree.

(c) Figure (2) shows the planes HAD and $ABCD$. X is a point lying on both planes such that the angle between the two planes is $\angle HXC$. Find AX in terms of a.

14B.39 HKCEE AM 1998 – II – 13

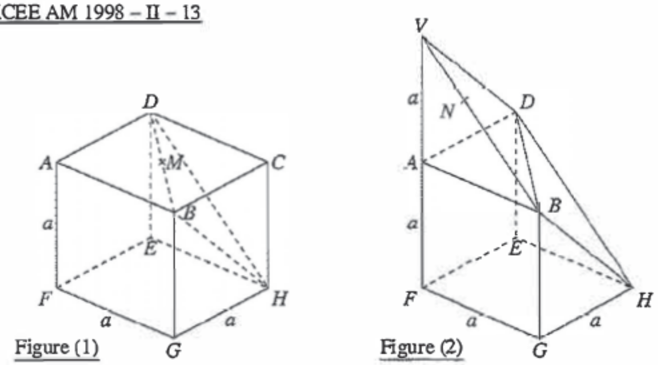


Figure (1) Figure (2)

(a) Figure (1) shows a solid cube $ABCDEFGH$ of side a. Let M be the mid point of BD.

(i) Find CM.

(ii) Find the angle between the lines CM and HM to the nearest degree.

(b) The tetrahedron $BCDH$ is cut off from the cube in (a) and is then placed on top of the solid $ABDEFGH$ as shown in Figure (2). The face BCD of the tetrahedron coincides with the face BAD of the solid $ABDEFGH$ such that vertex H of the tetrahedron moves to position V and vertex C coincides with A. The two faces BHD and BVD of the new solid lie on the same plane.

(i) Show that $\sin\angle FVH = \frac{\sqrt{3}}{3}$ and find the perpendicular distance from F to the face $BVDH$.

(ii) Let N be the point on VB such that DN and AN are both perpendicular to VB.

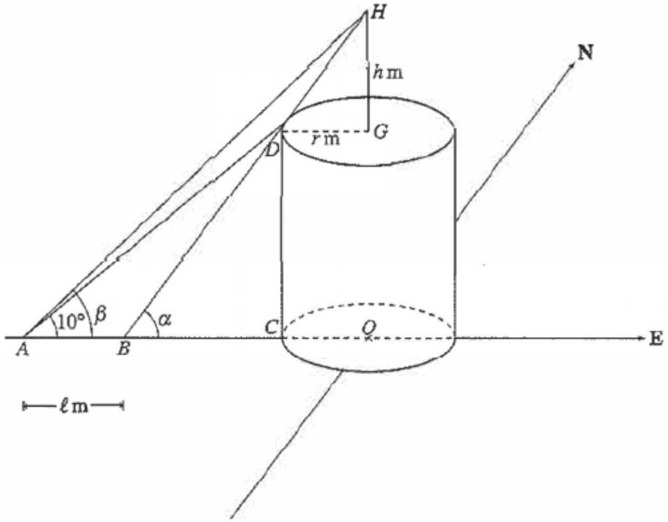
(1) Find DN.

(2) Find the angle between the faces BVD and BVA to the nearest degree.

(iii) A student says that the angle between the faces BHD and $ABGF$ is $\angle AND$. Explain briefly whether the student is correct.

14B.40 HKCEE AM 1999 – II – 11

The figure shows a right cylindrical tower with a radius of r m standing on horizontal ground. A vertical pole HG, h m in height, stands at the centre G of the roof of the tower. Let O be the centre of the base of the tower. C is a point on the circumference of the base of the tower due west of O and D is a point on the roof verticcally above C. A man stands at a point A due west of O. The angles of elevation of D and H from A are $10°$ and β respectively. The man walks towards the east to a point B where he can just see the top of the pole H as shown in the figure. (Note: If he moves forward, he can no longer see the pole.) The angle of elevation of H from B is α. Let $AB = \ell$ m.



(a) Show that $AD = \dfrac{\ell \sin\alpha}{\sin(\alpha - 10°)}$ m. Hence

(i) express CD in terms of ℓ and α,

(ii) show that $h = \dfrac{\ell \sin^2\alpha \sin(\beta - 10°)}{\sin(\alpha - 10°)\sin(\alpha - \beta)}$. (Hint: You may consider $\triangle ADH$.)

(b) **In this part, numerical answers should be given correct to two significant figures.**

Suppose $\alpha = 15°$, $\beta = 10.2°$ and $\ell = 97$.

(i) Find

(1) the height of the pole HG,

(2) the height and radius of the tower.

(ii) P is a point south-west of O. Another man standing at P can just see the top of the pole H. Find

(1) the distance of P from O,

(2) the bearing of B from P.

14B.41 HKCEE AM 2001 – 15

Figure (1)

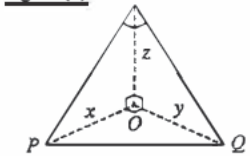
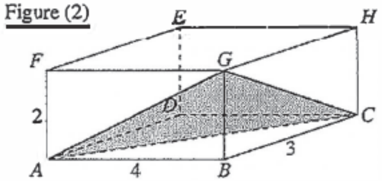


Figure (2)



(a) Figure (1) shows a pyramid $OPQR$. The sides OP, OQ and OR are of lengths x, y and z respectively, and they are mutually perpendicular to each other.

(i) Express $\cos\angle PRQ$ in terms of x, y and z.

(ii) Let S_1, S_2, S_3 and S_4 denote the areas of $\triangle OPR$, $\triangle OPQ$, $\triangle OQR$ and $\triangle PQR$ respectively. Show that $S_4{}^2 = S_1{}^2 + S_2{}^2 + S_3{}^2$.

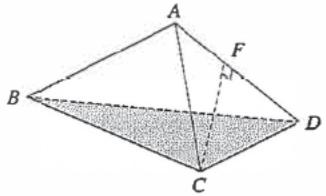
(b) Figure (2) shows a rectangular block $ABCDEFGH$. The lengths of sides AB, BC and AF are 4, 3 and 2 respectively. A pyramid $ABCG$ is cut from the block along the plane GAC.

(i) Find the volume of the pyramid $ABCG$.

(ii) Find the angle between the side AB and the plane GAC, giving your answer correct to the nearest degree.

14B.42 HKCEE AM 2002 17

The figure shows a tetrahedron $ABCD$ such that $AB = 28$, $CD = 30$, $AC = AD = 25$ and $BC = BD = 40$. F is the foot of perpendicular from C to AD.



(a) Find $\angle BFC$, giving your answer correct to the nearest degree.

(b) A student says that $\angle BFC$ represents the angle between the planes ACD and ABD.
Explain whether the student is correct or not.

14B.43 HKCEE AM 2003 18

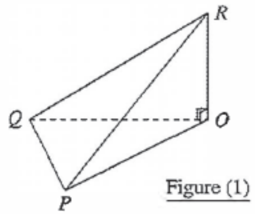


Figure (1)

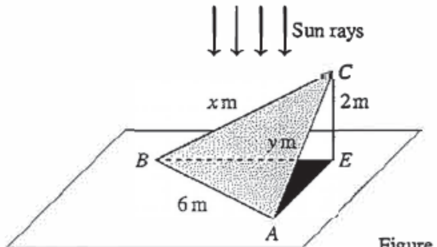


Figure (2)

(a) Figure (1) shows a tetrahedron $OPQR$ with RO perpendicular to the plane OPQ. Let θ be the angle between the planes RPQ and OPQ. Show that $\dfrac{\text{Area of } \triangle OPQ}{\text{Area of } \triangle RPQ} = \cos\theta$.

(b) In Figure (2), a pole of length 2 m is erected vertically at a point E on the horizontal ground. A triangular board ABC of area 12 m^2 is supported by the pole such that side AB touches the ground and vertex C is fastened to the top of the pole. $AB = 6$ m, $BC = x$ m and $CA = y$ m, where $6 > x > y$. The sun rays are vertical and cast a shadow of the board on the ground.

(i) Find the area of the shadow.

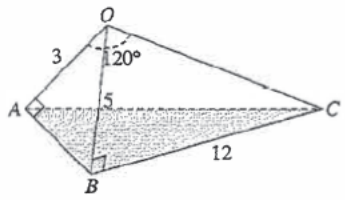
(ii) Two other ways of supporting the board with the pole are to fasten vertex A or B to the top of the pole with the opposite side touching the ground. Among these three ways determine which one will give the largest shadow.

14B.44 HKCEE AM 2004 – 11

In the figure, $OABC$ is a pyramid such that $OA = 3$, $OB = 5$, $BC = 12$, $\angle AOC = 120°$ and $\angle OAB = \angle OBC = 90°$.

(a) Find AC.

(b) A student says that angle between the planes OBC and ABC can be represented by $\angle OBA$. Determine whether the student is correct or not.



14B.45 HKCEE AM 2006 – 17

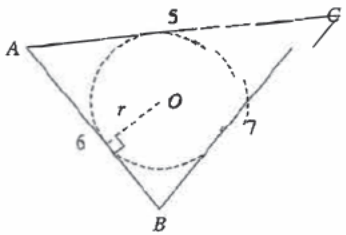


Figure (1)

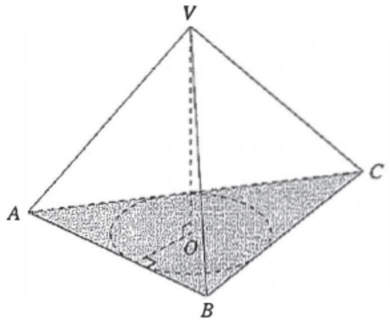


Figure (2)

(a) ABC is a triangle with $AB = 6$, $BC = 7$ and $CA = 5$. A circle is inscribed in the triangle (see Figure (1)). Let O be the centre of the circle and r be its radius.

 (i) Find the area of $\triangle ABC$.

 (ii) By considering the areas of $\triangle AOB$, $\triangle BOC$ and $\triangle COA$, show that $r = \frac{2\sqrt{6}}{3}$.

(b) $VABC$ is a tetrahedron with the $\triangle ABC$ described in (a) as the base (see Figure (2)). Furthermore, point O is the foot of perpendicular from V to the plane ABC. It is given that the angle between the planes VAB and ABC is 60°.

 (i) Find the volume of the tetrahedron $VABC$.

 (ii) Find the area of $\triangle VBC$.

 (iii) Find the angle between the side AB and the plane VBC, giving your answer correct to the nearest degree.

14B.46 HKCEE AM 2008 – 16

The figure shows a triangular pyramid $VABC$. The base of the pyramid is a right-angled triangle with $AB = 2$ cm and $\angle BAC = 90°$. $\triangle VAB$ and $\triangle VAC$ are equilateral triangles.

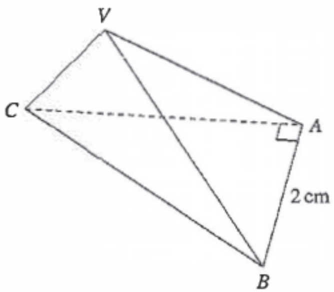
(a) Explain why the angle between the planes VAB and ABC **cannot** be represented by $\angle VAC$.

(b) Let D and E be the mid-points of AB and BC respectively.

 (i) Show that the angle between the planes VAB and ABC can be represented by $\angle VDE$.

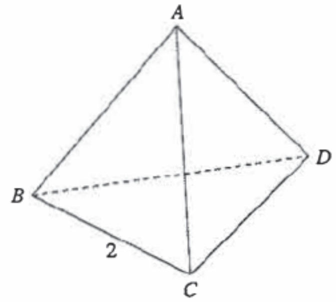
 (ii) Show that $\angle VED = 90°$.

(c) Find the distance between the point C and the plane VAB.



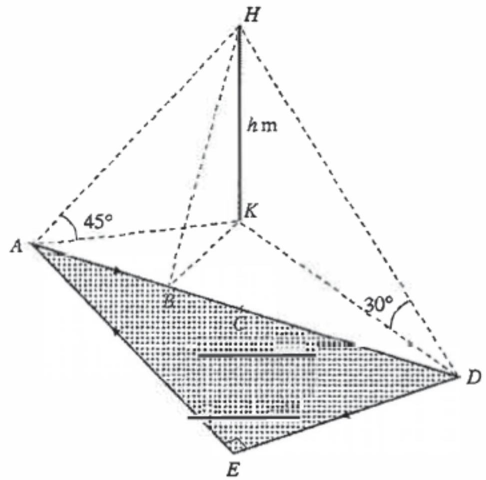
14B.47 HKCEE AM 2009 – 12

In the figure, $ABCD$ is a regular tetrahedron with length of each side 2. Find the angle between the planes ABC and BCD correct to the nearest degree.



14B.48 HKCEE AM 2009 – 18

The figure shows a park AED on a horizontal ground. The park is in the form of a right-angled triangle surrounded by a walking path with negligible width. Henry walks along the path at a constant speed. He starts from point A at 7:00 am. He reaches points B, C and D at 7:10 am, 7:15 am and 7:30 am respectively and returns to A via point E. The angles of elevation of H, the top of a tower outside the park, from A and D are 45° and 30° respectively. At point B, Henry is closest to the point K which is the projection of H on the ground. Let $HK = h$ m.



(a) Express DK in terms of h.

(b) Show that $AB = \sqrt{\frac{2}{3}}h$ m.

(c) Find the angle of elevation of H from C correct to the nearest degree.

(d) Henry returns to A at 8:10 am. It is known that the area of the park is 9450 m^2.

 (i) Find h.

 (ii) A vertical pole of length 3 m is located such that it is equidistant from A, D and E. Find the angle of elevation of H from the top of the pole correct to the nearest degree.

14B.49 HKCEE AM 2010 – 17

[Note: In this question, numerical answers may be given correct to 3 significant figures. You may use a ruler to tear off Figure (5) to help you if you attempt this question.]

Three faces of a tetrahedron (see Figure (4)) are formed by folding a triangular piece of paper ABC, where $AB = AC = 11$ cm, $\angle BAC = 120°$ and AD is an altitude (see Figure (1)), with the following steps.

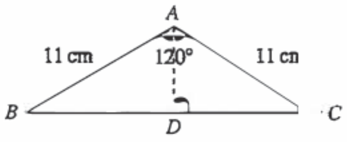


Figure (1)

Step 1: Fold AB over so that AB coincides with AD, then crease line AE (see Figure (2)).

(a) Calculate the length of AE and the area of $\triangle ABE$.

Step 2: Fold AC over so that AC coincides with AE, then crease line AF (see Figure (3)).

(b) Calculate the length of AF.

Step 3: Unfold the paper. Then fold the paper along AE and AF such that AB coincides with AC completely (see Figure (4)).

(c) It is known that the volume of the tetrahedron is 22.582 cm^3 (correct to 5 significant figures).
 (i) Find the angle between the line AF and the plane $\triangle ABE$ in the tetrahedron.
 (ii) Find the angle between the planes $\triangle ABE$ and $\triangle ABF$ in the tetrahedron.

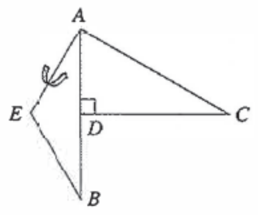


Figure (2)

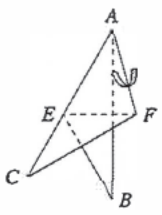


Figure (3)

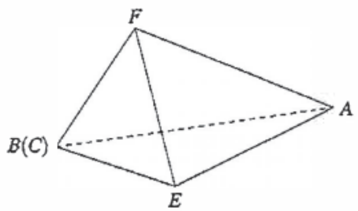


Figure (4)

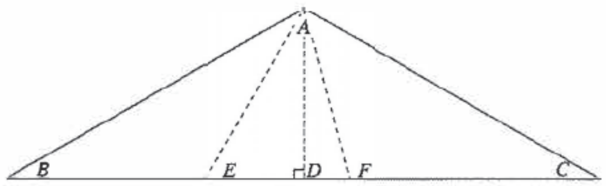


Figure (5)

14B.50 HKCEE AM 2011 13

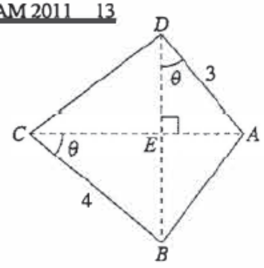


Figure (1)

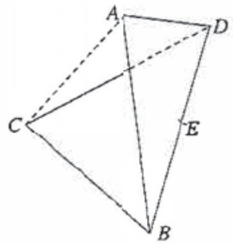


Figure (2)

In Figure (1), $ABCD$ is a quadrilateral with diagonals AC and BD perpendicular to each other and intersecting at E. It is given that $AD = 3$, $BC = 4$ and $\angle ADE = \angle BCE = \theta$, where $0° < \theta < 90°$.

(a) (i) Show that $AB = 5\sin\theta$.
 (ii) Express CD in terms of θ.

(b) The quadrilateral is folded along BD as shown in Figure (2). Let the planes ABD and BCD be Π_1 and Π_2 respectively. Let $\angle ABC = \alpha$. It is given that
 the angle between the lines AB and BC the angle between the planes Π_1 and Π_2.
 (i) By considering the length of AC, show that $\cos\alpha = \dfrac{4\sin\theta}{5 - 3\cos\theta}$.
 (ii) Prove that α is acute.
 (iii) Furthermore, it is given that
 the angle between the line AB and Π_2 = the angle between the line AD and Π_2.
 State with reason whether the angle between the line AC and Π_2 is greater than, less than or equal to the angle between the line AB and Π_2.

14B.51 HKDSE MA SP – I – 18

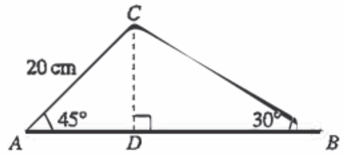


Figure (1)

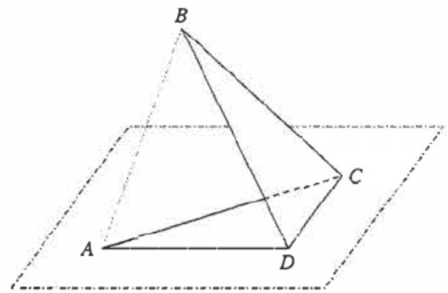


Figure (2)

In Figure (1), ABC is a triangular paper card. D is a point lying on AB such that CD is perpendicular to AB. It is given that $AC = 20$ cm, $\angle CAD = 45°$ and $\angle CBD = 30°$.

(a) Find, in surd form, BC and BD.

(b) The triangular paper card in Figure (1) is folded along CD such that $\triangle ACD$ lies on the horizontal plane as shown in Figure (2).
 (i) If the distance between A and B is 18 cm, find the angle between the plane BCD and the horizontal plane.
 (ii) Describe how the volume of the tetrahedron $ABCD$ varies when $\angle ADB$ increases from 40° to 140°. Explain your answer.

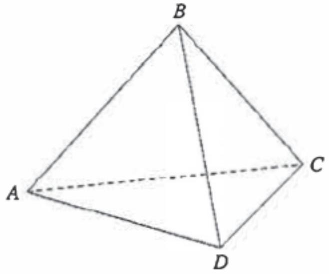
14B.52 HKDSE MA PP I 18

The figure shows a geometric model $ABCD$ in the form of a tetrahedron. It is found that $\angle ACB = 60°$, $AC = AD = 20$ cm, $BC = BD = 12$ cm and $CD = 14$ cm.

(a) Find the length of AB.

(b) Find the angle between the plane ABC and the plane ABD.

(c) Let P be a movable point on the slant edge AB. Describe how $\angle CPD$ varies as P moves from A to B. Explain your answer.



14B.53 HKDSE MA 2012 I – 18

Figure (1) shows a right pyramid $VABCD$ with a square base, where $\angle VAB = 72°$. The length of a side of the base is 20 cm. Let P and Q be the points lying on VA and VD respectively such that PQ is parallel to BC and $\angle PBA = 60°$. A geometric model is made by cutting off the pyramid $VPBCQ$ from $VABCD$ as shown in Figure (2).

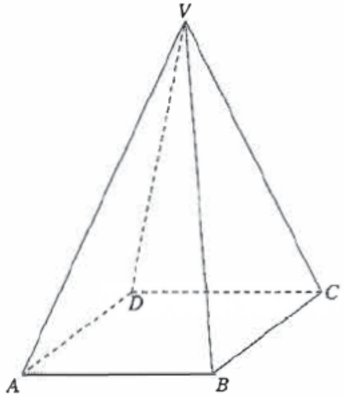


Figure (1)

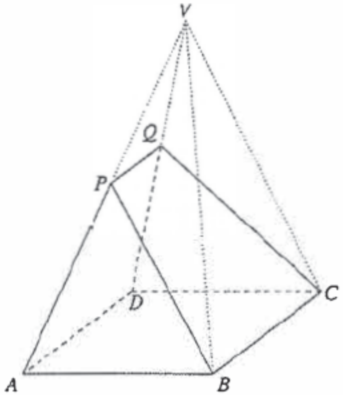


Figure (2)

(a) Find the length of AP.

(b) Let α be the angle between the plane $PBCQ$ and the base $ABCD$.

(i) Find α.

(ii) Let β be the angle between PB and the base $ABCD$. Which one of α and β is greater? Explain your answer.

14B.54 HKDSE MA 2013 – I – 18

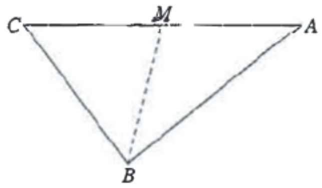


Figure (1)

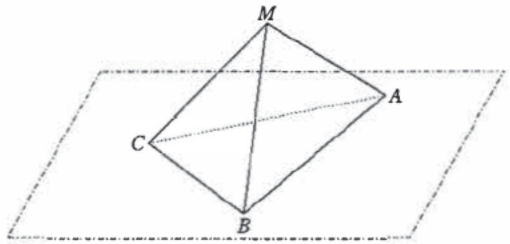


Figure (2)

(a) Figure (1) shows a piece of triangular paper card ABC with $AB = 28$ cm, $BC = 21$ cm and $AC = 35$ cm. Let M be a point lying on AC such that $\angle BMC = 75°$. Find

(i) $\angle BCM$,

(ii) CM.

(b) Peter folds the triangular paper card described in (a) along BM such that AB and BC lie on the horizontal ground as shown in Figure (2). It is given that $\angle AMC = 107°$.

(i) Find the distance between A and C on the horizontal ground.

(ii) Let N be a point lying on BC such that MN is perpendicular to BC. Peter claims that the angle between the face BCM and the horizontal ground is $\angle ANM$. Do you agree? Explain your answer.

14B.55 HKDSE MA 2014 I – 17

Figure (1) shows a solid pyramid $VABCD$ with a rectangular base, where $AB = 18$ cm, $BC = 10$ cm, $VB = VC = 30$ cm and $\angle VAB = \angle VDC = 110°$.

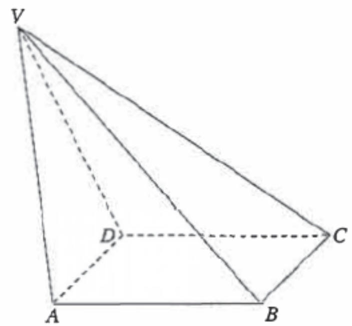


Figure (1)

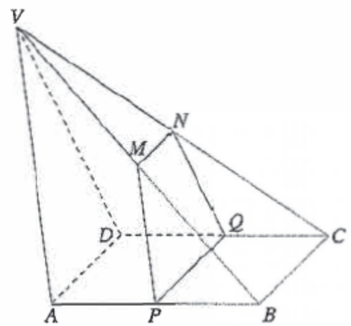


Figure (2)

(a) Find $\angle VBA$.

(b) P, Q, M and N are the mid points of AB, CD, VB and VC respectively. A geometric model is made by cutting off $PBCQNM$ from $VABCD$ as shown in Figure (2). A craftsman claims that the area of the trapezium $PQNM$ is less than 70 cm^2. Do you agree? Explain your answer.

14B.56 HKDSE MA 2015 – I 19

In Figure (1), $ABCDB'$ is a pentagonal paper card. It is given that $AB = AB' = 40$ cm, $BC = B'D = 24$ cm and $\angle ABC = \angle AB'D = 80°$.

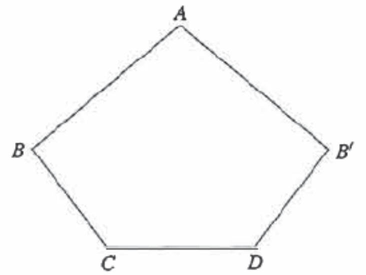


Figure (1)

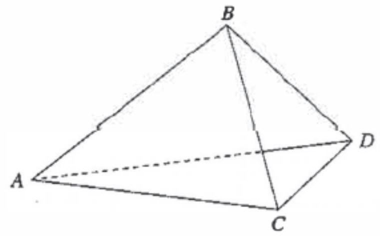


Figure (2)

(a) Suppose that $105° \le \angle BCD \le 145°$.
   (i) Find the distance between A and C.
   (ii) Find $\angle ACB$.
   (iii) Describe how the area of the paper card varies when $\angle BCD$ increases from 105° to 145°. Explain your answer.

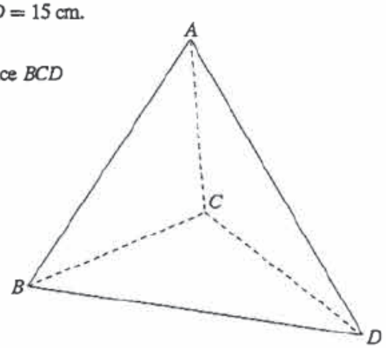
(b) Suppose that $\angle BCD = 132°$. The paper card in Figure (1) is folded along AC and AD such that AB and AB' join together to form a pyramid $ABCD$ as shown in Figure (2). Find the volume of the pyramid $ABCD$.

14B.57 HKDSE MA 2016 – I – 19

The figure shows a geometric model $ABCD$ in the form of a tetrahedron. It is given that $\angle BAD = 86°$, $\angle CBD = 43°$, $AB = 10$ cm, $AC = 6$ cm, $BC = 8$ cm and $BD = 15$ cm.

(a) Find $\angle ABD$ and CD.

(b) A craftsman claims that the angle between AB and the face BCD is $\angle ABC$. Do you agree? Explain your answer.

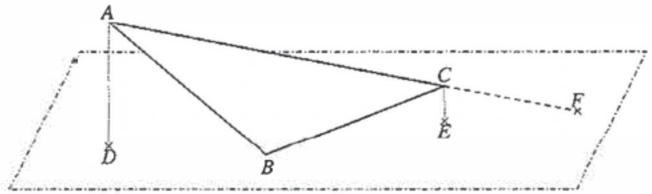


14B.58 HKDSE MA 2017 – I 19

ABC is a thin triangular metal sheet, where $BC = 24$ cm, $\angle BAC = 30°$ and $\angle ACB = 42°$.

(a) Find the length of AC.

(b) In the figure, the thin metal sheet ABC is held such that only the vertex B lies on the horizontal ground. D and E are points lying on the horizontal ground vertically below the vertices A and C respectively. AC produced meets the horizontal ground at the point F. A craftsman finds that $AD = 10$ cm and $CE = 2$ cm.
   (i) Find the distance between C and F.
   (ii) Find the area of $\triangle ABF$.
   (iii) Find the inclination of the thin metal sheet ABC to the horizontal ground.
   (iv) The craftsman claims that the area of $\triangle BDF$ is greater than 460 cm^2. Do you agree? Explain your answer.



14B.59 HKDSE MA 2018 I – 17

(a) In Figure (1), $ABCD$ is a paper card in the shape of a parallelogram. It is given that $AB = 60$ cm, $\angle ABD = 20°$ and $\angle BAD = 120°$. Find the length of AD.

(b) The paper card in Figure (1) is folded along BD such that the distance between A and C is 40 cm (see Figure (2)).
   (i) Find $\angle ABC$.
   (ii) Find the angle between the plane ABD and the plane BCD.

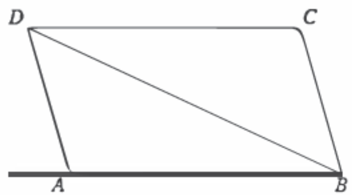


Figure (1)

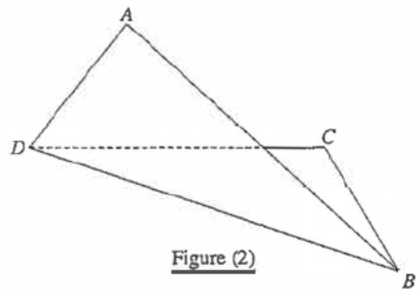


Figure (2)

14B.60 <u>HKDSE MA 2019 I – 18</u>

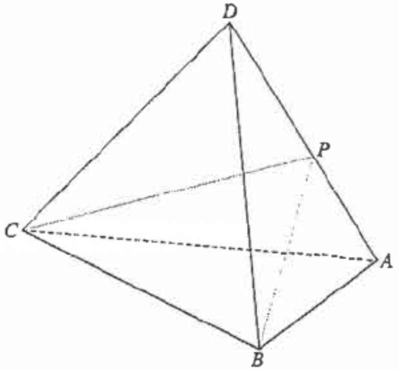
The figure shows a tetrahedron $ABCD$. Let P be a point lying on AD such that BP is perpendicular to AD. A craftsman finds that $AC = AD = CD = 13$ cm, $BC = 8$ cm, $BD = 12$ cm and $\angle ABD = 72°$.

(a) Find

(i) $\angle BAD$,

(ii) CP.

(b) The craftsman claims that $\angle BPC$ is the angle between the face ABD and the face ACD. Is the claim correct? Explain your answer.



14B.61 <u>HKDSE MA 2020 – I – 19</u>

$PQRS$ is a quadrilateral paper card, where $PQ = 60$ cm, $PS = 40$ cm, $\angle PQR = 30°$, $\angle PRQ = 55°$ and $\angle QPS = 120°$. The paper card is held with QR lying on the horizontal ground as shown in Figure 3.

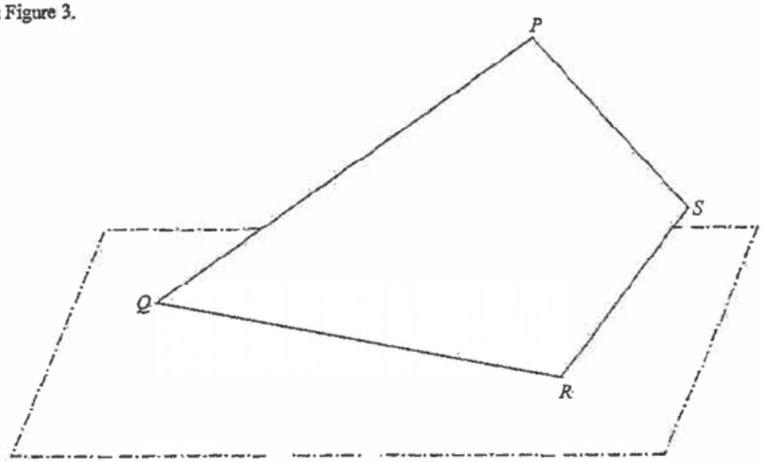


Figure 3

(a) Find the length of RS. (3 marks)

(b) Find the area of the paper card (2 marks)

(c) It is given that the angle between the paper card and the horizontal ground is $32°$.

(i) Find the shortest distance from P to the horizontal ground.

(ii) A student claims that the angle between RS and the horizontal ground is at most $20°$. Is the claim correct? Explain your answer.

(7 marks)

14 Applications of Trigonometry

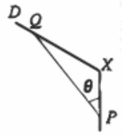
14A Two-dimensional applications

14A.1 HKCEE MA 1981(2/3) – I – 11

(a) Distance at noon $= \sqrt{24^2+9^2-2\cdot 24\cdot 9\cos 60^\circ}$
$= 21$ (km)

(b) At 4 p.m.,
Distance travelled by $P = 4.5\times 4 = 18$ (km)
$\Rightarrow\ PX = 24-18 = 6$ (km)
Distance travelled by $Q = 6\times 4 = 24$ (km)
$\Rightarrow\ QX = 24-9 = 15$ (km) [Q has gone past X.]
$\therefore$ Distance at 4 p.m. $= \sqrt{6^2+15^2-2\cdot 6\cdot 15\cos 60^\circ}$
$= \sqrt{171} = 13.1$ (km, 3 s.f.)

(c) $\theta = \cos^{-1}\dfrac{(\sqrt{171})^2+6^2-15^2}{2(\sqrt{171})(6)} = 96.59^\circ$
$\therefore$ Bearing $= 360^\circ - 96.59^\circ$
$= 263^\circ$
or N97°W (nearest deg)



14A.2 HKCEE MA 1982(3) – I – 2

$\angle A = \cos^{-1}\dfrac{4^2+5^2-7^2}{2\cdot 4\cdot 5} = 102^\circ$ (nearest deg)

14A.3 HKCEE MA 1985(A/B) – I – 13

(a) $DE^2 = BD^2+BE^2-2\cdot BD\cdot BE\cos\angle B$
$= (2-x)^2+x^2-\cancel{2}(2-x)(x)\cancel{\cos 60^\circ}$
$= 3x^2-6x+4$

(b) Area of $\triangle DEF = \dfrac{1}{2}DE\cdot DE\sin 60^\circ$
$= \dfrac{1}{2}(3x^2-6x+4)\cdot\dfrac{\sqrt{3}}{2}$
$= \dfrac{\sqrt{3}}{4}(3x^2-6x+4)$
$= \dfrac{3\sqrt{3}}{4}\left(x^2-2x+\dfrac{4}{3}\right)$
$= \dfrac{3\sqrt{3}}{4}\left(x^2-2x+1+\dfrac{1}{3}\right)$
$= \dfrac{3\sqrt{3}}{4}(x-1)^2+\dfrac{\sqrt{3}}{4}$
$\therefore$ Minimum area is attained when $x=1$.

(c) $\dfrac{3\sqrt{3}}{4}(x-1)^2+\dfrac{\sqrt{3}}{4} < \dfrac{\sqrt{3}}{3}$
$(x-1)^2 \le \dfrac{1}{9}$
$\dfrac{-1}{3} \le x-1 \le \dfrac{1}{3} \ \Rightarrow\ \dfrac{2}{3} \le x \le \dfrac{4}{3}$

14A.4 HKCEE MA 1989 – I – 6

(a) $\angle ABD = \angle ACD = 60^\circ$ ($\angle$s in the same segment)
$\angle BAD = 180^\circ-(60^\circ+40^\circ)$ (opp. $\angle$s, cyclic quad.)
$= 80^\circ$

(b) $\dfrac{BD}{\sin\angle BAD} = \dfrac{AD}{\sin\angle ABD}$
$BD = \dfrac{10\sin 80^\circ}{\sin 60^\circ} = 11.37$ (cm, 2 d.p.)

14A.5 HKCEE MA 1997 – I – 5

(a) $AC = \sqrt{3^2+4^2} = 5$
(b) $AD = \sqrt{5^2+6^2-2\cdot 5\cdot 6\cos 60^\circ} = \sqrt{31}$ (= 5.57, 3.s.f)
(c) Area $= \dfrac{1}{2}(5)(6)\sin 60^\circ = \dfrac{15\sqrt{3}}{2}$ (= 13.0, 3 s.f.)

14A.6 HKCEE MA 2000 – I – 13

(a) $\angle A = \angle ABC = \angle BCD$ (given)
$= (5-2)180^\circ \div 5$ ($\angle$ sum of polygon)
$= 108^\circ$
$\angle GCD = 90^\circ$ (property of square)
$\Rightarrow\ \angle BCG = 108^\circ - 90^\circ = 18^\circ$
$BC = CD = CG$ (given)
$\angle GBC = \angle BGC$ (base $\angle$s, isos. $\triangle$)
In $\triangle BCG$, $\angle GBC = (180^\circ - \angle BCG)\div 2$ ($\angle$ sum of $\triangle$)
$= 81^\circ$
$\angle ABP = 108^\circ - 81^\circ = 27^\circ$
$\angle APB = 180^\circ - \angle A - \angle ABP = 45^\circ$ ($\angle$ sum of $\triangle$)

(b) $AP = \dfrac{\sin\angle ABP}{\sin\angle APB}AB = \dfrac{\sin 27^\circ}{\sin 45^\circ}AB = 0.642AB$
$PE = AB \quad AP = (1-0.642)AB = 0.358AB < AP$
i.e. AP is longer.

14A.7 HKCEE MA 2001 – I – 9

$\dfrac{AB}{\sin 50^\circ} = \dfrac{8}{\sin(180^\circ-50^\circ-70^\circ)}$
$\Rightarrow\ AB = 7.0764 = 7.08$ (cm, 3 s.f.)
$\therefore$ Area $= \dfrac{1}{2}(8)(7.0764)\sin 70^\circ = 26.6$ (cm^2, 3 s.f.)

14B Three-dimensional applications

14B.1 HKCEE MA 1980(1/1*/3) – I – 9

(a) (i) In $\triangle PAC$, $x = \dfrac{h}{\tan\alpha}$
(ii) In $\triangle PBC$, $y = \dfrac{h}{\tan\beta}$

(b) In $\triangle ABC$,
$x^2+400^2 = y^2$
$\left(\dfrac{h}{\tan 60^\circ}\right)^2 + 160000 = \left(\dfrac{h}{\tan 30^\circ}\right)^2$
$\dfrac{h^2}{3} + 160000 = 3h^2$
$h^2 = 60000 \ \Rightarrow\ h = 245$ (3 s.f.)

14B.2 HKCEE MA 1982(1/2/3) – I – 8

(a) $8x+4y+9 = 69 \ \Rightarrow\ y = 15-2x$
$AC^2 = 9^2-y^2 \ \Rightarrow\ 2x^2 = 81-y^2$
$2x^2 = 81-(15-2x)^2$
$x^2-10x+24 = 0 \ \Rightarrow\ x = 4$ or 6
When $x=4$, $y = 15-2(4) = 7$
When $x=6$, $y = 15-2(6) = 3$

(b) $\angle ABC = \cos^{-1}\dfrac{y}{9} = \cos^{-1}\dfrac{7}{9} = 39^\circ$ (nrst deg)

14B.3 HKCEE MA 1983(A/B) – I – 13

(a) In $\triangle ACH$, $AC = \dfrac{50}{\tan 45^\circ} = 50$ (m)
In $\triangle BCH$, $BC = \dfrac{50}{\tan 30^\circ} = 50\sqrt{3}$ (m)
In $\triangle ABC$, $AB = \sqrt{(50)^2+(50\sqrt{3})^2} = 100$ (m)

(b) (i) $\dfrac{AC\cdot BC}{2} = \dfrac{CP\cdot AB}{2}$ (= Area of $\triangle ABC$)
$\Rightarrow\ CP = \dfrac{(50)(50\sqrt{3})}{100} = 25\sqrt{3} = 43.3$ (m, 3 s.f.)
(ii) Required $\angle = \angle HPC = \tan^{-1}\dfrac{HC}{CP} = 49^\circ$ (nrst deg)

14B.4 HKCEE MA 1984(A/B) – I – 13

(a) (i) In $\triangle ACH$,
$HA = 20\tan 15^\circ = 5.25898 = 5.36$ (m, 2 d.p.)
(ii) In $\triangle ABH$,
$AB = \dfrac{HA}{\tan 30^\circ} = 9.28203 = 9.28$ (m, 2 d.p.)

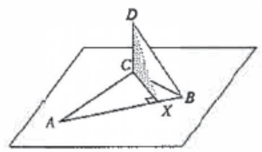
(b) Given: $\angle ABC = 90^\circ$ ($\angle$ in semi-circle)
(i) $BC = \sqrt{AC^2-AB^2} = 17.71564 = 17.72$ (m, 2 d.p.)
(ii) Area $= \dfrac{1}{2}AB\cdot BC = 82.22$ m^2 (2 d.p.)

14B.5 HKCEE MA 1985(A/B) – I – 8

(a) In $\triangle ABC$, $\dfrac{BC}{\sin 30^\circ} = \dfrac{100}{\sin(180^\circ-30^\circ-45^\circ)} = \dfrac{AC}{\sin 45^\circ}$
$\Rightarrow\ BC = 51.76381 = 51.8$ (m, 1 d.p.)
$AC = 73.20508 = 73.2$ (m, 1 d.p.)

(b) (i) In $\triangle BCD$, $CD = BC\tan 25^\circ = 24.13789$
$= 24.1$ (m, 1 d.p.)

(ii)



(1) In $\triangle CXB$, $CX = BC\sin 45^\circ = 36.60254$
$= 36.6$ (m, 1 d.p.)
(2) Required $\angle = \angle DXC$
$= \tan^{-1}\dfrac{CD}{CX} = 33^\circ$ (nrst deg)

14B.6 HKCEE MA 1986(A/B) – I – 10

(a) In $\triangle QRS$, $\dfrac{QS}{\sin 35^\circ} = \dfrac{500}{\sin(180^\circ-50^\circ-35^\circ)}$
$\Rightarrow\ QS = 287.88370$ (m)
$\therefore$ In $\triangle PQS$,
Required distance $= PS = QS\tan 15^\circ$
$= 77.13821 = 77.1$ (m, 3 s.f.)

(b) In $\triangle QRS$, $\dfrac{RS}{\sin 50^\circ} = \dfrac{500}{\sin 95^\circ} \ \Rightarrow\ RS = 384.48530$ (m)
$\therefore$ In $\triangle PRS$, Required $\angle = \angle PRS = \tan^{-1}\dfrac{PS}{RS}$
$= 11^\circ$ (nrst deg)

14B.7 HKCEE MA 1987(A/B) – I – 11

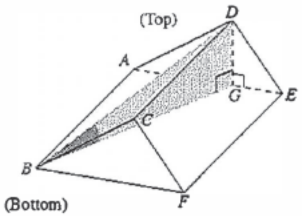
(a) In $\triangle ADE$, $AE = \sqrt{3^2+2^2-2\cdot 3\cdot 2\cos 80^\circ}$
$= 3.30397 = 3.304$ (cm, 3 d.p.)

(b) In $\triangle ADE$, $\angle DAE = \cos^{-1}\dfrac{AE^2+3^2-2^2}{2\cdot AE\cdot 3}$
$= 36.59365^\circ = 36.594^\circ$ (3 d.p.)

(c) In $\triangle ADG$, $DG = 3\sin\angle DAE$
$= 1.7884077 = 1.788$ (cm, 3 d.p.)

(d) In $\triangle ABD$, $BD = \sqrt{3^2+3^2}$
$= \sqrt{18} = 4.243$ (cm, 3 d.p.)

(e)



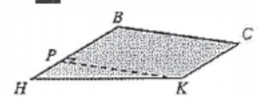
Required $\angle = \angle DBG = \sin^{-1}\dfrac{DG}{BD} = 24.931^\circ$ (3 d.p.)

14B.8 HKCEE MA 1988 – I – 13

(a) In $\triangle ABH$, $HB = \dfrac{3}{\tan\theta}$
In $\triangle DCK$, $KC = \dfrac{2}{\tan\theta}$

(b) (i) $S_1 = \dfrac{(2+3)(6)}{2} = 15$ (m^2)
(ii) $S_2 = \dfrac{(\frac{3}{\tan\theta}+\frac{2}{\tan\theta})(6)}{2} = \dfrac{15}{\tan\theta}$ (m^2)
$\therefore\ \dfrac{S_1}{S_2} = \dfrac{15}{\frac{15}{\tan\theta}} = \tan\theta$

(c)



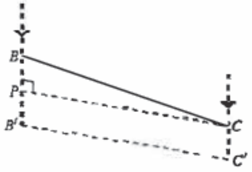
Let P be the foot of perpendicular from K to BH.
$PK = 6$ m, $PH = \dfrac{3}{\tan 30^\circ} - \dfrac{2}{\tan 30^\circ} = \sqrt{3}$ (m)
$\therefore\ HK = \sqrt{PK^2+PH^2} = \sqrt{39}$ (m)

14B.9 HKCEE MA 1989 – I – 10

(a) In $\Delta ABB'$, $AB' = 10\cos 45° = 5\sqrt{2}$ (m) (7.07 m, 3 s.f.)
In $\Delta ACC'$, $AC' = 10\cos 30° = 5\sqrt{3}$ (m) (8.66 m, 3 s.f.)

(b) In ΔABC, $BC = \sqrt{10^2 + 10^2} = \sqrt{200}$ (m) (14.1 m, 3 s.f.)
In $\Delta ABB'$, $BB' = 10\sin 45° = 5\sqrt{2}$ (m) (7.07 m, 3 s.f.)
In $\Delta ACC'$, $CC' = 10\sin 30° = 5$ (m)

(c)



Let P be the foot of perpendicular from C to BB'.
$BP = BB' - CC' = 5(\sqrt{2} - 1)$ m
$\therefore B'C' = PC$
$= \sqrt{BC^2 - BP^2}$
$= \sqrt{200 - 25(\sqrt{2} - 1)^2}$
$= \sqrt{125 + 50\sqrt{2}}$ (m) (14.0 m, 3 s.f.)

(d) In $\Delta AB'C'$,
$\angle B'AC' = \cos^{-1}\dfrac{AB'^2 + AC'^2 - B'C'^2}{2 \cdot AB' \cdot AC'}$
$= \cos^{-1}\dfrac{(5\sqrt{2})^2 + (5\sqrt{3})^2 - (125 + 50\sqrt{2})}{2(5\sqrt{2})(5\sqrt{3})}$
$= 125.2644° = 125°$ (3 s.f.)

Hence, Area $= \dfrac{1}{2}(AB')(AC')\sin\angle B'AC' = 25$ (m²)

14B.10 HKCEE MA 1990 – I – 10

(a) $\angle TAO = 30°$, $\angle TBO = 60°$
In ΔTOA, $OA = \dfrac{h}{\tan 30°} = \sqrt{3}h$ (m)
In ΔTOB, $OB = \dfrac{h}{\tan 60°} = \dfrac{h}{\sqrt{3}}$ (m)

(b) In ΔOAB,
$AB = \sqrt{OA^2 + OB^2 - 2 \cdot OA \cdot OB\cos(20° + 40°)}$
$= \sqrt{\dfrac{10}{3}h^2 - h^2} = \sqrt{\dfrac{7}{3}}h$ (m)
$\therefore h = 500 \div \sqrt{\dfrac{7}{3}} = 327.3268 = 327$ (m, 3 s.f.)

(c) $\angle OAB = \cos^{-1}\dfrac{OA^2 + 500^2 - OB^2}{2 \cdot OA \cdot 500}$
$= 19.1066° = 19°$ (nearest deg)
(i) N(20° + 19°)E = N39°E
(ii) S39°W

14B.11 HKCEE MA 1992 – I – 15

(a) In ΔABD, $BD = \sqrt{3^2 + 3^2} = \sqrt{18}$ (m)
In ΔBDE, $ED = \sqrt{BD^2 - BE^2} = \sqrt{14}$ (m)
In ΔABE, $AE = \sqrt{AB^2 - BE^2} = \sqrt{5}$ (m)

(b) In ΔADE, $\angle ADE = \cos^{-1}\dfrac{3^2 + 14 - 5}{2 \cdot 3 \cdot \sqrt{14}}$
$= 36.69923° = 36.7°$ (3 s.f.)

(c) Required $\angle = \angle BDE = \sin^{-1}\dfrac{BE}{BD}$
$= 28.12551° = 28.1°$ (3 s.f.)

(d) In ΔADC, $\angle ADC = 2\angle ADE = 73.39845°$
$AC = \sqrt{3^2 + 3^2 - 2 \cdot 3 \cdot 3\cos 73.39845°}$
$= 3.58569$ (m)

Denote the intersection of the diagonals of the square $ABCD$ by P. Since $BD \perp AC$ at P, the required angle is $\angle APC$ (in Figure (2)).

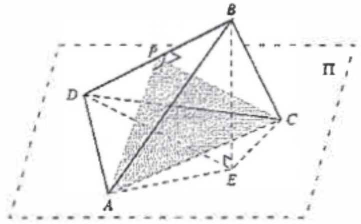


Figure (2)

$AP = PC = \dfrac{1}{2}BD = \dfrac{\sqrt{18}}{2}$

In ΔAPC, $\angle APC = \cos^{-1}\dfrac{(\frac{\sqrt{18}}{2})^2 + (\frac{\sqrt{18}}{2})^2 - 3.58569^2}{2(\frac{\sqrt{18}}{2})(\frac{\sqrt{18}}{2})}$
$= 115°$ (3 s.f.)

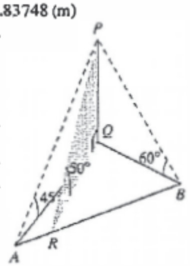
14B.12 HKCEE MA 1993 – I – 12

(a) (i) In ΔAPQ, $AQ = \dfrac{h}{\tan 45°} = h$ (m)
In ΔBPQ, $BQ = \dfrac{h}{\tan 60°} = \dfrac{h}{\sqrt{3}}$ (m)

(ii) In ΔABQ,
$100^2 = h^2 + \left(\dfrac{h}{\sqrt{3}}\right)^2 - 2(h)\left(\dfrac{h}{\sqrt{3}}\right)\cos 80°$
$10000 = \left(\dfrac{4}{3} - \dfrac{2\cos 80°}{\sqrt{3}}\right)h^2$
$h = 93.954854 = 94.0$ (3 s.f.)
$\angle QAB = \cos^{-1}\dfrac{AQ^2 + 100^2 - BQ^2}{2 \cdot AQ \cdot 100}$
$= 32.29019° = 32.3°$ (3 s.f.)

(b) In ΔPQR, $QR = \dfrac{h}{\tan 50°} = 78.83748$ (m)

(From A to B, the angle of elevation increases from 45° until it reaches the maximum. Supposing the max is reached at point M, R must lie between A and R as the angle of elevation between M and B must be larger than 60°. Since $\angle AMQ = 90°$, $\angle ARQ$ must be obtuse.)



Method 1
In ΔAQR, $AQ^2 + AR^2 - 2 \cdot AQ \cdot AR\cos\angle QAB = QR^2$
$AR^2 - 158.8501AR + 2612.1658 = 0$
$AR = 140.22$ (rej.) or 18.6 (m, 3 s.f.)

Method 2
In ΔAQR, $\dfrac{\sin\angle ARQ}{AQ} = \dfrac{\sin\angle QAR}{QR}$
$\sin\angle ARQ = \dfrac{h\sin 32.29019°}{\frac{h}{\tan 50°}}$
$\angle ARQ = 39.54201°$ (rej.) or $140.45799°$
$\Rightarrow \angle AQR = 180° - 32.29019° - 140.45799°$
$= 7.25182°$
$\therefore AR = \dfrac{QR\sin\angle AQR}{\sin\angle QAR} = 18.6$ (m, 3 s.f.)

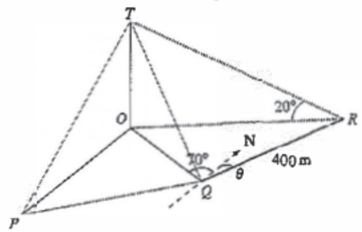
14B.13 HKCEE MA 1994 – I – 14

(a) In ΔOPQ, $\dfrac{OQ}{\sin 50°} = \dfrac{500}{\sin 70°} = \dfrac{OP}{\sin(180° - 50° - 70°)}$
$\Rightarrow OQ = 407.60373 = 408$ (m, 3 s.f.)
$OP = 460.80249 = 461$ (m, 3 s.f.)

(b) In ΔOPT, $h = OP\tan 30° = 266.04444 = 266$ (m, 3 s.f.)

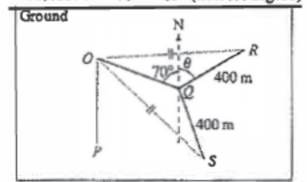
(c) In ΔOQT, Required $\angle = \angle OQT$
$= \tan^{-1}\dfrac{h}{OQ} = 33°$ (nrst deg)

(d) (i)



In ΔORT, $OR = \dfrac{h}{\tan 20°} = 730.9511$ (m)
In ΔOQR,
$\angle OQR = \cos^{-1}\dfrac{OQ^2 + QR^2 - OR^2}{2 \cdot OQ \cdot QR}$
$= \cos^{-1}\dfrac{407.60373^2 + 400^2 - 730.9511^2}{2 \cdot 407.60373 \cdot 400}$
$= 129.6674° = 130°$ (nearest degree)
$\therefore \theta = 129.6674° - 70° = 60°$ (nearest degree)

(ii)



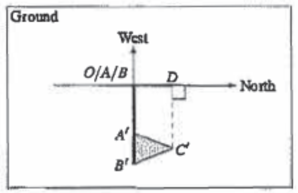
$\because \Delta OQR \cong \Delta OQS$ (SSS)
$\therefore \angle OQS = \angle OQR = 130°$
$\Rightarrow$ Bearing $= 360° - 70° - 130° = 160°$ (S20°E)

14B.14 HKCEE MA 1995 – I – 15

(a) In $\Delta OAA'$, $OA' = \dfrac{2}{\tan 30°} = 2\sqrt{3}$ (m) (3.46 m, 3 s.f.)
In $\Delta OBB'$, $OB' = \dfrac{2 + 0.6}{\tan 30°} = 2.6\sqrt{3}$
$\Rightarrow A'B' = 2.6\sqrt{3} - 2\sqrt{3} = 0.6\sqrt{3}$ (m) (1.04 m, 3 s.f.)

(b) In ΔABC, $\angle BAC = \cos^{-1}\dfrac{0.6^2 + 0.7^2 - 0.8^2}{2 \cdot 0.6 \cdot 0.7}$
$= 75.52249° = 75.5°$ (3 s.f.)
$\therefore OD = AC\sin\angle BAC = 0.67777 = 0.678$ (m, 3 s.f.)

(c)



As the height of $\Delta A'B'C'$ with $A'B'$ as base is also OD,
Area of shadow $= \dfrac{A'B' \cdot OD}{2} = 0.352$ m² (3 s.f.)

(d) (i) Let the angle of elevation be θ.
$\because A'B' = \dfrac{0.6}{\tan\theta}$
$\therefore \theta < 30° \Rightarrow \tan\theta < \tan 30° \Rightarrow \dfrac{0.6}{\tan\theta} > \dfrac{0.6}{\tan 30°}$
Thus, $A'B'$ will become longer.

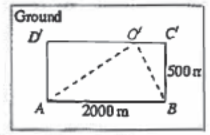
(ii) Since the area of the shadow is $\dfrac{A'B' \cdot OD}{2}$, when the angle of elevation is smaller, $A'B'$ is longer while OD is <u>unchanged</u>, the area of the shadow is <u>larger</u>.

14B.15 HKCEE MA 1996 – I – 15

(a) In ΔOBC, $BC = 1000\cos 60° = 500$ (m)
In $\Delta BCC'$, $CC' = 500\sin 30° = 250$ (m)

(b) $OO' = CC' = 250$ m
$\therefore$ In $\Delta OO'B$, Required $\angle = \angle OBO'$
$= \sin^{-1}\dfrac{250}{1000}$
$= 14.4775° = 14.5°$ (3 s.f.)

(c) *Method 1 to find O'A*
Denote the foot of perpendicular from D to the horizontal ground by D'.



In $\Delta OO'B$,
$O'B = \sqrt{1000^2 - 250^2}$
$= \sqrt{937500}$ (m)
In $\Delta BCC'$,
$BC' = 500\cos 30°$
$= 250\sqrt{3}$ (m)

$\therefore$ In $\Delta O'BC'$, $O'C' = \sqrt{O'B^2 - BC'^2} = \sqrt{750000}$ (m)
In $\Delta AO'D'$, $AD' = BC' = 250\sqrt{3}$ (m)
$D'O' = AB - O'C' = (2000 - \sqrt{750000})$ m
$AO' = \sqrt{AD'^2 + D'O'^2}$
$= \sqrt{4937500 - 4000\sqrt{750000}}$ m

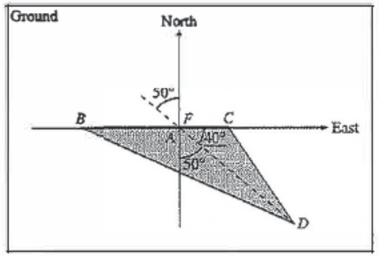
Method 2 to find O'A
In $\Delta OO'B$, $O'B = \sqrt{1000^2 - 250^2} = \sqrt{937500}$ (m)
In ΔOBC, $OC = 1000\sin 60° = 500\sqrt{3}$ (m)
$\Rightarrow O'C' = OC = 500\sqrt{3}$ m
$\therefore \cos\angle O'BA = \sin\angle O'BC' = \dfrac{O'C'}{O'B} = \dfrac{500\sqrt{3}}{\sqrt{937500}}\left(= \sqrt{\dfrac{4}{5}}\right)$
In $\Delta O'AB$,
$O'A = \sqrt{2000^2 + 937500 - 2 \cdot 2000 \cdot \sqrt{937500}\cos\angle O'BA}$
$= \sqrt{4937500 - 4000\sqrt{937500}\sqrt{\dfrac{4}{5}}}$
$= \sqrt{4937500 - 4000\sqrt{750000}}$

Hence... In $\Delta AO'T$,
$AT = \sqrt{AO'^2 + O'T^2}$
$= \sqrt{4937500 - 4000\sqrt{750000} + (250 + 50)^2}$
$= 1250.3593 = 1250$ (m, 3 s.f.)

(d) Time for Rt I $= \dfrac{1000}{0.3} + 60 = 3393$ (s)
Time for Rt II $= \dfrac{2000}{0.8} + \dfrac{1250.3593}{3.2} = 2891$ (s) < 3393 (s)
$\therefore$ Route II takes a shorter time.

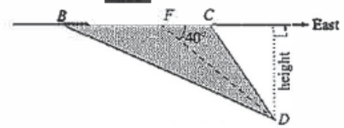
14B.16 HKCEE MA 1998 – I – 17

(The sun shining from N50°W is indicated in the diagram by $\angle CFD = 40°$.)



(a) In $\triangle ACF$, $AF = 4\sin 72° = 3.80423 = 3.80$ (m, 3 s.f.)
In $\triangle ADF$, $FD = \dfrac{AF}{\tan 35°} = 5.43300 = 5.43$ (m, 3 s.f.)

(b)



Height of $\triangle DBC$ with BC as base $= FD\sin 40°$
$= 3.49226$ m

$\therefore$ Area of shadow $= \dfrac{BC \cdot (FD\sin 40°)}{2} = 10.5$ (m², 3 s.f.)

(c) Area of shadow $= \dfrac{BC \cdot FD\sin(90° - x°)}{2} = \dfrac{BC \cdot FD}{2}\cos x°$
Since FD only depends on the $\angle$ of elevation (recall that $FD = \dfrac{AF}{\tan(\angle \text{ of elvn})}$),
$50 < x < 90 \Rightarrow \cos 50° > \cos x° > \cos 90°$
Hence the area becomes smaller.

14B.17 HKCEE MA 1999 – I – 18

$BD = DE = EF = FC = 6$ cm

(a) *Method 1 to find AD*
In $\triangle ABD$, $AD = \sqrt{24^2 + 6^2 - 2 \cdot 24 \cdot 6\cos 60°}$
$= \sqrt{468} = 21.6$ (cm, 3 s.f.)

Method 2 to find AD
In $\triangle ABE$ (before folding), $AE = \sqrt{24^2 - 12^2} = \sqrt{432}$ (cm)
In $\triangle ADE$, $AD = \sqrt{432 + 6^2} = \sqrt{468} = 21.6$ (cm, 3 s.f.)

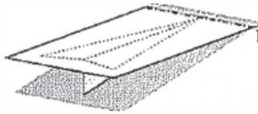
Method 1
$\angle BDA = \cos^{-1}\dfrac{BD^2 + AD^2 - AB^2}{2 \cdot BD \cdot AD} = 106.10211°$
$\therefore$ In $\triangle BCD$ (after folding),
$\angle BDC = 360° - 2(106.10211°) = 147.79577°$
$BC = \sqrt{6^2 + 6^2 - 2 \cdot 6 \cdot 6\cos 147.79577°}$
$= 11.52923 = 11.5$ (cm, 3 s.f.)

Method 2
Area of $\triangle ABD = \dfrac{1}{2}(6)(24)\sin 60° = 36\sqrt{3}$
Height of $\triangle ABD$ with base $AD = \dfrac{36\sqrt{3} \times 2}{AD} = \dfrac{72}{\sqrt{156}}$ (cm)
$\therefore$ $BC = 2 \times \dfrac{72}{\sqrt{156}} = 11.52923 = 11.5$ (cm, 3 s.f.)

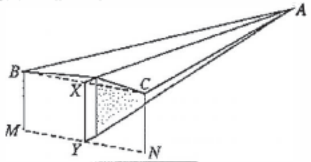
Method 3
In $\triangle ABD$, $\dfrac{\sin\angle BAD}{6} = \dfrac{\sin 60°}{AD} \Rightarrow \angle BAD = 13.89789°$
$\Rightarrow \angle BAC$ (after folding) $= 2\angle BAD = 27.79577°$
$\therefore$ In $\triangle ABC$ (after folding),
$BC = \sqrt{24^2 + 24^2 - 2 \cdot 24 \cdot 24\cos 27.79577°}$
$= 11.52923 = 11.5$ (cm, 3 s.f.)

(b) Required $\angle = \angle DAE = \tan^{-1}\dfrac{DE}{AE}$
$= \tan^{-1}\dfrac{6}{\sqrt{24^2 - 12^2}}$
$= 16.10211° = 16.1°$ (3 s.f.)

Note: Normally we need to look for the line of intersection of the 2 planes to locate the dihedral angle. In this problem, however, the planes intersect at only a point, and we could only assume that the aeroplane is positioned symmetrically, and that AE is perpendicular to the line of intersection.



(c) $BCNM$ is a rectangle. Suppose AD produced meets BC at X and AE produced meets MN at Y as shown.
Then $BM = XY = CN$.



In $\triangle ABX$, $AX = \sqrt{AB^2 - \left(\dfrac{BC}{2}\right)^2} = 23.2974$ cm
$\therefore$ In $\triangle AXY$,
$CN = XY = AX\sin\angle DAE = 6.46$ cm (3 s.f.)

14B.18 HKCEE MA 2000 – I – 17

(a) (i) $AD = \dfrac{h}{\sin 30°} = 2h$ (m)
$BD = \dfrac{h + OA}{\sin 60°} = \dfrac{10 + h}{\frac{\sqrt{3}}{2}} = \dfrac{2}{\sqrt{3}}(10 + h)$ (m)

(ii) In $\triangle OAB$, $AB = \sqrt{10^2 + 10^2} = \sqrt{200}$ (m)
In $\triangle ABD$,
$AB^2 = AD^2 + BD^2 - 2AD \cdot BD\cos 30°$
$200 = 4h^2 + \dfrac{4}{3}(10 + h)^2 - 2 \cdot 2h \cdot \dfrac{2}{\sqrt{3}}(10 + h) \cdot \dfrac{\sqrt{3}}{2}$
$200 = 4h^2 + \dfrac{4}{3}(100 + 20h + h^2) - 40h - 4h^2$
$0 = h^2 - 10h - 50$
$h = \dfrac{10 \pm \sqrt{100 + 200}}{2}$
$= 5 + 5\sqrt{3}$ or $5 - 5\sqrt{3}$ (rejected)
$= 13.66025 = 13.7$ (3 s.f.)

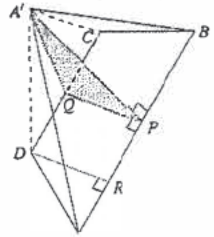
(b) (Similar approach as (a))
$AE = \dfrac{h}{\sin 25°} = 32.32291$ (m)
$AC = \sqrt{10^2 + 10^2 - 2 \cdot 10 \cdot 10\cos 20°} = 3.47296$ (m)
$\therefore$ $\dfrac{\sin\angle ACE}{AE} = \dfrac{\sin 5°}{AC}$
$\angle ACE = 54.2°$ or $126°$ (3 s.f.)

14B.19 HKCEE MA 2001 – I – 16

(a) Area of $BCDE = \triangle AFG - 2\triangle BCF - \triangle ABE$
$5\sqrt{3} = \dfrac{(6 + x)^2\sin 60°}{2} - x^2\sin 60° - \dfrac{(6)^2\sin 60°}{2}$
$= \dfrac{\sqrt{3}}{4}[(6 + x)^2 - 2x^2 - 6^2]$
$20 = 12x - x^2$
$0 = x^2 - 12x + 20$
$x = 10$ (rejected) or 2

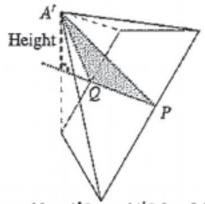
(b) (i) In $\triangle A'DE$, $A'D = \sqrt{6^2 + 2^2 - 2 \cdot 6 \cdot 2\cos 40°}$
$= 4.64919 = 4.65$ (3 s.f.)

(ii) Let P and Q be the mid-points of BE and CD respectively as shown. By symmetry, $A'P \perp BE$ and $QP \perp BE$. Hence, the required angle is $\angle A'PQ$.



Let R be the foot of perpendicular from DE to BE.
Then $PQ = RD = x\sin 60° = \sqrt{3}$ (cm).
In $\triangle A'EP$, $A'P = A'E\sin 60° = 3\sqrt{3}$ (cm)
In $\triangle A'DQ$, $DQ = CD \div 2 = 2$ cm
$\Rightarrow A'Q = \sqrt{A'D^2 - DQ^2} = 4.19701$ cm
$\therefore$ In $\triangle A'PQ$, $\angle A'PQ = \cos^{-1}\dfrac{PQ^2 + A'P^2 - A'Q^2}{2 \cdot PQ \cdot A'P}$
$= \cos^{-1}\dfrac{3 + 27 - 4.19701^2}{2 \cdot \sqrt{3} \cdot 3\sqrt{3}}$
$= 46.52332° = 46.5°$ (3 s.f.)

(iii)

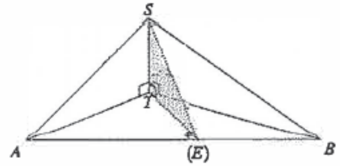


Height of pyramid $= A'P\cos\angle A'PQ = 3.77061$ cm
$\therefore$ Area $= \dfrac{1}{3} \times 5\sqrt{3} \times 3.77061 = 10.9$ (cm³)

14B.20 HKCEE MA 2002 – I – 14

(a) In $\triangle AST$, $AT = \dfrac{h}{\tan 20°}$; In $\triangle BST$, $BT = \dfrac{h}{\tan 15°}$.
In $\triangle ABT$, $\cos 30° = \dfrac{AT^2 + AB^2 - BT^2}{2AT \cdot BT}$
$\dfrac{900\sqrt{3}h}{\tan 20°} = \left(\dfrac{h}{\tan 20°}\right)^2 + 900^2 - \left(\dfrac{h}{\tan 15°}\right)^2$
$0 = 6.37957h^2 + 4282.8934h - 810000$
$h = 153.86177$ or -825 (rej)
$= 154$ (3 s.f.)

(b) (i) The shortest distance occurs when $TE \perp AB$ and thus $SE \perp AB$.



Method 1
In $\triangle AET$, $ET = AT\sin 30° = 211.3659$ (m)
In $\triangle EST$, $SE = \sqrt{ST^2 + ET^2} = 261.436$
$= 261$ (m, 3 s.f.)

Method 2
In $\triangle AST$, $SA = \dfrac{h}{\sin 20°} = 449.86172$ m
In $\triangle BST$, $SB = \dfrac{h}{\sin 15°} = 594.47623$ m
In $\triangle ABS$, $\angle SAB = \cos^{-1}\dfrac{SA^2 + AB^2 - SB^2}{2SA \cdot AB}$
$= 35.5313°$
$\therefore$ In $\triangle SAE$, $SE = SA\sin\angle SAB = 261$ m (3.s.f)

Method 3
In $\triangle AST$, $SA = \dfrac{h}{\sin 20°} = 449.86172$m
In $\triangle BST$, $SB = \dfrac{h}{\sin 15°} = 594.47623$ m
In $\triangle ABS$, let $s = \dfrac{SA + SB + 900}{2} = 972.1690$ m.
$\Rightarrow$ Area $= \sqrt{s(s - SA)(s - SB)(s - 900)}$
$= 117646.36$ (m²)
$\therefore$ $SE = \dfrac{\text{Area} \times 2}{AB} = 261$ m (3 s.f.)

(ii) At E as in (b)(i), $\angle SET = \tan^{-1}\dfrac{ST}{ET} = 36.1°$.
$\therefore$ From A to B, θ increases from 20° at A to 36.1° at E as in (b)(i), and then decreases to 15° at B (since SE is the 'line of greatest slope').

14B.21 HKCEE MA 2003 – I – 14

(a) In $\triangle OAC$, $\angle OAC = \cos^{-1}\dfrac{3^2 + 6^2 - 4^2}{2 \cdot 3 \cdot 6}$
$= 36.33606° = 36.3°$ (3 s.f.)

(b) (i) In $\triangle OBC$, $BC = 4\tan 40° = 3.35640$ (m)
In $\triangle BCD$, $CD = \dfrac{BC}{\tan 30°} = 5.81345 = 5.81$ (m, 3 s.f.)

(ii) In $\triangle ACD$, $\angle CAD = \cos^{-1}\dfrac{6^2 + 8^2 - CD^2}{2 \cdot 6 \cdot 8}$
$= 46.39976° = 46.4°$ (3 s.f.)

(iii) In $\triangle ACE$, $\dfrac{CE}{\sin\angle OAC} = \dfrac{6}{\sin\theta} \Rightarrow CE = \dfrac{3.55512}{\sin\theta}$
In $\triangle ADE$, $\dfrac{DE}{\sin(\angle CAD - \angle OAC)} = \dfrac{8}{\sin(180° - \theta)}$
$\Rightarrow DE = \dfrac{1.39794}{\sin\theta}$
$\therefore$ $CE + ED = CD$
$\Rightarrow \dfrac{3.55512}{\sin\theta} + \dfrac{1.39794}{\sin\theta} = 5.81345$
$\dfrac{4.95306}{\sin\theta} = 5.81345$
$\theta = 58.4°$ or $121.6°$ (rejected)

14B.22 HKCEE MA 2004 – I – 17

(a) (i) In $\triangle EFF'$, $FF' = 20\sin 30° = 10$ (m)

$$EF' = \frac{10}{\tan 30°} = 10\sqrt{3} \text{ (m)}$$

In $\triangle AFF'$, $AF' \frac{10}{\tan 60°} = \frac{10}{\sqrt{3}}$ (m)

In $\triangle AEF'$, $AE = \sqrt{AF'^2 + EF'^2}$

$$= \sqrt{\frac{1000}{3}} = 18.3 \text{ (m, 3 s.f.)}$$

(ii) In $\triangle AFF'$, $AF = \frac{FF'}{\sin 60°} = \frac{20}{\sqrt{3}}$ m

In $\triangle AEF$, $\angle AEF = \cos^{-1}\frac{AE^2 + EF^2 - AF^2}{2AE \cdot EF}$

$$= \cos^{-1}\frac{\frac{1000}{3} + 400 - \frac{400}{3}}{2 \cdot \sqrt{\frac{1000}{3}} \cdot 20}$$

$$= 34.75634° = 34.8° \text{ (3 s.f.)}$$

(b) In $\triangle BEF$, $\angle BEF = 180° - 34.75634° = 145.24366°$

$\angle FBE = 34.75634° - 20° = 14.75634°$

$$\frac{20}{\sin 14.75634°} = \frac{BE}{\sin 20°} = \frac{BF}{\sin 145.24366°}$$

$\Rightarrow$ $BE = 26.85576$ m, $BF = 44.76385$ m

$\therefore$ Time red car takes $= BE \div 2 = 13.4$ s

Time yellow car takes $= BF \div 3 = 14.9$ s > 13.4 s

$\therefore$ NO.

14B.23 HKCEE MA 2005 – I – 14

(a) In $\triangle BCE$, $BE = 120\sin 30° = 60$ (cm)

$CE = 120\cos 30° = 60\sqrt{3} = 104$ (cm, 3 s.f.)

(b) In $\triangle ABC$, $\angle C = 180° - 80° - 60° = 40°$

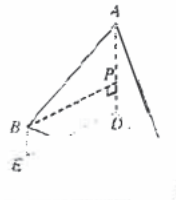
$$\therefore \frac{120}{\sin 60°} = \frac{AB}{\sin 40°} = \frac{AC}{\sin 80°}$$

$\Rightarrow$ $AB = 89.0673 = 89.1$ (cm, 3 s.f.)

$AC = 136.4590 = 136$ (cm, 3 s.f.)

(c) In $\triangle ACD$, $CD = \sqrt{AC^2 - AD^2} = 92.8496$ cm

In $ABED$, let P be on AD such that $BP \perp AD$.



$DE = PB = \sqrt{AB^2 - (AD - BE)^2} = 79.5800$ cm

$\therefore$ In $\triangle CDE$, $\angle CDE = \cos^{-1}\frac{CD^2 + DE^2 - CE^2}{2CD \cdot DE}$

$= 73.674°$

Shortest distance from C to DE

$= CQ$ in the figure

$= CD\sin\angle CDE = 89.1$ cm (3 s.f.)

14B.24 HKCEE MA 2006 – I – 17

(a) In $\triangle ABC$, $\cos\angle BAC = \frac{40^2 + 90^2 - 60^2}{2 \cdot 40 \cdot 90} = \frac{61}{72}$

In $\triangle ABD$, $AD = 40\cos\angle BAD = \frac{305}{9}$ (cm)

(b) (i) (1) $DC = 90 - \frac{305}{9} = \frac{505}{9}$ (cm)

In $\triangle ACD$,

$$\left(\frac{505}{9}\right)^2 = \left(\frac{305}{9}\right)^2 + AC^2 - 2\left(\frac{305}{9}\right)(AC)\cos 62°$$

$0 = AC^2 - 31.81974AC - 2000$

$AC = 63.37695$ or -31.6 (rejected)

63.4 (cm, 3 s.f.)

(2) Let $s = \frac{40 + 60 + 63.37695}{2} = 81.6885$ (cm)

Area of $\triangle ABC = \sqrt{s(s-40)(s-60)(s-63.37695)}$

$= 1162.961 = 1160$ (cm^2, 3 s.f.)

(3) For tetrahedron $ABCD$, note that BD is its height when $\triangle ACD$ is its base.

Area of $\triangle ACD = \frac{AD \cdot AC\sin 62°}{2} = 948.186$ cm^2

$\therefore$ Required height $= \frac{3 \times \text{Volume of } ABCD}{\text{Area of } \triangle ABC}$

$= \frac{\text{Area of } \triangle ACD \times BD}{\text{Area of } \triangle ABC}$

$= \frac{948.186 \times \sqrt{40^2 - \left(\frac{305}{9}\right)^2}}{1162.961}$

$= 17.3$ (cm, 3 s.f.)

(ii) Volume of $ABCD = \frac{1}{3}$(Area of $\triangle ACD$)(BD)

$= \frac{1}{3}AD \cdot DC \cdot BD\sin\angle ADC$

$\therefore$ Volume of $ABCD \propto \sin\angle ADC$

Thus, when $\angle ADC$ increases from 30° to 150°, the volume increases from $\frac{1}{3}AD \cdot DC \cdot BD \cdot \frac{1}{2} = 6734$ cm^3 to $\frac{1}{3}AD \cdot DC \cdot BD \cdot 1 = 13469$ cm^3 when $\angle ADC = 90°$, and then decreases back to 6734 cm^3.

14B.25 HKCEE MA 2007 – I – 16

(a) Let $s = \frac{5 + 6 + 9}{2} = 10$ (cm)

Area of $\triangle ABC = \sqrt{s(s-5)(s-6)(s-9)}$

$= \sqrt{200} = 14.1$ (cm^2, 3 s.f.)

$\therefore$ Volume of souvenir

$=$ Volume of prism $+$ Volume of pyramid

$= \sqrt{200} \times 20 + \frac{1}{3} \times \sqrt{200} \times (23 - 20)$

$= 21\sqrt{200} = 297$ (cm^3, 3 s.f.)

(b) Let P be the point on CD such that plane PEF is parallel to plane ABC as shown. $DP = 3$ cm, $EF = AB = 9$ cm, $FP = BC = 5$ cm, $EP = AC = 6$ cm

In $\triangle DFP$, $DF = \sqrt{3^2 + 5^2} = \sqrt{34}$ (cm)

In $\triangle DEP$, $DE = \sqrt{3^2 + 6^2} = \sqrt{45}$ (cm)

$\therefore$ In $\triangle DEF$, $\angle DFE = \cos^{-1}\frac{DF^2 + EF^2 - DE^2}{2DF \cdot EF}$

$= 48.16875° = 48.2°$ (3 s.f.)

$\therefore$ Required distance $= DF\sin\angle DFE = 4.3447$

$= 4.34$ cm (3 s.f.)

(c) Area of metal plane $= 4 \times 5 = 20$ (cm^2)

Area of $\triangle DEF = \frac{4.3447 \times 9}{2} = 19.6 < 20$ (cm^2)

$\therefore$ NO.

14B.26 HKCEE MA 2008 – I – 15

(a) In $\triangle BDH$, $\angle BHD = 50° - 35° = 15°$

$\angle BDH = 180° - 50° = 130°$

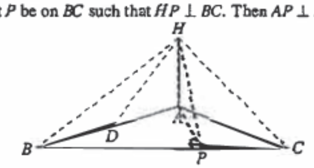
$$\frac{BH}{\sin 130°} = \frac{50}{\sin 15°} \Rightarrow BH = 147.98842 = 148 \text{ (m, 3 s.f.)}$$

(b) (i) In $\triangle BCH$, $\angle CBH = \cos^{-1}\frac{BC^2 + BH^2 - CH^2}{2BC \cdot BH}$

$= 37.81747° = 37.8°$ (3 s.f.)

(ii) In $\triangle ABH$, $AH = BH\sin 35° = 84.88267$ m

Let P be on BC such that $HP \perp BC$. Then $AP \perp BC$.



In $\triangle BHP$, $HP = BH\sin\angle CBH = 90.73880$ m

In $\triangle AHP$, Required $\angle = \angle HPA = \sin^{-1}\frac{AH}{HP}$

$= 69.3°$ (3 s.f.)

(iii) As the largest possible $\angle$ of elevation is $69.3° < 75°$, it is impossible.

14B.27 HKCEE MA 2009 – I – 17

(a) (i) In $\triangle BCD$, $CD = \sqrt{6^2 + 25^2 - 2 \cdot 6 \cdot 25\cos 57°}$

$= 22.30714 = 22.3$ (cm, 3 s.f.)

(ii) In $\triangle ABC$, $\frac{\sin\angle BAC}{25} = \frac{\sin 57°}{28}$

$\angle BAC = 48.48766°$ or $131.5°$ (rej.)

$= 48.5°$ (3 s.f.)

(iii) In $\triangle ABC$, $\angle ACB = 180° - 48.48766° - 57°$

$= 74.51234°$

Area of $\triangle ABC = \frac{1}{2}AC \cdot BC\sin 74.51234°$

$= 337.29079 = 337$ (cm^2, 3 s.f.)

(iv) Since $\triangle CDE \perp \triangle ABE$, we have $CE \perp \triangle ABE$.

In $\triangle BCE$, $CE = \sqrt{BC^2 - BE^2} = 7$ cm

In $\triangle ACE$, $AE = \sqrt{AC^2 - CE^2} = \sqrt{735}$ cm

In $\triangle ABC$, $\frac{AB}{\sin 74.51234°} = \frac{28}{57°}$

$AB = 32.17385$ (cm)

Let $s = \frac{AB + AE + BE}{2} = 41.64237$ cm

Area of $\triangle ABE = \sqrt{s(s - AB)(s - AE)(s - BE)}$

$= 317.9377$ (cm^2)

$\therefore$ Required dist $= \frac{3 \times \text{Volume of } ABCE}{\text{Area of } \triangle ABC}$

$= \frac{\text{Area of } \triangle ABE \times CE}{\text{Area of } \triangle ABC}$

$= \frac{337.29079 \times 7}{317.9377}$

$= 6.59835 = 6.60$ (cm, 3 s.f.)

(b) *Method 1* – Finding the angles explicitly

In $\triangle CDE$, $\angle CDE = \sin^{-1}\frac{CE}{CD} = 18.29°$

Denoting the distance from E to the ground (i.e. that found in (a)(iv)) by h cm and the angle between CE and the ground be θ,

$\theta = \sin^{-1}\frac{h}{DE} = 18.15° \neq 18.29°$

$\therefore$ NO.

Method 2 – Considering the projection of E

(If the student is correct, the projection of E on the ground would lie on CD.)

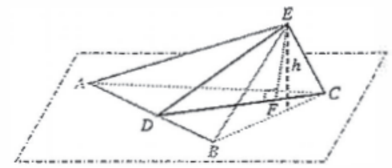
Let F be the projection of E onto CD.

$$EF = \frac{2 \times \text{Area of } \triangle CDE}{CD}$$

$$= \frac{CE \times DE}{CD}$$

$$= \frac{7 \times \sqrt{22.30714^2 - 7^2}}{22.30714} = 6.65 \neq 6.60 \text{ (cm)}$$

Hence, the projection of E onto the ground is *not* on CD, and thus the angle between DE and the ground is *not* the angle between DE and DC, i.e. $\angle CDE$. The student is disagreed.



(Remark: This diagram is for illustration only. In the real situation, the "h" is *behind* $\triangle CDE$, and would be too hard to visualise in the given diagram. But the key point is the same, that the dashed "h" is different from EF – in fact, h is shorter than EF since it is the *shortest* distance from E to the ground.)

14B.28 HKCEE MA 2010 – I – 15

(a) In $\triangle ABC$, $\angle CAB = 146° = 2 = 73°$,

$\angle ACB = 180° - 73° - 59° = 48°$

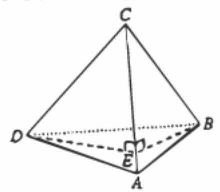
$$\frac{AB}{\sin 48°} = \frac{24}{\sin 73°} \Rightarrow AB = 18.65041 = 18.7 \text{ (cm, 3 s.f.)}$$

(b) (i) In $\triangle ABD$, $BD = \sqrt{AB^2 + AB^2 - 2 \cdot AB \cdot AB\cos 92°}$

$= 26.83196 = 26.8$ (cm, 3 s.f.)

(ii) Let the diagonals of the kite intersect at E.

Then $DE \perp AC$ and $BE \perp AC$.



In $\triangle BCE$, $BE = BC\sin\angle BCE = 17.83548$ (cm)

$DE = BE = 17.83548$ cm

In $\triangle BDE$, Required $\angle = \angle BED$

$= \cos^{-1}\frac{BE^2 + DE^2 - BD^2}{2BE \cdot DE}$

$= 97.6°$ (3 s.f.)

(iii) In $\triangle BCD$, $\angle BCD = \cos^{-1}\frac{BC^2 + CD^2 - BD^2}{2BC \cdot CD}$

$= 68.0°$

As P moves from A to E, $\angle BPD$ increases from 92° to 97.6°. As P moves from E to C, $\angle BPD$ decreases from 97.6° to 68.0°.

14B.29 HKCEE MA 2011 – I – 17

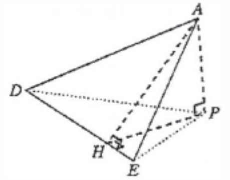
(a) (i) In $\triangle ABC$, $BC=\sqrt{20^2+30^2-2\cdot 20\cdot 30\cos 56^\circ}$
$= 25.07924 = 25.1$ (cm, 3 s.f.)

(ii) $\angle ACB = \cos^{-1}\dfrac{25.07924^2+30^2-20^2}{2\cdot 25.07924\cdot 30}$
$= 41.38645^\circ = 41.4^\circ$ (3 s.f.)

(iii) Required distance $= AC\sin\angle ACB - 4$
$= 15.83403 = 15.8$ (cm, 3.s.f.)

(iv) $\dfrac{DE}{BC}=\dfrac{\perp \text{ dist from } A \text{ to } DE}{\perp \text{ dist from } A \text{ to } BC}$
$DE=\dfrac{15.83403}{15.83403+4}\cdot 25.07924$
$= 20.02142 = 20.0$ (cm, 3 s.f.)

(b) (i) Let H be the point on DE such that $AB \perp DE$ and $PH \perp DE$.



$AH = 15.83403$ cm
$PH=\dfrac{2\times \text{Area of } \triangle PDE}{DE}=11.98716$ cm
$\therefore$ Required $\angle = \angle AHP$
$=\cos^{-1}\dfrac{PH}{AH}=40.8^\circ$ (3 s.f.)

(ii) Required distance $= AP$
$=\sqrt{AH^2-PH^2}=10.3$ cm (3 s.f.)

14B.30 HKCEE AM 1981 – II – 10

(a) In $\triangle BEF$, $\angle EBF=60^\circ$
$FE^2=k^2+(rk)^2-2\cdot k\cdot rk\cos 60^\circ$
$=k^2+r^2k^2-rk^2=(1-r+r^2)k^2$
$FG^2=\left(\frac{1}{2}FH\right)^2=\frac{1}{4}(HA^2+FA^2)$
$=\frac{1}{4}[2\times(k-rk)^2]=\dfrac{(1-r)^2k^2}{2}$

(b) In $\triangle EFG$, $EG=\sqrt{FE^2-FG^2}$
$=\sqrt{(1-r+r^2)k^2-\dfrac{1-2r+r^2}{2}k^2}$
$=\sqrt{\dfrac{1+r^2}{2}}k$
In $\triangle ACD$, $AC^2=AD^2+DC^2=2k^2$
$AN^2=\frac{1}{2^2}(2k^2)=\frac{1}{2}k^2$
In $\triangle AEN$, $EN=\sqrt{AE^2-AN^2}=\sqrt{k^2-\frac{1}{2}k^2}=\frac{1}{\sqrt{2}}k$
$\therefore \sin\theta=\dfrac{EN}{EG}=\dfrac{\frac{1}{\sqrt{2}}k}{\frac{\sqrt{1+r^2}}{\sqrt{2}}k}=\dfrac{1}{\sqrt{1+r^2}}$

(c) The inclination is θ.
$0<r<1 \Rightarrow 1<1+r^2<2$
$\Rightarrow 1>\sin\theta>\frac{1}{\sqrt{2}} \Rightarrow 90^\circ>\theta>45^\circ$
Hence, when r varies from 0 to 1, the inclination decreases from 90° to 45°.

14B.31 HKCEE AM 1983 – II – 8

$\angle CBF=\angle CFB=\theta$

(a) In $\triangle BCF$, $BF=2\times BC\cos\theta=2a\cos\theta$
In $\triangle FMN$, $MF=x\cos\theta$
$\therefore$ In $\triangle ABM$, $AM=\sqrt{AB^2+BM^2}$
$=\sqrt{a^2+(2a\cos\theta-x\cos\theta)^2}$
$=\sqrt{a^2+(2a-x)^2\cos^2\theta}$

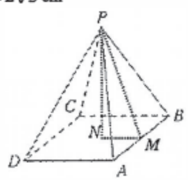
(b) In $\triangle ABF$, $AF=\sqrt{AB^2+BF^2}$
$=\sqrt{a^2+(2a\cos\theta)^2}=\sqrt{(1+4\cos^2\theta)a^2}$
$\therefore$ In $\triangle ANF$, $AN=\sqrt{AF^2-NF^2}$
$=\sqrt{(1+4\cos^2\theta)a^2-x^2}$

(c) In $\triangle FMN$, $NM=x\sin\theta$
In $\triangle AMN$, $AN^2=AM^2+NM^2$
$(1+4\cos^2\theta)a^2-x^2=a^2+(2a-x)^2\cos^2\theta+x^2\sin^2\theta$
$\cancel{(1+4\cos^2\theta)a^2}-x^2=\cancel{a^2+4a^2\cos^2\theta}-4ax\cos^2\theta$
$+x^2\cos^2\theta+x^2\sin^2\theta$
$4ax\cos^2\theta=x^2(\cos^2\theta+\sin^2\theta)+x^2$
$4ax\cos^2\theta=2x^2 \Rightarrow x=2a\cos^2\theta$

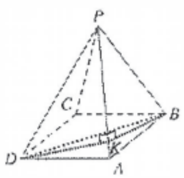
(d) $\frac{a}{2}=2a\cos^2\theta \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=60^\circ$
$NM=x\sin\theta=\frac{\sqrt{3}}{4}a$
$AM=\sqrt{a^2+(2a-x)^2\cos^2\theta}=\sqrt{a^2+\frac{9a^2}{4}\cdot\frac{1}{4}}=\frac{5}{4}a$
$\therefore$ Inclination $=\angle NAM=\tan^{-1}\dfrac{NM}{AM}=19^\circ$ (nrst deg)

14B.32 HKCEE AM 1991 – II – 6

(a) Let M and N be the mid-points of AB and CD respectively. Then $PM\perp AB$ and $PN\perp CD$.
In $\triangle APM$, $PM=AM\tan 60^\circ=2\sqrt{3}$ cm
In $\triangle MNP$,
Required $\angle=\angle PMN$
$=\cos^{-1}\dfrac{MN}{PM}$
$=\cos^{-1}\dfrac{2}{2\sqrt{3}}$
$=54.7^\circ$
(nearest 0.1°)

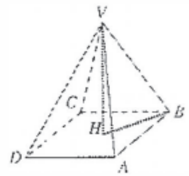


(b) Let K be on PA such that $DK\perp PA$. Then $BK\perp PA$.
In $\triangle ABD$,
$BD=\sqrt{4^2+4^2}$
$=\sqrt{32}$ (cm)
In $\triangle ADK$,
$DK=4\sin 60^\circ$
$=2\sqrt{3}$ (cm)
Similarly, $BK=2\sqrt{3}$ cm
In $\triangle BDK$,
Required $\angle=\angle BKD$
$=\cos^{-1}\dfrac{(2\sqrt{3})^2+(2\sqrt{3})^2-32}{2\cdot 2\sqrt{3}\cdot 2\sqrt{3}}$
$=109.5^\circ$ (nearest 0.1°)

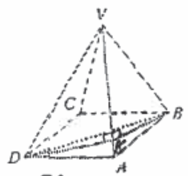


14B.33 HKCEE AM 1992 – II – 7

(a) Let H be the projection of V onto $ABCD$.
$BH=\frac{1}{2}BD$
$=\frac{1}{2}\sqrt{6^2+6^2}$
$=3\sqrt{2}$ (cm)
Required $\angle=\angle VBH$
$=\cos^{-1}\dfrac{3\sqrt{2}}{9}$
$=61.9^\circ$
(nearest 0.1°)



(b) Let K be on VA such that $BK\perp VA$. Then $DK\perp VA$.
$\angle VAB=\cos^{-1}\dfrac{\frac{1}{2}AB}{VA}$
$=70.5288^\circ$
$DK=BK=AB\sin\angle VAB$
$=5.6569$ cm
Required $\angle$
$=\angle BKD$
$=\cos^{-1}\dfrac{5.6569^2+5.6569^2-(2\cdot 3\sqrt{2})^2}{2\cdot 5.6569\cdot 5.6569}$
$=97.2^\circ$ (nearest 0.1°)



14B.34 HKCEE AM 1993 – II – 7

(a) $\angle VBA=\cos^{-1}\dfrac{\frac{1}{2}AB}{VB}=75.52249^\circ=75.5^\circ$ (3 s.f.)
$AD=AB\sin\angle VBA=11.61895=11.6$ (cm, 3 s.f.)

(b) $DC=AD=11.61895$ cm
Required $\angle=\angle ADC$
$=\cos^{-1}\dfrac{AD^2+DC^2-AC^2}{2AD\cdot DC}=62.2^\circ$ (3 s.f.)

14B.35 HKCEE AM 1994 – II – 12

(a) (i) In $\triangle ABC$, $\dfrac{AC}{\sin\beta}=\dfrac{100}{\sin(180^\circ-\alpha-\beta)}$
$AC=\dfrac{100\sin\beta}{\sin(\alpha+\beta)}$ (km)

(ii) In $\triangle ACP$, $PC=AC\tan\theta=\dfrac{100\sin\beta\tan\theta}{\sin(\alpha+\beta)}$ km

(b) (i) $AC=\dfrac{100\sin 30^\circ}{\sin(45^\circ+30^\circ)}$
$=51.76381=51.76$ (km, 2 d.p.)
$AC'=\dfrac{100\sin 43^\circ}{\sin(37^\circ+43^\circ)}$
$=69.25193=69.25$ (km, 2 d.p.)

(ii) $\angle CAC'=45^\circ-37^\circ=8^\circ$
In $\triangle ACC'$, $CC'=\sqrt{AC^2+AC'^2-2AC\cdot AC'\cos 8^\circ}$
$=19.38059=19.38$ (km, 2 d.p.)

(iii) $PC=\dfrac{100\sin 30^\circ\tan 20^\circ}{\sin(45^\circ+30^\circ)}=18.84049$ (km)
$P'C'=\dfrac{100\sin 43^\circ\tan 17^\circ}{\sin(37^\circ+43^\circ)}=21.17244$ (km)
$\therefore$ Increase in height $=P'C'-PC$
$=2.33195=2.33$ (km, 2 d.p.)

(iv) Required $\angle=\tan^{-1}\dfrac{2.33195}{19.38059}=6.86^\circ$ (2 d.p.)

14B.36 HKCEE AM 1995 – II – 7

(a) $\angle PQU=(180^\circ-42^\circ)\div 2=69^\circ$
$PU=10\sin 69^\circ=9.33580=9.34$ (cm, 3 s.f.)
$\angle PQR=180^\circ(5-3)\div 5=108^\circ$
$PR=\sqrt{10^2+10^2-2\cdot 10\cdot 10\cos 108^\circ}$
$=16.18034=16.2$ (cm, 3 s.f.)

(b) Required $\angle=\angle PUR$
$=\cos^{-1}\dfrac{PU^2+RU^2-PR^2}{2PU\cdot RU}=120^\circ$ (3 s.f.)

14B.37 HKCEE AM 1996 – II – 12

(a) $AD=AC\sin 30^\circ=1$, $DC=2\cos 30^\circ=\sqrt{3}$
$AB=\dfrac{AD}{\sin 45^\circ}=\sqrt{2}$, $BD=\dfrac{AD}{\tan 45^\circ}=1$

(b) (i) E is the mid-pt of AB (since $\triangle ABD$ is right-angled isosceles).
$\Rightarrow AE=DE=BE=\frac{\sqrt{2}}{2}$
$\therefore \theta=\angle DCE$
$\Rightarrow \sin\theta=\dfrac{DE}{DC}=\dfrac{\frac{\sqrt{2}}{2}}{\sqrt{3}}=\dfrac{\sqrt{6}}{6}$

(ii) $CE=\sqrt{CD^2-DE^2}=\sqrt{\frac{5}{2}}$
Hence, in $\triangle ACE$,
$\angle EAC=\cos^{-1}\dfrac{AE^2+AC^2-CE^2}{2AE\cdot AC}=45^\circ$

(iii) In $\triangle ABC$, $BC=\sqrt{2^2+2-2\cdot 2\cdot\sqrt{2}\cos 45^\circ}=\sqrt{2}$
In $\triangle BCD$, since $\angle ADC=\angle ADB=90^\circ$,
Required $\angle=\angle CDB=\cos^{-1}\dfrac{3+1-2}{2(\sqrt{3})(1)}$
$=55^\circ$ (nearest degree)

14B.38 HKCEE AM 1997 – II – 12

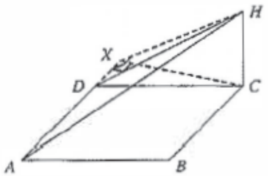
(a) (i) In $\triangle ABC$, $AC=\sqrt{AB^2+BC^2-2AB\cdot BC\cos\angle ABC}$
$=\sqrt{(3a)^2+(2a)^2-2(3a)(2a)\cos 120^\circ}$
$=\sqrt{9a^2+4a^2+6a^2}=\sqrt{19}a$

(ii) Required $\angle=\angle HMC$
$=\tan^{-1}\dfrac{HC}{MC}=25^\circ$ (nearest deg)

(b) (i) In $\triangle ABD$, $BD=\sqrt{(3a)^2+(2a)^2-2(3a)(2a)\cos 60^\circ}$
$=\sqrt{7}a$
Area of $\triangle BCD=\frac{1}{2}(3a)(2a)\sin 60^\circ=\frac{3\sqrt{3}}{2}a^2$
$\therefore CE=\dfrac{2\cdot\text{Area of }\triangle BCD}{BD}=\dfrac{3\sqrt{3}a^2}{\sqrt{7}a}=\dfrac{3\sqrt{21}}{7}a$

(ii) In $\triangle BCE$, $BE^2=BC^2-CE^2$
In $\triangle BCH$, $BH^2=BC^2+HC^2$
In $\triangle CEH$, $HE^2=HC^2+CE^2$
$\therefore HE^2+BE^2=(HC^2+CE^2)+(BC^2-CE^2)$
$=HC^2+BC^2=BH^2$
$\therefore HE\perp BD$
Hence, required $\angle=\angle HEC=\tan^{-1}\dfrac{HC}{CE}$
$=27^\circ$ (nearest deg)

(c) X is on AD extended such that $CX \perp AX$.



In $\triangle ACD$,

$\cos\angle DAC = \dfrac{(2a)^2+(\sqrt{19}a)^2-(3a)^2}{2(2a)(\sqrt{19}a)} = \dfrac{7}{2\sqrt{19}}$

$\therefore\ AX = AC\cos\angle XAC = \sqrt{19}a \cdot \dfrac{7}{2\sqrt{19}} = \dfrac{7}{2}a$

14B.39 HKCEE AM 1998 – II – 13

(a) (i) $CM = \frac{1}{2}AC = \frac{\sqrt{2}}{2}a$

(ii) Required $\angle = \angle CMH = \tan^{-1}\dfrac{HC}{CM} = 55°$ (nrst deg)

(b) (i) $FH = \sqrt{2}a,\ FV = 2a$

In $\triangle FVH$, $HV = \sqrt{(\sqrt{2}a)^2+(2a)^2} = \sqrt{6}a$

$\therefore\ \sin\angle FVH = \dfrac{FH}{HV} = \dfrac{\sqrt{2}a}{\sqrt{6}a} = \dfrac{\sqrt{3}}{3}$

Let the projection of F on $BVDH$ be P.

By symmetry, F lies on HV as shown.

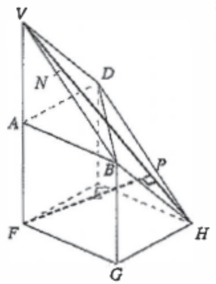
In $\triangle FVP$,

Required distance

$= FP$

$= FV\sin\angle FVP$

$= \dfrac{2\sqrt{3}}{3}a$



(ii) (1) Since $VB = BD = DV = \sqrt{2}a$, $\angle DVB = 60°$.

$\Rightarrow\ DN = \sqrt{2}a\sin 60° = \dfrac{\sqrt{6}}{2}a$

(2) *Method 1*

$AN = AB\sin 45° = \dfrac{\sqrt{2}}{2}a$

$\therefore$ Required $\angle = \angle AND$

$= \cos^{-1}\dfrac{AN^2+DN^2-AD^2}{2AN\cdot DN}$

$= \cos^{-1}\dfrac{\frac{1}{2}a^2+\frac{3}{2}a^2-a^2}{2\cdot\frac{\sqrt{2}}{2}a\cdot\frac{\sqrt{6}}{2}a}$

$= 55°$ (nearest degree)

Method 2

In fact, since AD is perpendicular to plane BVA, it is perpendicular to any line on plane BVA.

$\therefore$ Required $\angle = \angle AND$

$= \sin^{-1}\dfrac{AD}{DN} = 55°$ (nearest deg)

(iii) BHD and BVD is the same plane, and $ABGF$ and BVA is also the same plane. Hence the required angle is the same one as in (b)(ii)(2). $\therefore$ YES.

14B.40 HKCEE AM 1999 – II – 11

(a) In $\triangle ABD$, $\dfrac{AD}{\sin(180°-\alpha)} = \dfrac{\ell}{\sin(\alpha-10°)}$

$AD = \dfrac{\ell\sin\alpha}{\sin(\alpha-10°)}$ (m)

(i) In $\triangle ACD$, $CD = AD\sin 10° = \dfrac{\ell\sin\alpha\sin 10°}{\sin(\alpha-10°)}$ m

(ii) In $\triangle ADH$, $\dfrac{DH}{\sin(\beta-10°)} = \dfrac{AD}{\sin(\alpha-\beta)}$

$DH = \dfrac{\ell\sin\alpha\sin(\beta-10°)}{\sin(\alpha-10°)\sin(\alpha-\beta)}$ (m)

. In $\triangle DGH$, $h = DH\sin\alpha$

$= \dfrac{\ell\sin^2\alpha\sin(\beta-10°)}{\sin(\alpha-10°)\sin(\alpha-\beta)}$

(b) (i) (1) $HG = \dfrac{97\sin^2 15°\sin 0.2°}{\sin 5°\sin 4.8°}$

$= 3.11003 = 3.1$ (m, 2 s.f.)

(2) Height of tower $= \dfrac{97\sin 15°}{\sin 5°}$

$= 288.0527 = 290$ (m, 2 s.f.)

Radius of tower $= DH\cos\alpha$

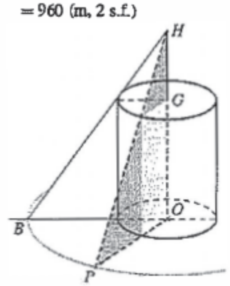
$= \dfrac{97\sin 15°\sin 0.2°\cos 15°}{\sin 5°\sin 4.8°}$

$= 11.60678 = 12$ (m, 2 s.f.)

(ii) (1) $PO = BO = \dfrac{h+CD}{\tan\alpha}$

$= 963.476$

$= 960$ (m, 2 s.f.)



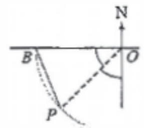
(2) $\angle OBP = (180° - 45°) \div 2$

$= 67.5$

$\therefore$ Bearing of B from P

N$(90° - 67.5°)$W

$=$ N22.5°W



14B.41 HKCEE AM 2001 – 15

(a) (i) $PR^2 = x^2+z^2,\ PQ^2 = x^2+y^2,\ QR^2 = y^2+z^2$

$\cos\angle PRQ = \dfrac{PR^2+QR^2-PQ^2}{2PR\cdot PQ}$

$= \dfrac{(x^2+z^2)+(y^2+z^2)-(x^2+y^2)}{2\sqrt{x^2+z^2}\sqrt{y^2+z^2}}$

$= \dfrac{z^2}{\sqrt{(x^2+z^2)(y^2+z^2)}}$

(ii) $S_1 = \dfrac{xz}{2},\ S_2 = \dfrac{xy}{2},\ S_3 = \dfrac{yz}{2}$

$\sin\angle PRQ = \sqrt{1-\cos^2\angle PRQ}$

$= \sqrt{1-\dfrac{z^4}{(x^2+z^2)(y^2+z^2)}}$

$= \sqrt{\dfrac{x^2y^2+x^2z^2+y^2z^2+z^4-z^4}{(x^2+z^2)(y^2+z^2)}}$

$= \sqrt{\dfrac{x^2y^2+x^2z^2+y^2z^2}{(x^2+z^2)(y^2+z^2)}}$

$\therefore\ S_4 = \dfrac{1}{2}PR\cdot QR\sin\angle PRQ$

$= \dfrac{1}{2}\sqrt{x^2+z^2}\sqrt{y^2+z^2}\sqrt{\dfrac{x^2y^2+x^2z^2+y^2z^2}{(x^2+z^2)(y^2+z^2)}}$

$= \dfrac{1}{2}\sqrt{x^2y^2+x^2z^2+y^2z^2}$

$\Rightarrow\ {S_4}^2 = \dfrac{x^2y^2}{4}+\dfrac{x^2z^2}{4}+\dfrac{y^2z^2}{4} = {S_2}^2+{S_1}^2+{S_3}^2$

(b) (i) Volume $= \dfrac{1}{3}\times\left(\dfrac{4\times 3}{2}\right)\times 2 = 4$

(ii) Height of pyramid with $\triangle GAC$ as base

$= \dfrac{3\times\text{Volume}}{\text{Area of }\triangle GAC}$

$= \dfrac{3\times 4}{\sqrt{(\frac{4\times 2}{2})^2+(\frac{4\times 3}{2})^2+(\frac{2\times 3}{2})^2}} = \dfrac{12}{\sqrt{61}}$

$\therefore$ Required $\angle = \sin^{-1}\dfrac{\frac{12}{\sqrt{61}}}{AB} = 23°$ (nearest degree)

14B.42 HKCEE AM 2002 – 17

(a) *Method 1 to find CF*

In $\triangle ACD$, $\cos\angle ADC = \dfrac{\frac{1}{2}CD}{AD} = \dfrac{3}{5}$ (since $\triangle ACD$ is isos.)

$\Rightarrow\ \sin\angle ADC = \dfrac{4}{5}$

$\therefore\ CF = CD\sin ADC = 24$

Method 2 to find CF

Area of $\triangle ACD = \dfrac{1}{2}\times 30\sqrt{25^2-(30\div 2)^2} = 300$

$\therefore\ CF = \dfrac{2\times\text{Area of }\triangle ACD}{AD} = \dfrac{2\times 300}{25} = 24$

Then...

In $\triangle ACF$, $AF = \sqrt{AC^2-CF^2} = 7$

In $\triangle ABD$, $\cos\angle BAD = \dfrac{28^2+25^2-40^2}{2\cdot 28\cdot 25} = \dfrac{-191}{1400}$

$\therefore$ In $\triangle ABF$, $BF = \sqrt{28^2+7^2-2\cdot 28\cdot 7\cos\angle BAD}$

$= 886.48 = 29.77381$

$\therefore$ In $\triangle BCF$, $\angle BFC = \cos^{-1}\dfrac{886.48+24^2-40^2}{2\cdot\sqrt{886.48}\cdot 24}$

$= 96°$ (nearest degree)

(b) *Method 1*

$AB^2 = 784$

$AF^2+BF^2 = 935.48 \neq AB^2$

$\therefore\ \angle AFB \neq 90°$

Method 2

$\angle AFB = \cos^{-1}\dfrac{AF^2+BF^2-AB^2}{2AF\cdot BF} = 69° \neq 90°$

Method 3

$\cos\angle BAD = \dfrac{-191}{4000} < 0$

$\Rightarrow\ \angle BAF > 90° \Rightarrow\ \angle AFB < 90°$

Hence

BF is not perpendicular to AD.

Thus, $\angle BFC$ is not the dihedral angle.

14B.43 HKCEE AM 2003 – 18

(a) Let S be on PQ such that $RS \perp PQ$ and $OS \perp PQ$.

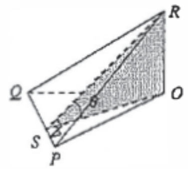
Then $\cos\theta = \dfrac{OS}{RS}$.

$\therefore\ \dfrac{\text{Area of }\triangle OPQ}{\text{Area of }\triangle RPQ}$

$= \dfrac{\frac{1}{2}\cdot OS\cdot PQ}{\frac{1}{2}\cdot RS\cdot PQ}$

$= \dfrac{OS}{RS}$

$= \cos\theta$



(b) (i) Let D be on AB such that $CD \perp AB$ and $ED \perp AB$.

$CD = \dfrac{2\times 12}{6} = 4$ (m)

$\angle$ between board and shadow $= \sin^{-1}\dfrac{2}{4} = 30°$

By (a)(i), Area of shadow $=$ (Area of board) $\cos 30°$

$= 12\cos 30° = 6\sqrt{3}$ (m²)

(ii) $\because$ AC is the longest side

$\therefore$ Height of $\triangle ABC$ from B to AC is the shortest.

Area of shadow $= 12\cos\phi$, where ϕ is the angle of inclination of the board.

Since $\sin\phi = \dfrac{2}{\text{Height of }\triangle ABC}$, ϕ is the smallest (i.e. $\cos\phi$ largest) when B is fastened to the pole.

$\therefore$ B fastened will give the largest shadow.

14B.44 HKCEE AM 2004 – 11

(a) In $\triangle OBC$, $OC = \sqrt{5^2+12^2} = 13$

In $\triangle OAC$, $AC = \sqrt{3^2+13^2-2\cdot 3\cdot 13\cos 12°}$

$= \sqrt{217}$ ($= 14.7$, 3 s.f.)

(b) In $\triangle OAB$, $AB = \sqrt{5^2-3^2} = 4$

In $\triangle ABC$,

Method 1 $AC^2 = 217$

$AB^2+BC^2 = 4^2+12^2 = 160 \neq AC^2$

$\therefore\ \angle ABC \neq 90°$

Method 2 $\angle ABC = \cos^{-1}\dfrac{4^2+12^2-217}{2\cdot 4\cdot 12} = 126° \neq 90°$

Hence, the student is not correct.

14B.45 <u>HKCEE AM 2006 – 17</u>

(a) (i) Let $s=\dfrac{5+6+7}{2}=9$

Area $=\sqrt{s(s-5)(s-6)(s-7)}$
$=\sqrt{216}$ $(=14.7,\ 3\text{ s.f.})$

(ii) Area of $\triangle ABC=\triangle AOB+\triangle BOC+\triangle COA$

$$\sqrt{216}=\frac{6r}{2}+\frac{7r}{2}+\frac{5r}{2}$$
$$r=\frac{\sqrt{216}}{9}\left(=\frac{2\sqrt{6}}{3}\right)$$

(b) (i) $VO=r\tan 60^\circ=2\sqrt{2}$

$\therefore$ Volume of $VABC=\dfrac{1}{3}\times\sqrt{216}\times 2\sqrt{2}$
$=8\sqrt{3}$ $(=13.9,\ 3\text{ s.f.})$

(ii) Height of $\triangle VBC$ from V to $BC=\sqrt{VO^2+r^2}=\sqrt{\dfrac{32}{3}}$

$\therefore$ Area of $\triangle VBC=\dfrac{1}{2}\times\sqrt{\dfrac{32}{3}}\times 7=\dfrac{14\sqrt{6}}{3}$
$(=11.4,\ 3\text{ s.f.})$

(iii) Height of pyramid from A to $\triangle VBC$

$=\dfrac{3\times\text{Volume of pyramid}}{\text{Area of }\triangle VBC}=\dfrac{3\times 8\sqrt{3}}{\frac{14\sqrt{6}}{3}}=\dfrac{18\sqrt{2}}{7}$

$\therefore$ Required $\angle=\sin^{-1}\dfrac{\frac{18\sqrt{2}}{7}}{6}=37^\circ$ (nearest degree)

14B.46 <u>HKCEE AM 2008 – 16</u>

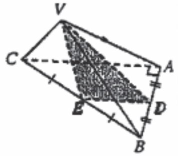
(a) Since VA is not perpendicular to AB, $\angle VAC$ is not the $\angle$ between the planes.

(b) (i) <u>*Criterion 1: $VD\perp AB$*</u>

$\because$ $\triangle VAB$ is equilateral and $BD=DA$
$\therefore$ $VD\perp AB$ (property of isos. $\triangle$)

<u>*Criterion 2: $ED\perp AB$*</u>

$\because$ $BD=DA$ and $BE=EC$
$\therefore$ $DE//AC$ (mid-pt thm)
$\Rightarrow$ $\angle EDB=\angle CAB=90^\circ$
(corr. $\angle$s, $DE//AC$)



<u>*Hence*</u>, the $\angle$ between VAB and ABC is $\angle VDE$.

(ii) $VA=VB=VC=AC=2$ cm
$ED=\dfrac{1}{2}AC=1$ cm
$BC=\sqrt{2^2+2^2}=\sqrt{8}$ (cm)
$VE=\sqrt{VB^2-(BC\div 2)^2}=\sqrt{2}$ cm
$VD=\sqrt{VA^2-(AB\div 2)^2}=\sqrt{3}$ cm
$\because$ $VD^2=3$
$VE^2+ED^2=2+1=3=VD^2$
$\therefore$ $\angle VED=90^\circ$

(c) Area of $\triangle ABC=\dfrac{1}{2}\times 2\times 2=2$ (cm^2)

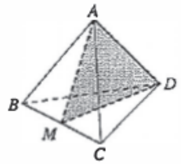
Volume of pyramid $=\dfrac{1}{3}\times$ Area of $\triangle ABC\times VE$
$=\dfrac{2\sqrt{2}}{3}$ cm^3

Area of $\triangle VAB=\dfrac{1}{2}\times 2\times 2\sin 60^\circ=\sqrt{3}$ (cm^2)

$\therefore$ Required dist. $=\dfrac{3\times\text{Volume of pyramid}}{\text{Area of }\triangle VAB}$
$=\dfrac{3\times\frac{2\sqrt{2}}{3}}{\sqrt{3}}=\dfrac{2\sqrt{6}}{3}$ $(=1.63\text{ cm},\ 3\text{ s.f.})$

14B.47 <u>HKCEE AM 2009 – 12</u>

Let M be the mid-point of BC.
$AM=AC\sin\angle ACB=\sqrt{3}$
$DM=\sqrt{3}$
$\therefore$ Required $\angle=\angle AMD$
$=\cos^{-1}\dfrac{3+3-2^2}{2\cdot\sqrt{3}\cdot\sqrt{3}}$
$=71^\circ$ (nearest deg)



14B.48 <u>HKCEE AM 2009 – 18</u>

(a) In $\triangle DHK$, $DK=\dfrac{h}{\tan 30^\circ}=\sqrt{3}h$ (m)

(b) In $\triangle AHK$, $AK=\dfrac{h}{\tan 45^\circ}=h$ (m)

From the time taken, $BD=2AB$.
Since B is the closest point on AD to K, $KB\perp AD$.
In $\triangle ABK$, $BK^2=AK^2-AB^2=h^2-AB^2$
In $\triangle BDK$, $BK^2=DK^2-BD^2=3h^2-4AB^2$
$\therefore$ $h^2-AB^2=3h^2-4AB^2$
$3AB^2=2h^2$
$AB=\sqrt{\dfrac{2}{3}}h$ (m)

(c) $BC=\dfrac{1}{2}AB=\dfrac{1}{\sqrt{6}}h$ m

$BK=\sqrt{h^2-AB^2}=\dfrac{1}{\sqrt{3}}h$ m

In $\triangle BCK$, $CK=\sqrt{BK^2+BC^2}=\dfrac{1}{\sqrt{2}}h$ m

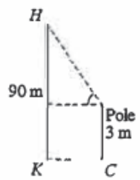
$\therefore$ Required $\angle=\angle HCK=\tan^{-1}\dfrac{HK}{CK}=55^\circ$ (nearest deg)

(d) (i) $AD=3AB=\sqrt{6}h$ m (30 mins)

$$AE+ED=4AB=\frac{4\sqrt{6}}{3}h\text{ m}\quad(40\text{ mins})$$
$$(AE+ED)^2=\frac{32}{3}h^2$$
$$AE^2+ED^2+2AE\cdot ED=\frac{32}{3}h^2$$
$$AD^2+2(9450\times 2)=\frac{32}{3}h^2$$
$$6h^2+37800=\frac{32}{3}h^2$$
$$h^2=8100\ \Rightarrow\ h=90$$

(ii) The pole is to be located at the circumcentre of $\triangle ADE$.
Since it is a right-angled triangle, the circumcentre is the mid-point of its hypotenuse.
i.e. The pole is located at C.

$\therefore$ Required $\angle$ of elevation
$=\tan^{-1}\dfrac{HK-3}{CK}$
$=\tan^{-1}\dfrac{87}{\frac{1}{\sqrt{2}}(90)}$
$=54^\circ$ (nearest degree)

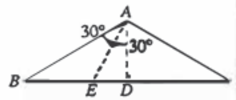


14B.49 <u>HKCEE AM 2010 – 17</u>

(a) In $\triangle ABD$, $AD=11\cos 60^\circ=5.5$ (cm)

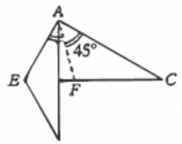
In $\triangle AED$, $AE=\dfrac{AD}{\cos 30^\circ}=\dfrac{11}{\sqrt{3}}=6.35$ (cm, 3 s.f.)

$\therefore$ Area of $\triangle ABE$
$=\dfrac{1}{2}\cdot 11\cdot\dfrac{11}{\sqrt{3}}\sin 30^\circ$
$=\dfrac{121}{4\sqrt{3}}$
$=17.5$ (cm^2, 3 s.f.)



<u>Figure (1)</u>

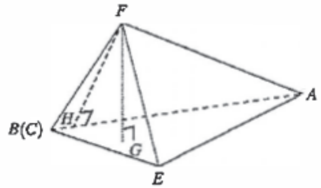
(b) $\angle FAC=(120^\circ-30^\circ)\div 2$
$=45^\circ$
$\angle ACF=(180^\circ-120^\circ)\div 2$
$=30^\circ$



<u>Figure (2)</u>

In $\triangle ACF$, $\dfrac{AF}{\sin 30^\circ}=\dfrac{11}{\sin(180^\circ-45^\circ-30^\circ)}$
$AF=5.69402=5.69$ (cm, 3 s.f.)

(c)



(i) Let G be the projection of F onto $\triangle ABE$.

$FG=\dfrac{3\times\text{Volume of tetrahedron}}{\text{Area of }\triangle ABE}=3.87899$ cm

$\therefore$ Required $\angle=\angle FAG$
$=\sin^{-1}\dfrac{3.87899}{AF}$
$=42.94060^\circ=42.9^\circ$ (3 s.f.)

(ii) Let H be the projection of F onto AB. Then, since GH is the projection of FH onto $\triangle ABE$, the required angle is $\angle FHG$.
In $\triangle AFH$, $FH=AF\sin\angle FAC=4.02628$ cm
$\therefore$ Required $\angle=\angle FHG=\sin^{-1}\dfrac{FG}{FH}=74.5^\circ$ (3 s.f.)

14B.50 <u>HKCEE AM 2011 – 13</u>

(a) (i) In $\triangle ADE$, $AE=3\sin\theta$
In $\triangle BCE$, $BE=4\sin\theta$
$\therefore$ In $\triangle ABE$, $AB=\sqrt{AE^2+BE^2}=5\sin\theta$

(ii) $CD=\sqrt{DE^2+CE^2}=\sqrt{(3\cos\theta)^2+(4\cos\theta)^2}$
$=5\cos\theta$

(b) (i) In $\triangle ABC$, $AC^2=AB^2+BC^2-2AB\cdot BC\cos\alpha$
$=25\sin^2\theta+16-40\sin\theta\cos\alpha$

In $\triangle AEC$, $AC^2=AE^2+EC^2-2AE\cdot EC\cos\alpha$
$=9\sin^2\theta+16\cos^2\theta-24\sin\theta\cos\theta\cos\alpha$

$\therefore$ $25\sin^2\theta+16-40\sin\theta\cos\alpha$
$=9\sin^2\theta+16\cos^2\theta-24\sin\theta\cos\theta\cos\alpha$
$16\sin^2\theta+16(1-\cos^2\theta)=8\sin\theta(5-3\cos\theta)\cos\alpha$
$32\sin^2\theta=8\sin\theta(5-3\cos\theta)\cos\alpha$
$\cos\alpha=\dfrac{4\sin\theta}{5-3\cos\theta}$

(ii) $\because$ $\sin\theta>0$ and $5-3\cos\theta\geq 2>0$
$\therefore$ $\cos\alpha=\dfrac{4\sin\theta}{5-3\cos\theta}>0\ \Rightarrow\ \alpha$ is acute.

(iii) From the given info, since the distance between A and Π_2 is the same,

$AB=AD\ \Rightarrow\ 5\sin\theta=3\ \Rightarrow\ \sin\theta=\dfrac{3}{5}$
$\Rightarrow\ \cos\theta=\dfrac{4}{5}$

$\Rightarrow\ \cos\alpha=\dfrac{4(\frac{3}{5})}{5-3(\frac{4}{5})}=\dfrac{12}{13}$

$AC=\sqrt{25\sin^2\theta+16-40\sin\theta\cos\alpha}$
$=\sqrt{\dfrac{37}{13}}<3=AB$

Hence, the angle between AC and Π_2 is greater than the angle between AB and Π_2.

14B.51 <u>HKDSE MA SP – I – 18</u>

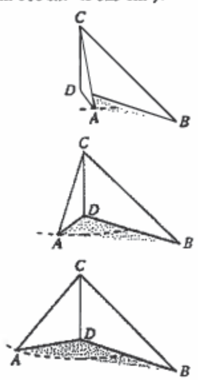
(a) In $\triangle ACD$, $CD=20\sin 45^\circ=10\sqrt{2}$ (cm)
$AD=20\cos 45^\circ=10\sqrt{2}$ (cm)

In $\triangle BCD$, $BC=\dfrac{CD}{\sin 30^\circ}=20\sqrt{2}$ cm
$BD=\dfrac{CD}{\tan 30^\circ}=10\sqrt{6}$ cm

(b) (i) In $\triangle ABD$, Required $\angle=\cos^{-1}\dfrac{AD^2+BD^2-AB^2}{2AD\cdot BD}$
$=\cos^{-1}\dfrac{200+600-324}{2\cdot 10\sqrt{2}\cdot 10\sqrt{6}}$
$=46.60321^\circ$
$=46.6^\circ$ (3 s.f.)

(ii) $\because$ $CD\perp AD$ and $CD\perp BD$
$\therefore$ $CD\perp$ Plane ABD
$\Rightarrow$ Volume of $ABCD=\dfrac{1}{3}\times$ Area of $\triangle ABD\times CD$
$=\dfrac{1}{6}AD\cdot BD\cdot CD\sin\angle ADB$
$\Rightarrow$ Volume of $ABCD\propto\sin\angle ADB$

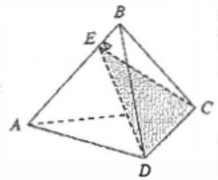
Hence, when $\angle ADB$ increases from 40° to 90°, the volume increases (from 525 cm^3 to 816 cm^3); when $\angle ADB$ increases from 90° to 140°, the volume decreases (from 816 cm^3 to 525 cm^3).



14B.52 <u>HKDSE MA PP – I – 18</u>

(a) In $\triangle ABC$, $AB = \sqrt{20^2 + 12^2 - 2\cdot 20\cdot 12\cos 60^\circ}$
$= \sqrt{304} = 17.4$ (cm, 3 s.f.)

(b) Let E be on AB such that $CE \perp AB$. Since $\triangle ABC$ and $\triangle ABD$ are congruent, $DE \perp AB$ as well.



In $\triangle ABC$,

$CE = \dfrac{2\times \text{Area of } \triangle ABC}{AB}$
$= \dfrac{2\times\frac{1}{2}\cdot 12\cdot 20\sin 60^\circ}{\sqrt{304}}$
$= 11.92079$ (cm)

$DE = CE = 11.92079$ cm

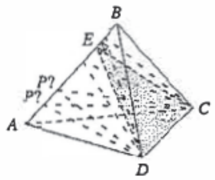
$\therefore$ In $\triangle CDE$,

Required $\angle = \angle CED$
$= \cos^{-1}\dfrac{CE^2 + DE^2 - CD^2}{2CE\cdot DE}$
$= \cos^{-1}\dfrac{11.92079^2 + 11.92079^2 - 14^2}{2\cdot 11.92079\cdot 11.92079}$
$= 71.9^\circ$ (3s.f.)

(c) $\angle CAD = \cos^{-1}\dfrac{20^2+20^2-14^2}{2\cdot 20\cdot 20} = 41.0^\circ$

$\angle CBD = \cos^{-1}\dfrac{12^2+12^2-14^2}{2\cdot 12\cdot 12} = 71.4^\circ$

As P moves from A to B, $\angle CPD$ increases from 41.0° to $\angle CED = 71.9^\circ$ at E and then decreases to 71.4°.

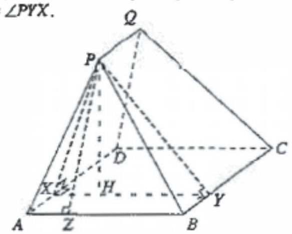


14B.53 <u>HKDSE MA 2012 – I – 18</u>

(a) In $\triangle ABP$, $\angle APB = 180^\circ - 72^\circ - 60^\circ = 48^\circ$

$\dfrac{AP}{\sin 60^\circ} = \dfrac{20}{\sin 48^\circ} \Rightarrow AP = 23.30704 = 23.3$ (cm, 3 s.f.)

(b) Since the pyramid is square-based and right, all lateral faces are congruent. Thus, all their base angles are 72°.
Let X, Y, Z and H be the projections of P on AD, BC, AB and $ABCD$ respectively. Then PXY is perpendicular to $ABCD$. (This is assumed by the symmetry without proof.)
$\therefore\ \alpha = \angle PYX$.



(i) <u>*Method 1* – Use $\triangle PXY$ to find α</u>

In $\triangle ABP$, $\dfrac{BP}{\sin 72^\circ} = \dfrac{20}{\sin 48^\circ}$
$\Rightarrow BP = 25.595456$ (cm)

By the symmetry of the pyramid, $PQCB$ and $PQDA$ are isosceles trapeziums.

In $\triangle APX$, $PX = AP\sin 72^\circ = 22.166315$ cm
$AX = AP\cos 72^\circ = 7.202272$ cm
$\Rightarrow PQ = AD \quad 2AX = 5.595456$ cm
In $\triangle BPY$, $BY = AX = 7.202272$ cm
$PY = \sqrt{PB^2 - BY^2} = 24.561242$ cm
$\therefore$ In $\triangle PXY$, $XY = AB = 20$ cm
$\Rightarrow \alpha = \cos^{-1}\dfrac{XY^2 + PY^2 - PX^2}{2XY\cdot PY} = 58.6^\circ$ (3 s.f.)

<u>*Method 2* – Use $\triangle PHY$ to find α</u>

In $\triangle APZ$, $AZ = AP\cos 72^\circ = 7.202272$ cm
$PZ = AP\sin 72^\circ = 22.166315$ cm
In $\triangle APX$, $AX = AP\cos 72^\circ = 7.202272$ cm
$\Rightarrow$ In $\triangle PHZ$, $HZ = AX = 7.202272$ cm
$PH = \sqrt{PZ^2 - HZ^2} = 20.963606$ cm
$\therefore$ In $\triangle PHY$, $HY = ZB = AB - AZ = 12.797728$ cm
$\alpha = \tan^{-1}\dfrac{PH}{HY} = 58.6^\circ$ (3 s.f.)

<u>*Method 3*</u>

In $\triangle ABP$, $\dfrac{AP}{\sin 60^\circ} = \dfrac{BP}{\sin 72^\circ} \Rightarrow AP = \dfrac{BP\sin 60^\circ}{\sin 72^\circ}$

In $\triangle ABX$, $AX = AP\cos 72^\circ$
$= \dfrac{BP\sin 60^\circ}{\sin 72^\circ}\cos 72^\circ$
$= \dfrac{BP\sin 60^\circ}{\tan 72^\circ}$

In $\triangle BPZ$, $BZ = BP\cos 60^\circ$

In $\triangle PHY$, $HY = BZ = BP\cos 60^\circ$
$PY = \dfrac{HY}{\cos\alpha} = \dfrac{BP\cos 60^\circ}{\cos\alpha}$

$\therefore$ In $\triangle BPY$, $BY = AX = \dfrac{BP\sin 60^\circ}{\tan 72^\circ}$
$BP^2 = BY^2 + PY^2$
$= \dfrac{BP^2\sin^2 60^\circ}{\tan^2 72^\circ} + \dfrac{BP^2\cos^2 60^\circ}{\cos^2\alpha}$
$\dfrac{\cos^2 60^\circ}{\cos^2\alpha} = 1 - \dfrac{\sin^2 60^\circ}{\tan^2 72^\circ}$
$\dfrac{\cos^2\alpha}{\cos^2 60^\circ} = \dfrac{\tan^2 72^\circ}{\tan^2 72^\circ - \sin^2 60^\circ}$
$\cos\alpha = \sqrt{\dfrac{\tan^2 72^\circ\cos^2 60^\circ}{\tan^2 72^\circ - \sin^2 60^\circ}} \Rightarrow \alpha = 58.6^\circ$

<u>*Method 4*</u>

In $\triangle ABP$, $\dfrac{\tan 72^\circ}{\tan 60^\circ} = \dfrac{\frac{PL}{AL}}{\frac{PL}{BL}} = \dfrac{BL}{AL}$

Similarly, in $\triangle PXY$, $\dfrac{\tan\theta}{\tan\alpha} = \dfrac{YH}{XH}$
$= \dfrac{BL}{AL} = \dfrac{\tan 72^\circ}{\tan 60^\circ}$
$\Rightarrow \tan\alpha = \dfrac{\tan 60^\circ}{\tan 72^\circ}\tan\theta$

In $\triangle APZ$, $AZ = AP\cos 72^\circ$
In $\triangle APX$, $PX = AP\sin 72^\circ$
In $\triangle PHX$, $HX = AZ = AP\cos 72^\circ$
$\therefore \cos\theta = \dfrac{HX}{PX} = \dfrac{AP\cos 72^\circ}{AP\sin 72^\circ} = \dfrac{1}{\tan 72^\circ}$
$\Rightarrow \tan\theta = \sqrt{\tan^2 72^\circ - 1}$
Hence, $\tan\alpha = \dfrac{\tan 60^\circ}{\tan 72^\circ}\tan\theta$
$= \dfrac{\tan 60^\circ}{\tan 72^\circ}\sqrt{\tan^2 72^\circ - 1} \Rightarrow \alpha = 58.6^\circ$

(ii) $\sin\alpha = \dfrac{PH}{PY}$, $\sin\beta = \dfrac{PH}{PB}$
$\because PY < PB$
$\therefore \dfrac{PH}{PY} > \dfrac{PH}{PB} \Rightarrow \sin\alpha > \sin\beta \Rightarrow \alpha > \beta$

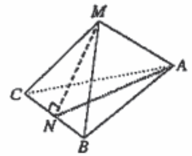
14B.54 <u>HKDSE MA 2013 – I – 18</u>

(a) (i) In $\triangle ABC$, $\angle BCM = \cos^{-1}\dfrac{21^2 + 35^2 - 28^2}{2\cdot 21\cdot 35}$
$= 53.13010^\circ = 53.1^\circ$ (3 s.f.)

(ii) In $\triangle BCM$, $\angle CBM = 51.86990^\circ$
$\dfrac{CM}{\sin 51.86990^\circ} = \dfrac{21}{\sin 75^\circ}$
$CM = 17.10155 = 17.1$ (cm, 3 s.f.)

(b) (i) $AM = 35 - 17.101545 = 17.89845$ (cm)
In $\triangle ACM$,
$AC = \sqrt{AM^2 + CM^2 - 2AM\cdot CM\cos\angle AMC}$
$= 28.13898 = 28.1$ (cm, 3 s.f.)

(ii) In $\triangle CMN$, $CN = CM\cos\angle MCN$
$= 17.10155\cos 53.13030^\circ$
$= 10.26093$ (cm)
$\Rightarrow BN = 21 - 10.26093 = 10.73907$ (cm)
In $\triangle ABC$, $\angle ABC = \cos^{-1}\dfrac{28^2 + 21^2 - 28.13898^2}{2\cdot 28\cdot 21}$
$= 68.38516^\circ$



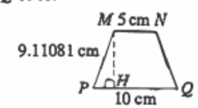
<u>*Method 1* – Check whether $AN \perp BC$</u>
In $\triangle ABN$, $AN = \sqrt{AB^2 + BN^2 - 2AB\cdot BN\cos\angle ABC}$
$= 26.03454$ cm
$\because AB^2 = 784$
$AN^2 + BN^2 = 681 \neq AB^2$
$\therefore \angle ANB \neq 90^\circ$
i.e. $\angle ANM$ is not the described angle. Disagreed.

<u>*Method 2* – Check if N is the projection of A onto BC</u>
Suppose the projection of A onto BC is P.
In $\triangle ABP$, $BP = AB\cos\angle ABC = 10.31423$ cm $\neq BN$.
i.e. N is not the projection of A onto BC.
Hence, $\angle ANM$ is not the described angle. Disagree.

14B.55 <u>HKDSE MA 2014 – I – 17</u>

(a) In $\triangle VAB$, $\dfrac{\sin\angle AVB}{18} = \dfrac{\sin 110^\circ}{30}$
$\angle AVB = 34.32008^\circ$ or 145.7° (rej.)
$\therefore \angle VBA = 180^\circ - 110^\circ - 34.32008^\circ$
$= 35.67992^\circ = 35.7^\circ$ (3 s.f.)

(b) In $\triangle VAB$, $VA = \sqrt{18^2 + 30^2 - 2\cdot 18\cdot 30\cos 35.67992^\circ}$
$= 18.22161$ cm
In $\triangle VBC$, $\because VM = MB$ and $VN = NC$
$\therefore MN = \frac{1}{2}BC = 5$ cm (mid-pt theorem)
Similarly, $MP = \frac{1}{2}VA = 9.11081$ cm
Let the projection of M onto PQ be H.
In $\triangle MPH$,
$PH = (PQ - MN) \div 2$
$= 2.5$ cm
$MH = \sqrt{MP^2 - PH^2}$
$= 8.7611$ cm



$\therefore$ Area of $PQNM = \dfrac{(5+10)(8.7611)}{2} = 65.7 < 70$ (cm^2)
$\therefore$ The craftsman is agreed.

14B.56 <u>HKDSE MA 2015 – I – 19</u>

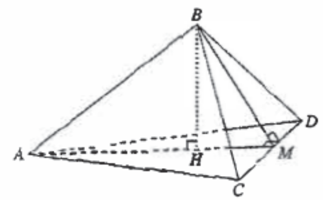
(a) (i) In $\triangle ABC$, $AC = \sqrt{40^2 + 24^2 - 2\cdot 40\cdot 24\cos 80^\circ}$
$= 42.92546 = 42.9$ (cm, 3 s.f.)

(ii) $\dfrac{\sin\angle ACB}{40} = \dfrac{\sin 80^\circ}{42.92546}$
$\angle ACB = 66.59082^\circ$ or 113° (rej.) $= 66.6^\circ$ (3 s.f.)

(iii) Note how the given information had fixed the areas of $\triangle ABC$ and $\triangle AB'D$. Hence, the only varying part of the paper card is $\triangle ACD$.

Area of $\triangle ABC$ = Area of $\triangle AB'D$
$= \frac{1}{2}(40)(24)\sin 80^\circ = 472.71$ (cm^2),
which is a constant.
Area of $\triangle ACD = \frac{1}{2}AC^2\sin\angle ACD$
$= 921.30\sin(\angle BCD - 66.6^\circ)$
$\because 105^\circ \le \angle BCD \le 145^\circ$
$\therefore 38.4^\circ \le \angle ACD \le 78.4^\circ$
Hence, as $\angle BCD$ increases from 105° to 145°, the area of the paper card increases.
(from $472.71\times 2 + 921.30\sin 38.4^\circ = 1518$ (cm^2)
to $472.71\times 2 + 921.30\sin 78.4^\circ = 1848$ (cm^2))

(b) Let the projection of B onto ACD be H and the mid-point of CD be M. By symmetry, we have $BM \perp CD$, $AM \perp CD$ and H lying on AM.



$\angle ACD = 132^\circ - 66.59082^\circ = 65.40918^\circ$
In $\triangle ACM$, $AM = AC\sin(132^\circ - 65.40918^\circ)$
$= 39.39231$ (cm)
$CM = AC\cos(132^\circ - 65.40918^\circ)$
$= 17.86279$ (cm)
In $\triangle BCM$, $BM = \sqrt{BC^2 - CM^2} = 16.02875$ cm
In $\triangle ABM$, $\angle BAM = \cos^{-1}\dfrac{AB^2 + AM^2 - BM^2}{2AB\cdot AM}$
$= 23.2791^\circ$
$\Rightarrow BH = AB\sin\angle BAH = 15.8084$ cm
$\therefore$ Vol of pyramid $= \frac{1}{3}(921.30\sin 65.40918^\circ)(15.8084)$
$= 4410$ (cm^3, 3 s.f.)

14B.57 HKDSE MA 2016 – I – 19

(a) In $\triangle ABD$, $\dfrac{\sin\angle ADB}{10}=\dfrac{\sin 86^\circ}{15}$

$\angle ADB = 41.68560^\circ$ or 138.3° (rej.)

$\Rightarrow \angle ABD = 180^\circ - 86^\circ - 41.68560^\circ$

$= 52.31440^\circ = 52.3^\circ$ (3 s.f.)

In $\triangle BCD$, $CD=\sqrt{8^2+15^2 \quad 2\cdot 8\cdot 15\cos 43^\circ}$

$= 10.65247 = 10.7$ (cm, 3 s.f.)

(b) We need to verify $AC \perp BC$ and $AC \perp CD$.

In $\triangle ABC$, $AC^2+BC^2 = 6^2+8^2$

$= 100 = AB^2$

$\therefore AC \perp BC$

In $\triangle ABD$, $AD^2 = AB^2+BD^2 \quad 2AB\cdot BD\cos\angle ABD$

$= 141.60$

$\because AC^2+CD^2 = 149.48 \neq AD^2$

$\therefore AC$ is not perpendicular to CD.

Since C is not the projection of A onto BCD, $\angle ABC$ is not the described angle. The craftsman is disagreed.

14B.58 HKDSE MA 2017 – I – 19

(a) In $\triangle ABC$, $\angle B = 180^\circ - 30^\circ - 42^\circ = 108^\circ$

$\dfrac{AC}{\sin 108^\circ}=\dfrac{24}{\sin 30^\circ} \Rightarrow AC = 45.65071 = 45.7$ (cm, 3 s.f.)

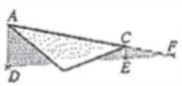
(b) (i) $\because \triangle ADF \sim \triangle CEF$

$\therefore \dfrac{10}{2}=\dfrac{45.65071+CF}{CF}$

$4CF = 45.65071$

$CF = 11.41268$

$= 11.4$ (cm, 3 s.f.)



(ii) *Method 1*

In $\triangle ABC$, $\dfrac{AB}{\sin 42^\circ}=\dfrac{24}{\sin 30^\circ} \Rightarrow AB = 32.11827$ (cm)

Area of $\triangle ABF = \dfrac{1}{2}AB\cdot AF\sin\angle FAB$

$= 458.1943 = 458$ (cm^2, 3 s.f.)

Method 2

Area of $\triangle ABF$

$=$ Area of $\triangle ABC$ + Area of $\triangle FBC$

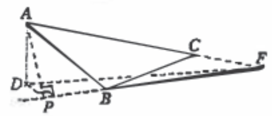
$= \dfrac{1}{2}AC\cdot BC\sin\angle ACB + \dfrac{1}{2}CF\cdot BC\sin(180^\circ - \angle ACB)$

$= \dfrac{1}{2}(AC+CF)BC\sin\angle ACB = 458$ cm^2 (3 s.f.)

(iii) In $\triangle FBC$, $BF=\sqrt{BC^2+CF^2-2BC\cdot CF\cos\angle BCF}$

$= 33.36690$ (cm)

(Or use $\triangle ABF$ to find BF.)



Let the projection of A onto BF be P.

$AP = \dfrac{2\times \text{Area of }\triangle ABF}{BF} = 27.46400$ cm

$\therefore$ Inclination $= \sin^{-1}\dfrac{AD}{AP} = 21.4^\circ$ (3 s.f.)

(iv) Since P is also the projection of D onto BF,

Area of $\triangle BDF = \dfrac{1}{2}BF\cdot DP$

$< \dfrac{1}{2}BF\cdot AP = 458 < 460$

. The craftsman is disagreed.

14B.59 HKDSE MA 2018 – I – 17

(a) In $\triangle ABD$, $\dfrac{AD}{\sin 20^\circ}=\dfrac{60}{\sin(180^\circ-120^\circ-20^\circ)}$

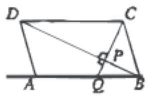
$AD = 31.92533 = 31.9$ (cm, 3 s.f.)

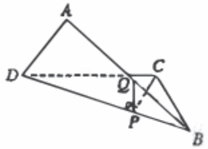
(b) (i) In $\triangle ABC$, $\angle ABC = \cos^{-1}\dfrac{AB^2+BC^2-AC^2}{2AB\cdot BC}$

$= \cos^{-1}\dfrac{60^2+31.92533^2-40^2}{2\cdot 60\cdot 31.92533}$

$= 37.99208^\circ = 38.0^\circ$ (3 s.f.)

(ii) Let P be on BD such that $CP \perp BD$, and CP extended meet AB at Q (in Figure (1)). Then the angle between ABD and BCD in Figure (2) is $\angle CPQ$.





In $\triangle BCP$, $BP = BC\cos 40^\circ = 24.45622$ cm

In $\triangle BPQ$, $BQ = \dfrac{BP}{\cos 20^\circ} = 26.02577$ cm

In $\triangle BCQ$, $CQ = \sqrt{BQ^2+BC^2-2BQ\cdot BC\cos\angle QBC}$

$= 19.67077$ cm

$\therefore$ In $\triangle CPQ$,

Required $\angle = \angle CPQ = \cos^{-1}\dfrac{PQ^2+CP^2-CQ^2}{2PQ\cdot CP}$

$= 71.9^\circ$ (3 s.f.)

14B.60 HKDSE MA 2019 – I – 18

(a) (i) In $\triangle ABD$, $\dfrac{\sin\angle BAD}{12}=\dfrac{\sin 72^\circ}{13}$

$\angle BAD = 61.38987^\circ$ or 118.6° (rej.)

$= 61.4^\circ$ (3 s.f.)

(ii) $\angle ADB = 180^\circ - 72^\circ - 61.38987^\circ = 46.61013^\circ$

$\Rightarrow DP = BD\cos\angle ADB = 8.24351$ cm

In $\triangle CDP$,

$CP = \sqrt{12^2+8.24351^2-2\cdot 12\cdot 8.24351\cos 60^\circ}$

$= 11.39253 = 11.4$ (cm, 3 s.f.)

(b) Since $BP \perp AD$, we need to check whether $CP \perp AD$.

In $\triangle CDP$, $CD^2 = 169$

$CP^2+DP^2 = 197.7 \neq 169 = CD^2$

Hence, $\angle CPD \neq 90^\circ$, and thus $\angle BPC$ is not the angle between ABD and ACD. The claim is disagreed.

14B.61 HKDSE MA 2020 – I – 19

19a

$$\frac{PR}{\sin\angle PQR}=\frac{PQ}{\sin\angle PRQ}$$

$$\frac{PR}{\sin 30^\circ}=\frac{60}{\sin 55^\circ}$$

$$PR \approx 36.62323766 \text{ cm}$$

$\angle PQR + \angle PRQ + \angle QPR = 180^\circ$ ($\angle$ sum of Δ)

$30^\circ + 55^\circ + \angle QPR = 180^\circ$

$\angle QPR = 95^\circ$

$\angle QPR + \angle RPS = \angle QPS$

$95^\circ + \angle RPS = 120^\circ$

$\angle RPS = 25^\circ$

$RS^2 = PR^2 + PS^2 - 2(PR)(PS)\cos\angle RPS$

$RS^2 = 36.62323766^2 + 40^2 - 2(36.62323766)(40)\cos 25^\circ$

$RS = 16.90879944$ cm

$RS = 16.9$ cm (corr. to 3 sig. fig.)

b

The area of the paper card

$\dfrac{1}{2}(PQ)(PR)\sin\angle QPR + \dfrac{1}{2}(PR)(PS)\sin\angle RPS$

$= \dfrac{1}{2}(60)(36.62323766)\sin 95^\circ + \dfrac{1}{2}(36.62323766)(40)\sin 25^\circ$

≈ 1404.069236

$= 1400$ cm^2 (corr. to 3 sig. fig.)

ci

Let A be the perpendicular foot of P on the line passing through Q and R and O be the projection of P on the horizontal ground.

Then, $\angle OAP = 32^\circ$.

$\sin\angle PQA = \dfrac{PA}{PQ}$

$\sin 30^\circ = \dfrac{PA}{60}$

$PA = 30$ cm

$\sin\angle OAP \quad \dfrac{OP}{PA}$

$\sin 32^\circ = \dfrac{OP}{30}$

$OP = 30\sin 32^\circ$ cm

$OP = 15.9$ cm (corr. to 3 sig. fig.)

ii

Produce PS and QR to intersect at the point B.

$\angle PQB + \angle PBQ + \angle QPB = 180^\circ$ ($\angle$ sum of Δ)

$30^\circ + \angle PBQ + 120^\circ = 180^\circ$

$\angle PBQ = 30^\circ$

$\angle PBQ = \angle PQB$

$PQ \quad PB$ (sides opp. eq. $\angle$s)

$PS + SB = PQ$

40 cm $+ SB = 60$ cm

$SB = 20$ cm

Let C be the perpendicular foot of S on AB.

$\angle PBA = \angle SBC$ (common $\angle$)

$\angle PAB = \angle SCB$ (by construction)

$\angle APB = \angle CSB$ (3rd $\angle$ of Δ)

$\Delta PAB \sim \Delta SCB$ (AAA)

$\dfrac{PA}{SC}=\dfrac{PB}{SB}$ (corr. sides, $\sim\Delta$s)

$\dfrac{30\sin 32^\circ}{SC}=\dfrac{60}{20}$

$SC = 10\sin 32^\circ$ cm

$\sin\angle SRC \quad \dfrac{SC}{RS}$

$\sin\angle SRC \quad \dfrac{10\sin 32^\circ}{16.90879944}$

$\angle SRC \approx 18.26416068^\circ$

The angle between RS and the horizontal ground $\approx 18.26416068^\circ$

The angle between RS and the horizontal ground $< 20^\circ$

Hence, the student's claim is agreed with.

15 Mensuration

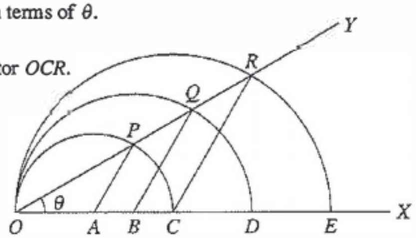
15A Lengths and areas of plane figures

15A.1 HKCEE MA 1980(1/1*/3) I 10 (To continue as **12A.1**.)

A, B and C are three points on the line OX such that $OA = 2$, $OB = 3$ and $OC = 4$. With A, B, C as centres and OA, OB, OC as radii, three semi circles are drawn as shown in the figure. A line OY cuts the three semi-circles at P, Q, R respectively.

(a) If $\angle YOX = \theta$, express $\angle PAX$, $\angle QBX$ and $\angle RCX$ in terms of θ.

(b) Find the following ratios:
area of sector OAP : area of sector OBQ : area of sector OCR.



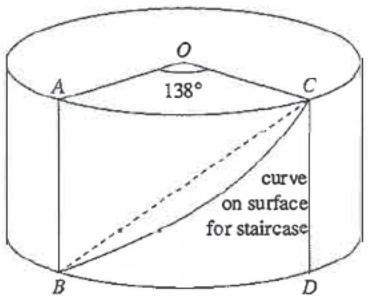
15A.2 (HKCEE MA 1981(1/2/3) – I – 12)

The figure shows a cylinder 10 metres high and 10 metres in radius used for storing coal gas. AB and CD are two vertical lines on the curved surface of the cylinder. The arc AC subtends an angle of 138° at the point O, which is the centre of the top of the cylinder.

(a) Inside the cylinder, a straight pipe runs from B to C. Calculate the length of the pipe BC correct to 3 significant figures.

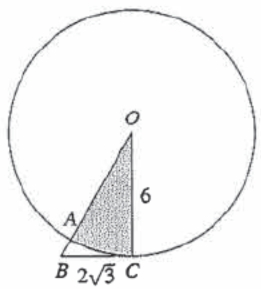
(b) Calculate the area of the curved surface $ABDC$ bounded by the minor arcs AC, BD and the lines AB, CD.

(c) A staircase from B to C is built along the shortest curve on the curved surface $ABDC$. Find the length of the curve.



15A.3 HKCEE MA 1982(1/2/3) – I – 4

In the figure, the circle, centre O and radius 6, touches the straight line BC at C. $BC = 2\sqrt{3}$. OAB is a straight line. Find the area of the shaded sector in terms of π.



15A.4 HKCEE MA 1982(1/2/3) – I – 9

(In this question, answers should be given in surd form.)

In Figures (1) and (2), $ABCDEF$ is a regular hexagon with $AB = 1$.

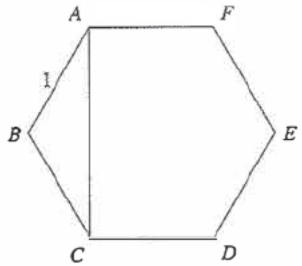


Figure (1)

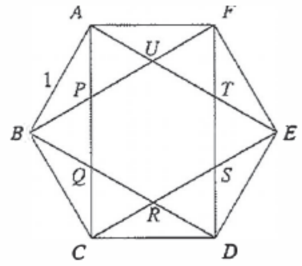


Figure (2)

(a) Calculate the area of the hexagon in Figure (1) and the length of its diagonal AC.

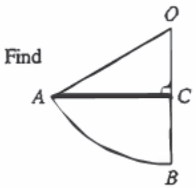
(b) In Figure (2), $PQRSTU$ is another regular hexagon formed by the diagonals of $ABCDEF$.
 (i) Calculate the length of PQ.
 (ii) Calculate the area of the hexagon $PQRSTU$.

15A.5 HKCEE MA 1983(A/B) I 5

In the figure, O is the centre of the sector OAB. $OA = 30$, $CB = 15$ and $AC \perp OB$. Find

(a) $\angle AOC$,

(b) the length of the arc AB in terms of π.



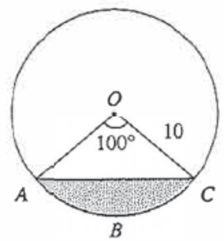
15A.6 HKCEE MA 1988 – I – 5

In the figure, ABC is a circle with centre O and radius 10. $\angle AOC = 100°$. Calculate, correct to 2 decimal places,

(a) the area of sector $OABC$,

(b) the area of $\triangle OAC$,

(c) the area of segment ABC.

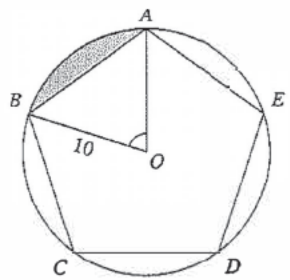


15A.7 HKCEE MA 1992 – I – 7

In the figure, $ABCDE$ is a regular pentagon inscribed in a circle with centre O and radius 10.

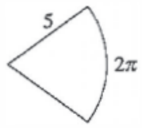
(a) Find $\angle AOB$ and the area of triangle OAB.

(b) Find the area of the shaded part in the figure.



15A.8 HKCEE MA 1994 – I – 2(d)

In the figure, find the area of the sector.

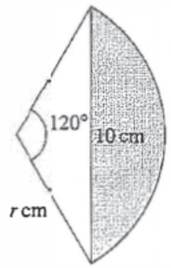


15A.9 HKCEE MA 1999 – I – 9

The figure shows a sector.

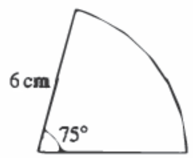
(a) Find r.

(b) Find the area of the shaded region.



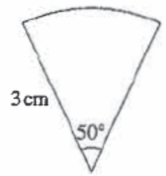
15A.10 HKCEE MA 2000 – I – 3

Find the area of the sector in the figure.



15A.11 HKCEE MA 2001 – I – 3

Find the perimeter of the sector in the figure.



15A.12 HKCEE MA 2004 – I – 9

In the figure, the area of the sector is $162\pi \text{ cm}^2$.

(a) Find the radius of the sector.

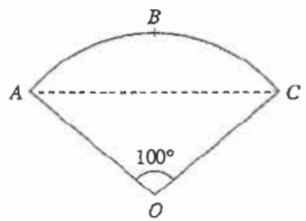
(b) Find the perimeter of the sector in terms of π.

15A.13 HKCEE MA 2005 – I – 9

In the figure, $OABC$ is a sector with $\overset{\frown}{ABC} = 10\pi \text{cm}$.

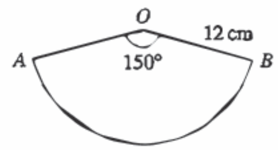
(a) Find OA.

(b) Find the area of segment ABC.



15A.14 HKCEE MA 2006 – I – 4

In the figure, the radius of the sector OAB is 12 cm. Find the length of $\overset{\frown}{AB}$ in terms of π.

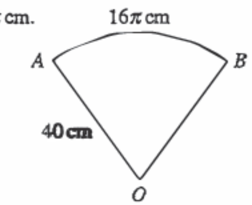


15A.15 HKCEE MA 2007 – I – 9

In the figure, the radius of the sector AOB is 40 cm. It is given that $\overset{\frown}{AB} = 16\pi$ cm.

(a) Find $\angle AOB$.

(b) Find the area of the sector AOB in terms of π.



15A.16 HKDSE MA 2015 – I – 9

The radius and the area of a sector are 12 cm and $30\pi \text{ cm}^2$ respectively.

(a) Find the angle of the sector.

(b) Express the perimeter of the sector in terms of π.

15B Volumes and surface areas of solids

15B.1 HKCEE MA 1983(A/B) – I – 8

The solid in Figure (1) is made up of two parts. The lower part is a right circular cylinder of height h cm and radius r cm; the upper part is a hemisphere of the same radius r cm. The two parts are of the same volume.

(a) Find the ratio $r : h$.

(b) Figure (2) shows a section of the solid through the axis of the cylinder. The perimeter of this section is 136 cm.

 (i) Calculate r to 2 significant figures.

 (ii) Calculate the total external surface area (including the base) of the solid in cm^2 to 1 significant figure.

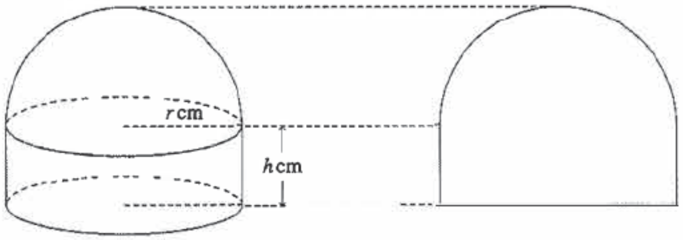


Figure (1) Figure (2)

15B.2 HKCEE MA 1984(A/B) I 12

In the figure, all vertical cross-sections of the solid that are parallel to $APBCQD$ are identical. $ABCD$, $BRTC$ and $ABRS$ are squares, each of side 20 cm. P is the mid point of AB. CQD is a circular arc with centre P and radius PC.

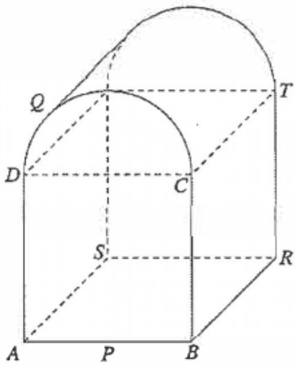
(In this question, give your answers correct to 1 decimal place.)

(a) Find $\angle CPD$.

(b) Find, in cm, the length of the arc CQD.

(c) Find, in cm^2, the area of the cross section $APBCQD$.

(d) Find, in cm^2, the total surface area of the solid.



15B.3 HKCEE MA 1985(A/B) I 11

Figure (1) shows a solid right circular cone. O is the vertex and P is a point on the circumference of the base. The area of the curved surface is 135π cm^2 and the radius of the base is 9 cm.

(a) (i) Find the length of OP.

 (ii) Find the height of the cone.

(b) The cone in Figure (1) is cut into two portions by a plane parallel to its base. The upper portion is a cone of base radius 3 cm. The lower portion is a frustum of height x cm.

 (i) Find the value of x.

 (ii) A right cylindrical hole of radius 3 cm is drilled through the frustum (see Figure (2)). Find the volume of the solid which remains in the frustum. (Give your answer in terms of π.)

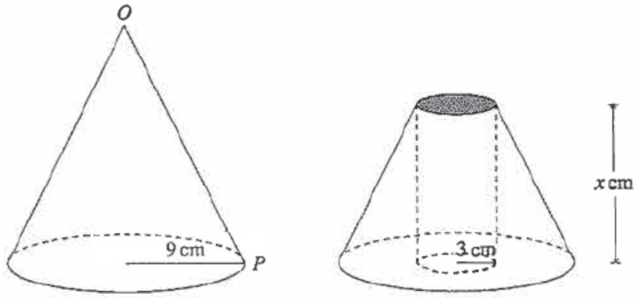


Figure (1) Figure (2)

15B.4 HKCEE MA 1986(A/B) I – 12

Figure (1) shows a solid consisting of a right circular cone and a hemisphere with a common base which is a circle of radius 6. The volume of the cone is equal to $\frac{4}{3}$ of the volume of the hemisphere.

(a) (i) Find the height of the cone.

 (ii) Find the volume of the solid. (Leave your answer in terms of π.)

(b) (i) The solid is cut into two parts. The upper part is a right circular cone of height y and base radius x as shown in Figure (2). Find $\frac{x}{y}$.

 (ii) If the two parts in (b)(i) are equal in volume, find y, correct to 1 decimal place.

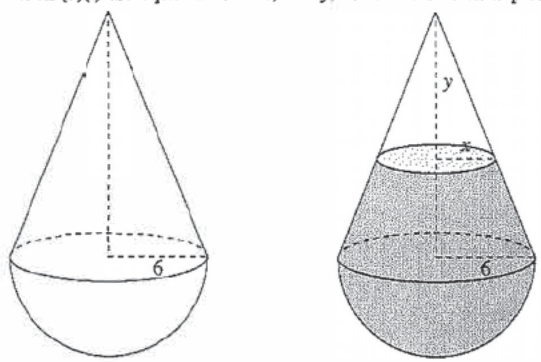


Figure (1) Figure (2)

15B.5 HKCEE MA 1989 – I – 11

Figure (1) shows a rectangular swimming pool 50 m long and 20 m wide. The floor of the pool is an inclined plane. The depth of water is 10 m at one end and 2 m at the other.

(a) Find the volume of water in the pool in m^3.

(b) Water in the pool is now pumped out through a pipe of internal radius 0.125 m. Water flows in the pipe at a constant speed of 3 m/s.

(i) Find the volume of water, in m^3, REMAINING in the pool when the depth of water is 8 m at the deeper end.

(ii) Find the volume of water pumped out in 8 hours, correct to the nearest m^3.

(iii) Let h metres be the depth of water at the deeper end after 8 hours (see Figure (2)). Find the value of h, correct to 1 decimal place.

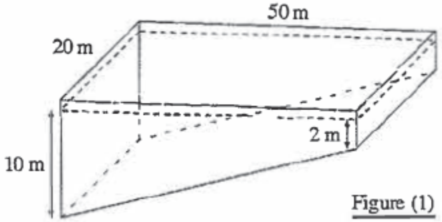


Figure (1)

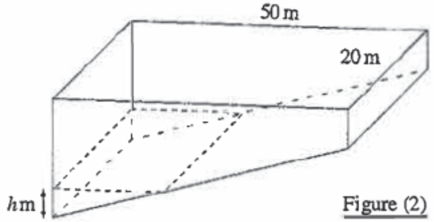


Figure (2)

15B.6 HKCEE MA 1990 – I – 11

(To continue as 4B.8.)

A solid right circular cylinder has radius r and height h. The volume of the cylinder is V and the total surface area is S.

(a) (i) Express S in terms of r and h.

(ii) Show that $S = 2\pi r^2 + \frac{2V}{r}$.

15B.7 HKCEE MA 1991 – I – 11

Figure (1) shows a metal bucket. Its slant height AB is 60 cm. The diameter AD of the base is 40 cm and the diameter BC of the open top is 80 cm. The curved surface of the bucket is formed by the thin metal sheet $ABB'A'$ shown in Figure (2), where $\overset{\frown}{ADA'}$ and $\overset{\frown}{BCB'}$ are arcs of concentric circles with centre O.

(a) Find OA and $\angle AOA'$.

(b) Find the area of the metal sheet $ABB'A'$, leaving your answer in terms of π.

(c) There is an ant at the point A on the outer curved surface of the bucket. Find the shortest distance for it to crawl along the outer **curved surface** of the bucket to reach the point C.

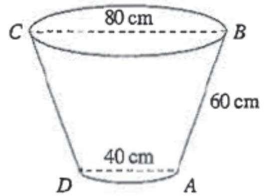


Figure (1)

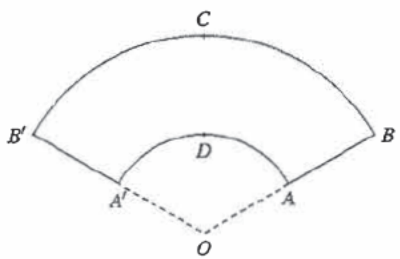


Figure (2)

15B.8 (HKCEE MA 1993 – I – 9)

The figure shows a right circular cylinder. O is the centre and r is the radius of its top face. A chord AB divides the area of the top face in the ratio 4 : 1 and subtends an angle α at O. C is a point on the minor arc AB.

(a) (i) Find the area of the sector $OACB$ in terms of r and α.

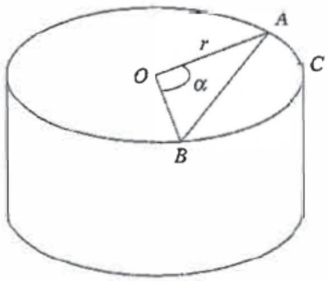
(ii) Find the area of the segment ACB in terms of r and α.

(iii) Show that $\sin\alpha = \left(\frac{\alpha}{180°} - \frac{2}{5}\right)\pi$.

(iv) *[Out of syllabus]*

(v) *[Out of syllabus: The result $\alpha \approx 121°$ is obtained.]*

(b) The cylinder is cut along AB into 2 parts by a plane perpendicular to its top face. Find the ratio of the curved surface areas of the two parts in the form $k : 1$, where $k > 1$.



15B.9 HKCEE MA 1994 – I – 10

Figure (1) shows the longitudinal section of a right cylindrical water tank of base radius 2 m and height 3 m. The tank is filled with water to a depth of 1.5 m.

(a) Express the volume of water in the tank in terms of π.

(b) If a solid sphere of radius 0.6 m is put into the tank and is completely submerged in water, the water level rises by h metres. Find h (see Figure (2)).

(c) A solid sphere of radius r m is put into the tank and is just submerged in water (see Figure (3)).

(i) Show that $2r^3 - 12r + 9 = 0$.

(ii) *[Out of syllabus]*

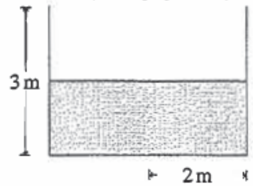


Figure (1)

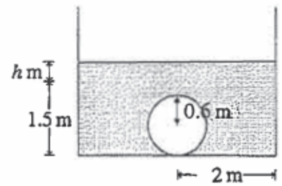


Figure (2)

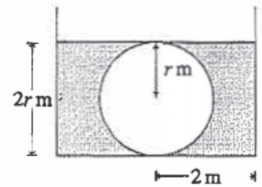


Figure (3)

15B.10 HKCEE MA 1995 – I – 13

A right cylindrical vessel of base radius 4 cm and height 11 cm is placed on a horizontal table. A right conical vessel of base radius 6 cm and height 12 cm is placed, with its axis vertical, in the cylindrical vessel. The conical vessel is full of water and the cylindrical vessel is empty. Figure (1) shows the longitudinal sections of the two vessels where A is the vertex of the conical vessel.

(a) Find, in terms of π, the volume of water in the conical vessel.

(b) The vertex A is d cm from the base of the cylindrical vessel. Use similar triangles to find d.

(c) Supposer water leaks out from the conical vessel through a small hole at the vertex A into the cylindrical vessel.

  (i) Find, in terms of π, the volume of water that has leaked out when the water level in the cylindrical vessel reaches the vertex A.

  (ii) If $104\pi\,\text{cm}^3$ of water has leaked out and the water level in the cylindrical vessel is h cm above the vertex A (see Figure (2)), show that $h^3 - 192h + 672 = 0$.

  (iii) ***[Out of syllabus]***

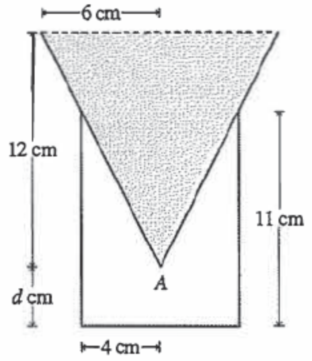


Figure (1)

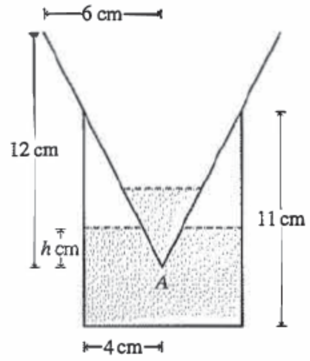


Figure (2)

15B.11 HKCEE MA 1996 I 8

Figure (1) shows a paper cup in the form of a right circular cone of base radius 5 cm and height 12 cm.

(a) Find the capacity of the paper cup.

(b) If the paper cup is cut along the slant side AB and unfolded to become a sector as shown in Figure (2), find

  (i) the area of the sector;

  (ii) the angle of the sector.

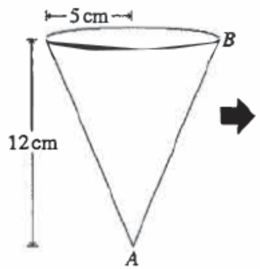


Figure (1)

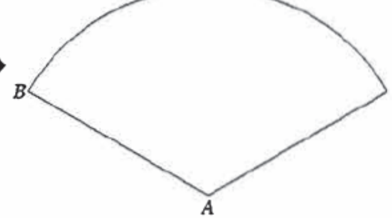


Figure (2)

15B.12 HKCEE MA 1997 I 12

Figure (1) shows a greenhouse $VABCD$ in the shape of a right pyramid with a square base of side 6 m. M is the mid-point of BC and VN is the height of the pyramid. Each of the triangular faces makes an angle θ with the square base.

(a) (i) Express VN and VM in terms of θ.

  (ii) Find the capacity and total surface area of the greenhouse (excluding the base) in terms of θ.

(b) Figure (2) shows another greenhouse in the shape of a right cylinder with base radius r m and height h m. It is known that both the base areas and the capacities of the two greenhouses are equal.

  (i) Express r in terms of π.

  (ii) Express h in terms of θ.

  (iii) If the total surface areas of the two greenhouses (excluding the bases) are equal, show that $3 + \sqrt{\pi}\tan\theta = \dfrac{3}{\cos\theta}$.

  (iv) ***[Out of syllabus]***

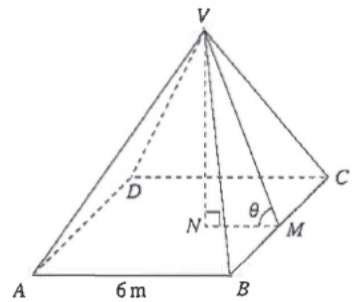


Figure (1)

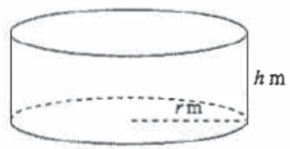
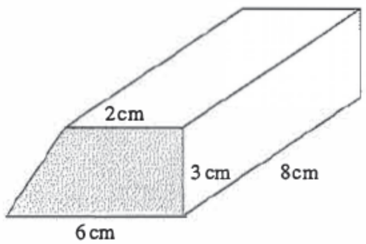


Figure (2)

15B.13 HKCEE MA 1998 – I – 1

The figure shows a right prism, the cross section of which is a trapezium. Find the volume of the prism.



15B.14 HKCEE MA 1999 – I – 13

In Figure (1), a piece of wood in the form of an inverted right circular cone is cut into two portions by a plane parallel to its base. The upper portion is a frustum with height 10 cm, and the radii of the two parallel faces are 9 cm and 4 cm respectively. The pen stand shown in Figure (2) is made from the frustum by drilling a hole in the middle. The hole consists of a cylindrical upper part of radius 5 cm and a hemispherical lower part of the same radius. The depth of the hole is 9 cm.

(a) Find, in terms of π, the capacity of the hole.

(b) Find, in terms of π, the volume of wood in the pen-stand.

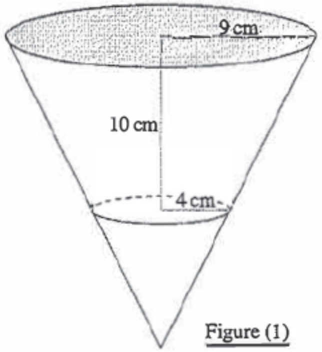


Figure (1)

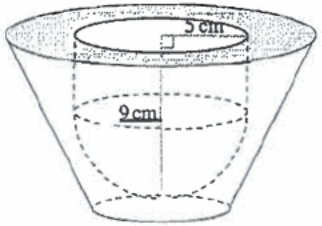


Figure (2)

15B.15 HKCEE MA 2002 – I – 15

(a) Figure (1) shows two vessels of the same height 24 cm, one in the form of a right circular cylinder of radius 6 cm and the other a right circular cone of radius 9 cm. The vessels are held vertically on two horizontal platforms, one of which is 5 cm higher than the other. To begin with, the cylinder is empty and the cone is full of water. Water is then transferred into the cylinder from the cone until the water in both vessels reaches the same horizontal level. Let h cm be the depth of water in the cylinder.

  (i) Show that $h^3+15h^2+843h-13699=0$.

  (ii) *[Out of syllabus; the result $h=11.8$ (cor. to 1 d.p.) is obtained.]*

(b) Figure (2) shows a set up which is modified from the one in Figure (1). The lower part of the cone is cut off and sealed to form a frustum of height 19 cm. The two vessels are then held vertically on the same horizontal platform. To begin with, the cylinder is empty and the frustum is full of water. Water is then transferred into the cylinder from the frustum until the water in both vessels reaches the same horizontal level. Find the depth of water in the cylinder.

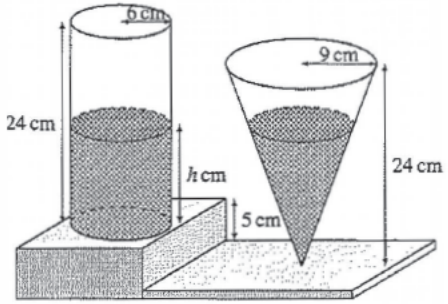


Figure (1)

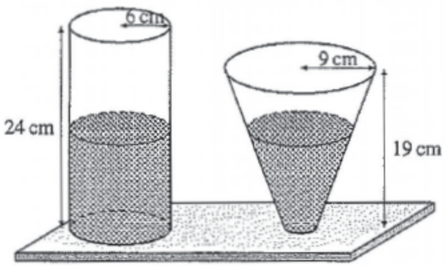
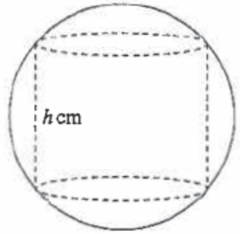


Figure (2)

15B.16 HKCEE MA 2004 – I – 14

In the figure, a solid right circular cylinder of height h cm and volume V cm^3 is inscribed in a thin hollow sphere of radius 12 cm.



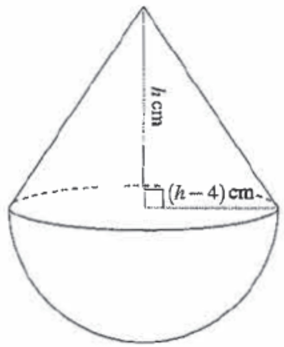
(a) Prove that $V=144\pi h-\frac{\pi}{4}h^3$.

(b) *[Out of syllabus]*

(c) If the volume of the cylinder is 286π cm^3, find the exact height(s) of the cylinder.

15B.17 HKCEE MA 2005 – I – 12

The figure shows a solid consisting of a right circular cone and a hemisphere with a common base. The height and the base radius of the cone are h cm and $(h-4)$ cm respectively. It is known that the volume of the cone is equal to the volume of the hemisphere.



(a) Find h.

(b) Find the total surface area of the solid correct to the nearest cm^2.

(c) If the solid is cut into two identical parts, find the increase in the total surface area correct to the nearest cm^2.

15B.18 HKCEE MA 2009 – I – 13

(a) The height and the base radius of an inverted right circular conical container are 18 cm and 12 cm respectively.

  (i) Find the capacity of the circular conical container in terms of π.

  (ii) Figure (1) shows a frustum which is made by cutting off the lower part of the container. The height of the frustum is 6 cm. Find the volume of the frustum in terms of π.

(b) Figure (2) shows a vessel which is held vertically. The vessel consists of two parts with a common base: the upper part is the frustum shown in Figure (1) and the lower part is a right circular cylinder of height 10 cm. Some water is poured into the vessel. The vessel now contains 884π cm^3 of water.

  (i) Find the depth of water in the vessel.

  (ii) If a piece of metal of volume 1000 cm^3 is then put into the vessel and the metal is totally immersed in the water, will the water overflow? Explain your answer.

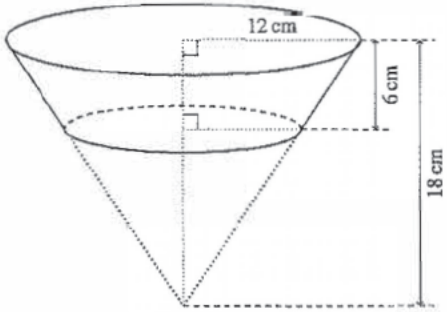


Figure (1)

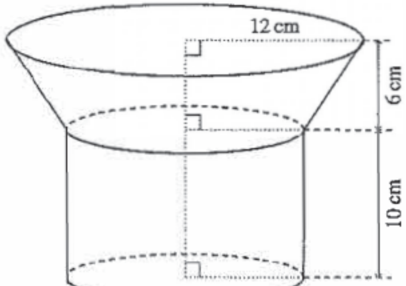


Figure (2)

15B.19 HKCEE MA 2011 – I – 13

Figure (1) shows the thin paper sector $OXYZ$ of area $2880\pi\ \text{mm}^2$. By joining OX and OZ together, $OXYZ$ is folded to form an inverted right circular conical container as shown in Figure (2).

(a) Find the length of OX.

(b) Find the height of the container

(c) Suppose that the container is held vertically. If water of volume $150\ \text{cm}^3$ is poured into the container, will the water overflow? Explain your answer.

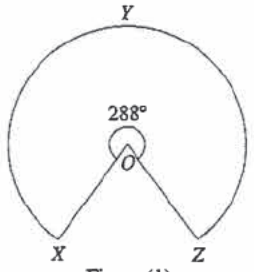


Figure (1)

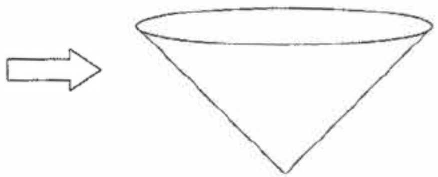


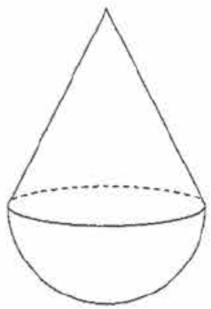
Figure (2)

15B.20 HKDSE MA SP – I – 6

The figure shows a solid consisting of a hemisphere of radius r cm joined to the bottom of a right circular cone of height 12 cm and base radius r cm. It is given that the volume of the circular cone is twice the volume of the hemisphere.

(a) Find r.

(b) Express the volume of the solid in terms of π.

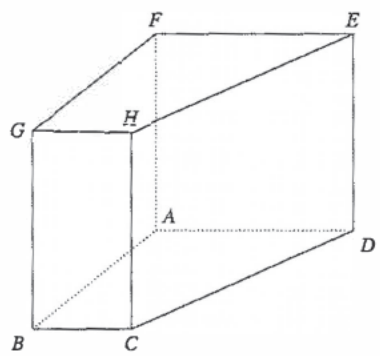


15B.21 HKDSE MA 2012 – I – 9

In the figure, the volume of the solid right prism $ABCDEFGH$ is $1020\ \text{cm}^3$. The base $ABCD$ of the prism is a trapezium, where AD is parallel to BC. It is given that $\angle BAD = 90°$, $AB = 12\ \text{cm}$, $BC = 6\ \text{cm}$ and $DE = 10\ \text{cm}$. Find

(a) the length of AD,

(b) the total surface area of the prism $ABCDEFGH$.



15B.22 HKDSE MA 2012 I – 12

Figure (1) shows a solid metal right circular cone of base radius 48 cm and height 96 cm.

(a) Find the volume of the circular cone in terms of π.

(b) A hemispherical vessel of radius 60 cm is held vertically on a horizontal surface. The vessel is fully filled with milk.

(i) Find the volume of the milk in the vessel in terms of π.

(ii) The circular cone is now held vertically in the vessel as shown in Figure (2). A craftsman claims that the volume of the milk remaining in the vessel is greater than $0.3\ \text{m}^3$. Do you agree? Explain your answer.

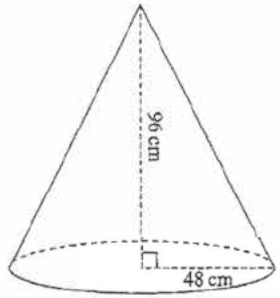


Figure (1)

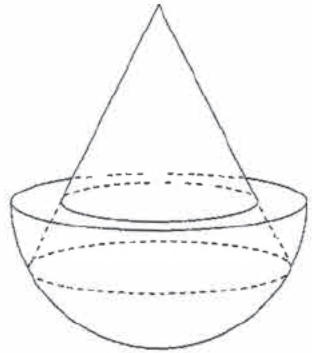


Figure (2)

15B.23 HKDSE MA 2020 – I – 12

The height and the base radius of a solid right circular cone are 36 cm and 15 cm respectively. The circular cone is divided into three parts by two planes which are parallel to its base. The heights of the three parts are equal. Express, in terms of π,

(a) the volume of the middle part of the circular cone; (3 marks)

(b) the curved surface area of the middle part of the circular cone. (3 marks)

15C Similar plane figures and solids

15C.1 HKCEE MA 1981(1/2/3) I – 1

The capacities of two spherical tanks are in the ratio $27:64$. If 72 kg of paint is required to paint the outer surface of the smaller tank, then how many kilograms of paint would be required to paint the outer surface of the bigger tank?

15C.2 HKCEE MA 1987(A/B) I 9

Figure (1) shows a test-tube consisting of a hollow cylindrical tube joined to a hemisphere bowl of the same radius. The height of the cylindrical tube is h cm and its radius is r cm. The capacity of the test-tube is $108\pi\text{cm}^3$. The capacity of the hemispherical part is $\frac{1}{6}$ of the whole test tube.

(a) (i) Find r and h.

(ii) The test tube is placed upright and water is poured into it until the water level is 4 cm beneath the rim as shown in Figure (2). Find the volume of the water. (Leave your answer in terms of π.)

(b) The water in the test-tube is poured into a right circular conical vessel placed upright as shown in Figure (3). If the depth of water is half the height of the vessel, find the capacity of the vessel. (Leave your answer in terms of π.)

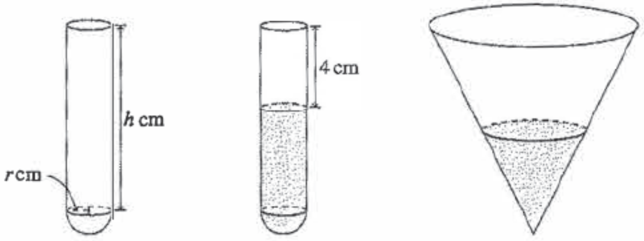


Figure (1) Figure (2) Figure (3)

15C.3 HKCEE MA 1992 – I – 12

Figure (1) shows a vertical cross-section of a separating funnel with a small tap at its vertex. The funnel is in the form of a right circular cone of base radius 9 cm and height 20 cm. It contains oil and water (which do not mix) of depths 5 cm and 10 cm respectively, with the water at the bottom.

(a) (i) Find the capacity of the separating funnel in terms of π.

(ii) Find the ratios volume of water : total volume of oil and water : capacity of the funnel.
Hence, or otherwise, find the ratios volume of water : volume of oil : capacity of the funnel.

(b) All the water in the funnel is drained through the tap into a glass tube of height 15 cm. The glass tube consists of a hollow cylindrical upper part of radius 3 cm and a hollow hemispherical lower part of the same radius, as shown in Figure (2). Find the depth of the water in the glass tube.

(c) After all the water has been drained into the glass tube, find the depth of the oil remaining in the funnel.

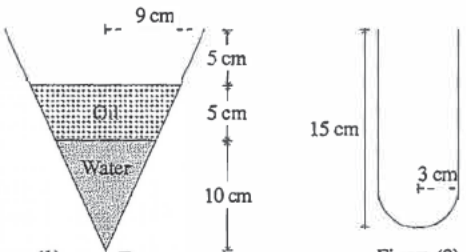


Figure (1) Figure (2)

15C.4 HKCEE MA 1994 – I – 2(e)

The ratio of the radii of two spheres is $2:3$. Find the ratio of their volumes.

15C.5 HKCEE MA 1997 – I – 7

(To continue as 8C.8.)

The ratio of the volumes of two similar solid circular cones is $8:27$.

(a) Find the ratio of the height of the smaller cone to the height of the larger cone.

15C.6 HKCEE MA 2000 – I 8

On a map of scale $1:5000$, the area of the passenger terminal of the Hong Kong International Airport is 220cm^2. What is the actual area, in m^2, occupied by the terminal on the ground?

15C.7 HKCEE MA 2002 – I – 6

The radius of a circle is 8 cm. A new circle is formed by increasing the radius by 10%.

(a) Find the area of the new circle in terms of π.

(b) Find the percentage increase in the area of the circle.

15C.8 HKCEE MA 2002 I – 11

(Continued from 8C.13.)

The area of a paper bookmark is A cm^2 and its perimeter is P cm. A is a function of P. It is known that A is the sum of two parts, one part varies as P and the other part varies as the square of P. When $P=24$, $A=36$ and when $P=18$, $A=9$.

(a) Express A in terms of P.

(b) (i) The best-selling paper bookmark has an area of 54cm^2. Find the perimeter of this bookmark.

(ii) The manufacturer of the bookmarks wants to produce a gold miniature similar in shape to the best selling paper bookmark. If the gold miniature has an area of 8cm^2, find its perimeter.

15C.9 HKCEE MA 2003 I – 13

Sector OCD is a thin metal sheet. The sheet $ABCD$ is formed by cutting away sector OBA from sector OCD as shown in Figure (1).
It is known that $\angle COD = x°$, $AD = BC = 24\text{cm}$, $OA = OB = 56\text{cm}$ and $\overparen{CD} = 30\pi\text{cm}$.

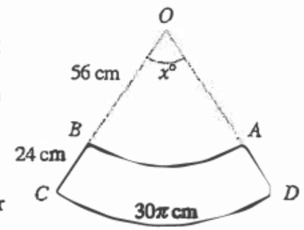


Figure (1)

(a) (i) Find x.

(ii) Find, in terms of π, the area of $ABCD$.

(b) Figure (2) shows another thin metal sheet $EFGH$ which is similar to $ABCD$. It is known that $FG = 18$ cm.

(i) Find, in terms of π, the area of $EFGH$.

(ii) By joining EH and FG together, $EFGH$ is then folded to form a hollow frustum of base radius r cm as shown in Figure (3). Find r.

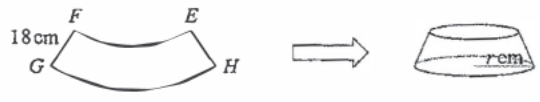


Figure (2) Figure (3)

15C.10 HKCEE MA 2006 – I 13

In Figure (1), the frustum of height 8 cm is made by cutting off a right circular cone of base radius 3 cm from a solid right circular cone of base radius 6 cm. Figure (2) shows the solid X formed by fixing the frustum onto a solid hemisphere of radius 6 cm. The solid Y in Figure (3) is similar to X. The ratio of the surface area of X to the surface area of Y is $4:9$.

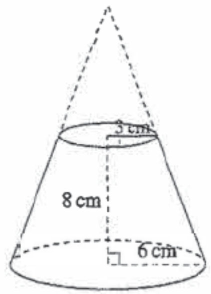
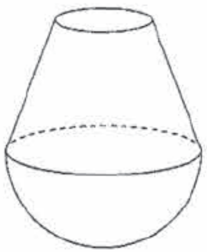
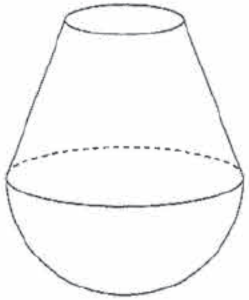


Figure (1)



Solid X

Figure (2)

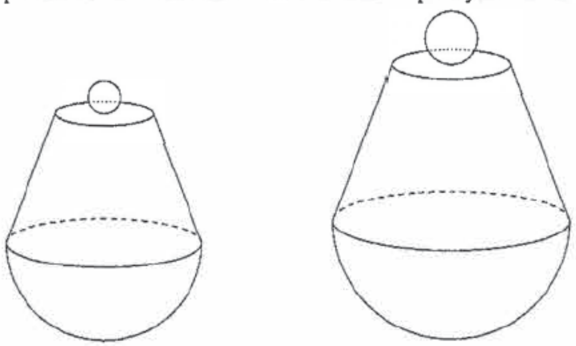


Solid Y

Figure (3)

(a) Find the volume of X and the volume of Y. Give your answers in terms of π.

(b) In Figure (4), the solid X' is formed by fixing a solid sphere of radius 1 cm onto the centre of the top circular surface of X while another solid Y' is formed by fixing a solid sphere of radius 2 cm onto the centre of the top circular surface of Y. Are X' and Y' similar? Explain your answer.

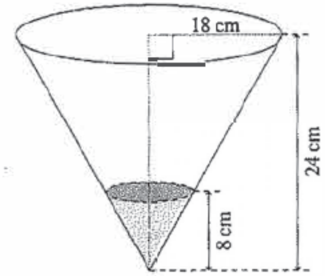


Solid X' Solid Y'

Figure (4)

15C.11 HKCEE MA 2007 – I 11

The figure shows an inverted right circular conical vessel which is held vertically. The height and the base radius of the vessel are 24 cm and 18 cm respectively. The vessel contains some water and the depth of the water is 8 cm.



(a) Find the volume of water contained in the vessel in terms of π.

(b) (i) Find the area of the wet curved surface of the vessel in terms of π.

(ii) Another inverted right circular conical vessel with height 36 cm and base radius 27 cm is held vertically. This bigger vessel and the vessel shown in the figure contain the same volume of water. Find the area of the wet curved surface of the bigger vessel in terms of π.

15C.12 HKCEE MA 2008 – I – 13

In Figure (1), sector $OABC$ is a thin metal sheet. By joining OA and OC together, $OABC$ is folded to form a right ciruclar cone X as shown in Figure (2). It is given that $OA = 20$ cm.

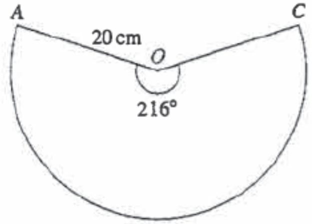


Figure (1)

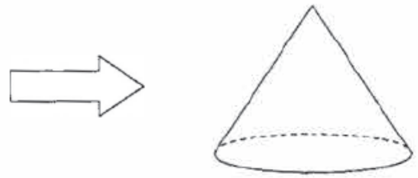


Figure (2)

(a) Find the base radius and the height of X.

(b) Find the volume of X in terms of π.

(c) In Figure (3), sector $PDEF$ is another thin metal sheet. By joining PD and PF together, $PDEF$ is folded to form another right circular cone Y as shown in Figure (4). It is given that $PD = 10$ cm. Are X and Y similar? Explain your answer.

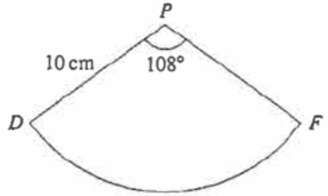


Figure (3)

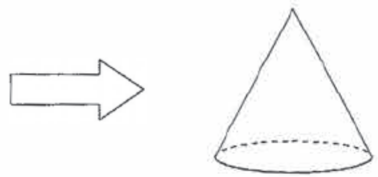


Figure (4)

15C.13 HKCEE MA 2010 – I – 13

In Figure (1), $ABCDEF$ is a wooden block in the form of a right prism. It is given that $AB = AC = 17$ cm, $BC = 16$ cm and $CD = 20$ cm.

(a) Find the area of $\triangle ABC$.

(b) Find the volume of the wooden block $ABCDEF$.

(c) The plane $PQRS$ which is parallel to the face $BCDF$ cuts the wooden block $ABCDEF$ into two blocks $APQRES$ and $BCQPSFDR$ as shown in Figure (2). It is given that $PQ = 4$ cm.

  (i) Find the volume of the wooden block $APQRES$.

  (ii) Are the wooden blocks $APQRES$ and $ABCDEF$ similar? Explain your answer.

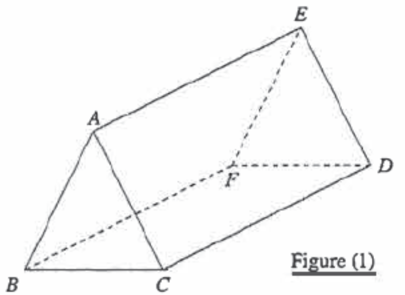


Figure (1)

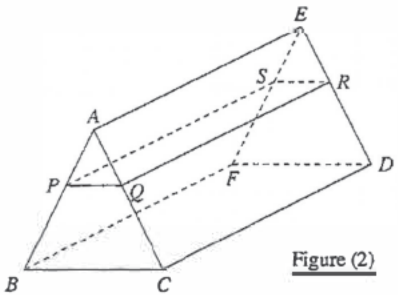


Figure (2)

15C.14 HKDSE MA 2012 – I – 11 (Continued from 8C.23.)

Let \$$C$ be the cost of painting a can of surface area A m^2. It is given that C is the sum of two parts, one part is a constant and the other part varies as A. When $A = 2$, $C = 62$; when $A = 6$, $C = 74$.

(a) Find the cost of painting a can of surface area 13 m^2.

(b) There is a larger can which is similar to the can described in (a). If the volume of the larger can is 8 times that of the can described in (a), find the cost of painting the larger can.

15C.15 HKDSE MA 2013 – I – 13

In a workshop, 2 identical solid metal right circular cylinders of base radius R cm are melted and recast into 27 smaller identical solid right circular cylinders of base radius r cm and height 10 cm. It is given that the base area of a larger circular cylinder is 9 times that of a smaller one.

(a) Find

  (i) $r : R$,

  (ii) the height of a larger circular cylinder.

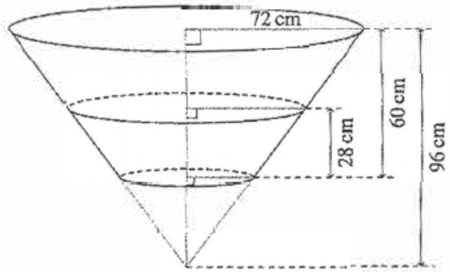
(b) A craftsman claims that a smaller circular cylinder and a larger circular cylinder are similar. Do you agree? Explain your answer.

15C.16 HKDSE MA 2014 – I – 14

The figure shows a vessel in the form of a frustum which is made by cutting off the lower part of an inverted right circular cone of base radius 72 cm and height 96 cm. The height of the vessel is 60 cm. The vessel is placed on a horizontal table. Some water is now poured into the vessel. John finds that the depth of water in the vessel is 28 cm.

(a) Find the area of the wet curved surface of the vessel in terms of π.

(b) John claims that the volume of water in the vessel is greater than 0.1 m^3. Do you agree? Explain your answer.



15C.17 HKDSE MA 2016 – I – 11

An inverted right circular conical vessel contains some milk. The vessel is held vertically. The depth of milk in the vessel is 12 cm. Peter then pours 444π cm^3 of milk into the vessel without overflowing. He now finds that the depth of milk in the vessel is 16 cm.

(a) Express the final volume of milk in the vessel in terms of π.

(b) Peter claims that the final area of the wet curved surface of the vessel is at least 800 cm^2. Do you agree? Explain your answer.

15C.18 HKDSE MA 2017 – I – 12

A solid metal right prism of base area 84 cm^2 and height 20 cm is melted and recast into two similar solid right pyramids. The bases of the two pyramids are squares. The ratio of the base area of the smaller pyramid to the base area of the larger pyramid is 4 : 9.

(a) Find the volume of the larger pyramid.

(b) If the height of the larger pyramid is 12 cm, find the total surface area of the smaller pyramid.

15C.19 HKDSE MA 2018 – I – 14

A right circular cylindrical container of base radius 8 cm and height 64 cm and an inverted right circular conical vessel of base radius 20 cm and height 60 cm are held vertically. The container is fully filled with water. The water in the container is now poured into the vessel.

(a) Find the volume of water in the vessel in terms of π.

(b) Find the depth of water in the vessel.

(c) If a solid metal sphere of radius 14 cm is then put into the vessel and the sphere is totally immersed in the water, will the water overflow? Explain your answer.

15C.20 HKDSE MA 2019 – I – 9

The sum of the volumes of two spheres is 324π cm^3. The radius of the larger sphere is equal to the diameter of the smaller sphere. Express, in terms of π,

(a) the volume of the larger sphere;

(b) the sum of the surface areas of the two spheres.

15 Mensuration

15A Lengths and areas of plane figures

15A.1 HKCEE MA 1980(1/1*/3) I – 10

(a) $\angle PAX = 2\theta$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
Similarly, $\angle QBX = \angle RCX = 2\theta$

(b) Areas of sector $OAP : OBQ : OCR = (OA : OB : OC)^2$
$= 4 : 9 : 16$

(c) $\cos\angle RCX = \frac{CD}{CR} = \frac{2}{4} = \frac{1}{2} \Rightarrow 2\theta = 60° \Rightarrow \theta = 30°$

15A.2 (HKCEE MA 1981(1/2/3) – I – 12)

(a) $AC = 10\sin(138° \div 2) \times 2 = 18.6716$ (m)
$\therefore BC = \sqrt{AB^2 + AC^2} = 21.2$ (m, 3 s.f.)

(b) Area of $ABDC = \frac{138°}{360°} \times$ C.S.A. of cylinder
$= \frac{138°}{360°} \times 2\pi(10)(10) = 241$ (cm², 3 s.f.)

(c) (Imagine the curved $ABDC$ is straightened.)
Length of curve $= \sqrt{AB^2 + (\widehat{AC})^2} = 26.1$ m (3 s.f.)

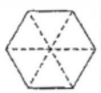
15A.3 HKCEE MA 1982(1/2/3) – I – 4

$\angle BOC \quad \tan^{-1}\frac{2\sqrt{3}}{6} = 30°$

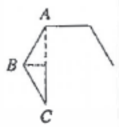
$\therefore$ Areaw $= \frac{30°}{360°} \times \pi(6)^2 = 3\pi$

15A.4 HKCEE MA 1982(1/2/3) – I – 9

(a) Divide the hexagon into 6 equal parts.
Area of hexagon $= 6 \times \frac{1}{2} \times 1 \times 1 \times \sin 60°$
$\frac{3\sqrt{3}}{2}$



$AC = 2 \times AB\sin 60° = \sqrt{3}$



(b) (i) In $\triangle ABC$, $\angle BAC = 30°$
Similarly, $\angle ABF = 30°$
$\Rightarrow AP = BP$ and, similarly, $BQ = QC$
Besides, $\angle PBQ = 120° - 30° - 30° = 60°$
Hence, $\triangle BPQ$ is equilateral. $\Rightarrow AP = PQ = QC$
$\Rightarrow PQ = \frac{1}{3}AC = \frac{\sqrt{3}}{3}$

(ii) Area $= 6 \times \frac{1}{2} \times \left(\frac{\sqrt{3}}{3}\right)^2 \sin 60° = \frac{\sqrt{3}}{2}$

15A.5 HKCEE MA 1983(A/B) – I – 5

(a) $OC = OB - CB = 15$
$\angle AOC = \cos^{-1}\frac{OC}{OA} = 60°$

(b) $\widehat{AB} = \frac{60°}{360°} \times 2\pi(30) = 20\pi$

15A.6 HKCEE MA 1988 – I – 5

(a) Area of $OABC = \frac{100°}{360°} \times \pi(10)^2 = 87.27$ (2 d.p.)

(b) Area of $\triangle OAC = \frac{1}{2}(10)^2\sin 100° = 49.24$ (2 d.p.)

(c) Area of $ABC = 87.27 - 49.24 = 38.03$ (2 d.p.)

15A.7 HKCEE MA 1992 – I – 7

(a) $\angle AOB = 360° \div 5 = 72°$
Area of $\triangle OAB = \frac{1}{2}(10)^2\sin 72° = 47.533 = 47.6$ (3 s.f.)

(b) Shaded area $= \frac{72°}{360°} \times \pi(10)^2 - 47.553 = 15.3$ (3 s.f.)

15A.8 HKCEE MA 1994 – I 2(d)

Method 1

$\angle$ subtended $= \frac{2\pi}{2\pi(5)} \times 360° = 72°$

$\therefore$ Area of sector $\frac{72°}{360°} \times \pi(5)^2 = 5\pi = 15.7$ (3 s.f.)

Method 2

Area of sector $=$ Area of circle $\times \frac{\text{Arc length}}{\text{Circumference}}$
$= \pi(5)^2 \times \frac{2\pi}{2\pi(5)} = 5\pi = 15.7$ (3 s.f.)

15A.9 HKCEE MA 1999 – I – 9

(a) $r\sin 60° = 5 \Rightarrow r = \frac{10}{\sqrt{3}} = 5.77$ (3 s.f.)

(b) Area $= \frac{120°}{360°} \times \pi r^2 - \frac{1}{2}r^2\sin 120° = 20.5$ (cm², 3 s.f.)

15A.10 HKCEE MA 2000 – I – 3

Area $= \frac{75°}{360°} \times \pi(6)^2 = 2.20$ (cm², 3 s.f.)

15A.11 HKCEE MA 2001 – I – 3

Perimeter $= \frac{50°}{360°} \times 2\pi(3) + 3 + 3 = 8.62$ (cm, 3 s.f.)

15A.12 HKCEE MA 2004 – I – 9

(a) Let r cm be the radius.
$\frac{80°}{360°} \times \pi r^2 = 162\pi \Rightarrow r = 27$
$\therefore$ The radius is 27 cm.

(b) Perimeter $= \frac{80°}{360°} \times 2\pi(27) + 27 \times 2 = 91.7$ (cm, 3 s.f.)

15A.13 HKCEE MA 2005 – I – 9

(a) $\frac{100°}{360°} \times 2\pi(OA) = 10\pi \Rightarrow OA = 18$ (cm)

(b) Area = Area of sector OAC Area of $\triangle OAC$
$= \frac{100°}{360°} \times \pi(18)^2 \quad \frac{1}{2}(18)^2\sin 100°$
$= 123$ (cm², 3 s.f.)

15A.14 HKCEE MA 2006 – I – 4

$\widehat{AB} = \frac{150°}{360°} \times 2\pi(12) = 10\pi$ (cm)

15A.15 HKCEE MA 2007 – I – 9

(a) $\frac{\angle AOB}{360°} \times 2\pi(40) = 16\pi \Rightarrow \angle AOB = 72°$

(b) Area $= \frac{72°}{360°} \times \pi(40)^2 = 320\pi$ (cm²)

15A.16 HKDSE MA 2015 – I – 9

(a) $\frac{\text{Angle}}{360°} \times \pi(12)^2 = 30\pi \Rightarrow$ Angle $= 75°$

(b) Perimeter $= \frac{75°}{360°} \times 2\pi(12) + 12 \times 2 = 5\pi + 24$ (cm)

15B Volumes and surface areas of solids

15B.1 HKCEE MA 1983(A/B) – I – 8

(a) Volume of cylinder = Volume of hemisphere
$\pi r^2 h = \frac{4}{3}\pi r^3 \div 2$
$h = \frac{2}{3}r \Rightarrow r : h = 3 : 2$

(b) (i) $\because h = \frac{2}{3}r$
$\therefore 136 = \frac{1}{2}(2\pi r) + h + (2r) + h$
$136 = \pi r + 2r + 2\left(\frac{2}{3}r\right)$
$r = 136 \div \left(\pi + \frac{10}{3}\right) = 21$ (2 s.f.)

(ii) Total external s.a.
$= 4\pi r^2 \div 2 + 2\pi rh + \pi r^2$
$= 2\pi(21)^2 + 2\pi(21)\left(\frac{2}{3}\cdot 21\right) + \pi(21)^2$
$= 6000$ (cm², 1 s.f.)

15B.2 HKCEE MA 1984(A/B) – I – 12

(a) Suppose E is the mid-point of CD.
$\angle CPD = 2\angle CPE$
$= 2\tan^{-1}\frac{CE}{PE}$
$= 2\tan^{-1}\frac{20 \div 2}{20} = 53.1301° = 53.1°$ (1 d.p.)

(b) $PC = \sqrt{PE^2 + CE^2} = \sqrt{500}$
$\therefore \widehat{CQD} = \frac{53.1301°}{360°} \times 2\pi(\sqrt{500})$
$= 20.73495 = 20.7$ (cm, 1 d.p.)

(c) Area of $APBCQD$
$=$ Area of sector $PCD + 2 \times$ Area of $\triangle PBC$
$= \frac{53.1301°}{360°} \times \pi(\sqrt{500})^2 + 2 \times \frac{20 \times 10}{2}$
$= 431.8$ (cm², 1 d.p.)

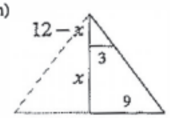
(d) T.S.A. $= BR \times$ Perimeter of $APBCQD$
$= 20 \times (20 \times 3 + \widehat{CQD}) = 1614.7$ (cm², 1 d.p.)

15B.3 HKCEE MA 1985(A/B) – I 11

(a) (i) $135\pi = \pi\ 9 \cdot OP \Rightarrow OP = 15$ (cm)

(ii) Height $= \sqrt{15^2 - 9^2} = 12$ (cm)

(b) (i) By $\sim\triangle$s, $\frac{12 - x}{x} = \frac{3}{9} = \frac{1}{3}$
$3(12 - x) = x$
$x = 8$



(ii) *Method 1*
Volume remained
$=$ Big cone $-$ Small cone $-$ Cylinder
$= \frac{1}{3}\pi(9)^2(12) - \frac{1}{3}\pi(3)^2(12 - 8) - \pi(3)^2(8)$
$= 240\pi$ (cm³)

Method 2

Vol of frustum $=$ Vol of big cone $\times \left[1 \quad \left(\frac{3}{9}\right)^3\right]$
$= \frac{1}{3}\pi(9)^2(12) \times \left(1 \quad \frac{1}{27}\right)$
$= 324\pi \times \frac{26}{27} = 312\pi$ (cm³)

$\therefore$ Vol remained $= 312\pi - \pi(3)^2(8) = 240\pi$ (cm³)

15B.4 HKCEE MA 1986(A/B) – I – 12

(a) (i) Let h be the height of the cone.

$\frac{1}{3}\pi(6)^2(h)=\frac{4}{3}\cdot\left[\frac{4}{3}\pi(6)^3\div 2\right]$

$12\pi h=\frac{4}{3}(144\pi) \Rightarrow h=16$

$\therefore$ The height of the cone is 16.

(ii) $\text{Vol}=144\pi+144\pi\times\frac{4}{3}=336\pi$

(b) (i) By $\sim\Delta$s, $\frac{x}{y}=\frac{6}{16}=\frac{3}{8}$

(ii) $\frac{1}{3}\pi x^2y=336\pi\div 2$

$\frac{1}{3}\pi\left(\frac{3}{8}y\right)^2 y=168\pi$

$\frac{3}{64}\pi y^3=168\pi \Rightarrow y=15.3$ (1 d.p.)

15B.5 HKCEE MA 1989 – I – 11

(a) Vol of water $=\frac{(10+2)\times 50}{2}\times 20=6000$ (m^3)

(b) (i) (The cross-section would change from a trapezium to a triangle.)

Vol of water remaining $=\frac{8\times 50}{2}\times 20=4000$ (m^3)

(ii) Vol of water through pipe in 1 second

$=\pi(0.125)^2(3)=0.046875\pi$ (m^3)

$\therefore$ Vol of water pumped in 8 hours

$=0.046875\pi\times 8\times 60\times 60$

$=1350\pi=4241$ (m^3, nearest m^3)

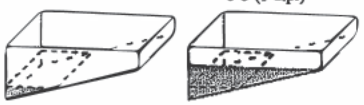
(iii) Vol of water remaining after 8 hours

$=6000-1350\pi=1758.8499$ (m^3)

Since the cross-section right-angled Δs are similar,

$\frac{1758.8499}{4000}=\left(\frac{h\text{ m}}{8\text{ m}}\right)^2 \Rightarrow h=\sqrt{\frac{1758.8499}{4000}}\times 8$

$=5.3$ (1 d.p.)



15B.6 HKCEE MA 1990 – I – 11

(a) (i) $S=2\pi r^2+2\pi rh$

(ii) $V=\pi r^2h \Rightarrow h=\frac{V}{\pi r^2}$

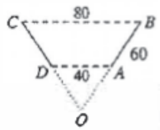
$\therefore S=2\pi r^2+2\pi r\left(\frac{V}{\pi r^2}\right)=2\pi r^2+\frac{2V}{r}$

15B.7 HKCEE MA 1991 I – 11

(a) By $\sim\Delta$s, $\frac{OA}{OA+60}=\frac{40}{80}=\frac{1}{2}$

$2OA=OA+60$

$OA=60$ (cm)



In Figure (2), $\widehat{ADA'}$ = Base $\odot^{ce}$ of bucket

$=2\pi(40\div 2)=40\pi$ (cm)

$\therefore \frac{\angle AOA'}{360^\circ}\times 2\pi(OA)=40\pi$

$\angle AOA'=\frac{40\pi}{120\pi}\cdot 360^\circ=120^\circ$

(b) Area of $ABB'A'=\frac{120^\circ}{360^\circ}\left[\pi(60+60)^2-\pi(60)^2\right]$

$=3600\pi$ (cm^2)

(c) The shortest path is AC in Figure (2).

Method 1

Since $OA=OB$ and $\angle AOC=120^\circ\div 2=60^\circ$, $\triangle OBC$ is equilateral.

$\therefore$ Required path $=OC\sin 60^\circ=120\cdot\frac{\sqrt{3}}{2}=60\sqrt{3}$ (cm)

Method 2

Required path $=\sqrt{OA^2+OC^2-2OA\cdot OC\cos\angle AOC}$

$=\sqrt{60^2+120^2-2\cdot 60\cdot 120\cos 60^\circ}$

$=\sqrt{10800}$ (cm)

15B.8 (HKCEE MA 1993 I 9)

(a) (i) Area of sector $OACB=\frac{\alpha}{360^\circ}\times\pi r^2$

(ii) Area of segment $ACB=\frac{\alpha}{360^\circ}\times\pi r^2-\frac{1}{2}r^2\sin\alpha$

(iii) $\because$ A of segment $ACB=\frac{1}{5}$(A of circle)

$\therefore \left(\frac{\alpha\pi}{360^\circ}-\frac{\sin\alpha}{2}\right)r^2=\frac{1}{5}(\pi r^2)$

$\frac{\alpha\pi}{360^\circ}-\frac{\sin\alpha}{2}=\frac{1}{5}$

$\sin\alpha=\left(\frac{\alpha}{180^\circ}-\frac{2}{5}\right)\pi$

(b) Required ratio $=\frac{\text{major }\widehat{AB}}{\text{minor }\widehat{AB}}=\frac{360^\circ\quad\alpha}{\alpha}=1.98:1$ (3 s.f.)

15B.9 HKCEE MA 1994 – I – 10

(a) Vol of water $=\pi(2)^2(1.5)=6\pi$ (m^3)

(b) $\pi(2)^2h=\frac{4}{3}\pi(0.6)^3$

$h=\frac{\frac{4}{3}\pi(0.6)^3}{4\pi}=0.072$

(c) (i) $\cancel{\pi(2)^2}(2r\quad 1.5)=\frac{\cancel{4}}{3}\cancel{\pi}r^3$

$2r-1.5=\frac{1}{3}r^3 \Rightarrow 2r^3-12r+9=0$

15B.10 HKCEE MA 1995 – I – 13

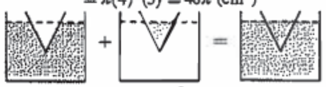
(a) Vol of water $=\frac{1}{3}\pi(6)^2(12)=144\pi$ (cm^3)

(b) Consider (the cross-section of) the entire conical vessel and (the cross-section of) the part of the conical vessel inside the cylindrical vessel.

$\frac{6}{12}=\frac{4}{11-d} \Rightarrow 11-d=8 \Rightarrow d=3$

(c) (i) Vol leaked = Vol of water in cylindrical vessel

$=\pi(4)^2(3)=48\pi$ (cm^3)

(ii) 

$104\pi+\frac{1}{3}\pi\left(\frac{h}{2}\right)^2(h)=\pi(4)^2(3+h)$

$1248+h^3=192(3+h)$

$h^3-192h+672=0$

15B.11 HKCEE MA 1996 – I – 8

(a) Cap of cup $=\frac{1}{3}\pi(5)^2(12)=100\pi=314$ (cm^3, 3 s.f.)

(b) (i) Area of sector = C.S.A. of cone

$=\pi(5)\sqrt{5^2+12^2}$

$=\pi(5)(13)=65\pi=204$ (cm^2, 3 s.f.)

(ii) $\frac{\angle\text{ of sector}}{360^\circ}\times\pi(13)^2=65\pi$

$\angle$ of sector $=138^\circ$ (3 s.f.)

15B.12 HKCEE MA 1997 – I – 12

(a) (i) In $\triangle VMN$, $NM=6\div 2=3$ (m)

$VN=NM\tan\theta=3\tan\theta$ (m)

$VM=\frac{NM}{\cos\theta}=\frac{3}{\cos\theta}$ (m)

(ii) $\text{Cap}=\frac{1}{3}\times 6\times 6\times 3\tan\theta=36\tan\theta$ (m^3)

T.S.A. $=4\times\frac{6\times\frac{3}{\cos\theta}}{2}=\frac{36}{\cos\theta}$ (m^2)

(b) (i) $6\times 6=\pi r^2 \Rightarrow r=\frac{6}{\sqrt{\pi}}$

(ii) $\pi r^2h=36\tan\theta \Rightarrow (36)h=36\tan\theta \Rightarrow h\quad\tan\theta$

(iii) $2\pi rh+\pi r^2=\frac{36}{\cos\theta}$

$2\pi\left(\frac{6}{\sqrt{\pi}}\right)(\tan\theta)+(36)=\frac{36}{\cos\theta}$

$12\sqrt{\pi}\tan\theta+36=\frac{36}{\cos\theta}$

$\sqrt{\pi}\tan\theta+3=\frac{3}{\cos\theta}$

15B.13 HKCEE MA 1998 – I – 1

Volume $=\frac{(2+6)\times 3}{2}\times 8=96$ (cm^3)

15B.14 HKCEE MA 1999 – I – 13

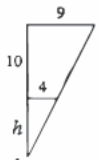
(a) Capacity of hole $=\frac{4}{3}\pi(5)^3\times\frac{1}{2}+\pi(5)^2(9-5)$

Capacity of hole $=\frac{550}{3}\pi$ (cm^3)

(b) By $\sim\Delta$s, $\frac{h}{h+10}=\frac{4}{9}$

$9h=4h+40$

$h=8$



$\therefore$ Vol of frustum $=\frac{1}{3}\pi(9)^2(10+8)-\frac{1}{3}\pi(4)^2(8)$

$=\frac{1330}{3}\pi$ (cm^3)

$\therefore$ Vol of wood $=\frac{1330}{3}\pi-\frac{550}{3}\pi=260\pi$ (cm^3)

15B.15 HKCEE MA 2002 – I – 15

(a) (i) Total vol of water $=\frac{1}{3}\pi(9)^2(24)=648\pi$ (cm^3)

Vol of water remained in cone $=648\pi\times\left(\frac{h+5}{24}\right)^3$

$=\frac{3}{64}\pi(h+5)^3$

$\therefore \frac{3}{64}\pi(h+5)^3=648\pi-\pi(6)^2h$

$(h+5)^3=13824-768h$

$h^3+15h^2+75h+125=13824-768h$

$h^3+15h^2+843h-13699=0$

(b) (The final situation in Figure (2) is the same as Figure (1) with the lowest 5 cm removed.)

Depth of water = 11.8 cm

15B.16 HKCEE MA 2004 I 14

(a) Base radius of cylinder $=\sqrt{12^2-\left(\frac{h}{2}\right)^2}=\sqrt{144-\frac{h^2}{4}}$

$\therefore V=\pi\left(144-\frac{h^2}{4}\right)(h)=144\pi h-\frac{\pi}{4}h^3$

(c) $144\pi h-\frac{\pi}{4}h^3=286\pi \Rightarrow h^3-576h+1144=0$

Since $(2)^3-576(2)+1144=0$, $h-2$ is a factor.

$(h-2)(h^2+2h-572)=0$

$h=2$ or $\frac{-2\pm\sqrt{4+2288}}{2}$

$=2$ or $\sqrt{573}-1$ or $-\sqrt{573}-1$ (rj.)

Hence, the height is 2 cm or $(\sqrt{573}-1)$ cm.

15B.17 HKCEE MA 2005 – I – 12

(a) $\frac{1}{3}\pi(h-4)^2h=\frac{4}{3}\pi(h-4)^3\div 2$

$h=2(h-4) \Rightarrow h=8$

(b) T.S.A. $=\pi(h-4)\sqrt{h^2+(h-4)^2}+4\pi(h-4)^2\div 2$

$=\pi(8)\sqrt{8^2+4^2}+2\pi(4)^2$

$=325$ (cm^2, nearest cm^2)

(c) Increase $=2\times(\triangle+\text{semi-circle})$

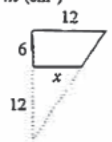
$=2\times\left[\frac{8\times 8}{2}+\frac{\pi(4)^2}{2}\right]=114$ (cm^2, nrst cm^2)

15B.18 HKCEE MA 2009 I – 13

(a) (i) Capacity $=\frac{1}{3}\pi(12)^2(18)=864\pi$ (cm^3)

(ii) By $\sim\Delta$s, $\frac{x}{12}=\frac{18-6}{18}$

$x=8$



$\therefore$ Vol of frustum $=864\pi-\frac{1}{3}\pi(8)^2(12)$

$=608\pi$ (cm^3)

(b) (i) Cap of cylinder $=\pi(8)^2(10)=640\pi$ (cm^3)

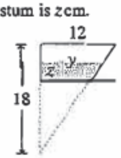
$\therefore$ Vol of water in the frustum part

$=884\pi-640\pi=244\pi$ (cm^3)

Suppose the depth of water in the frustum is z cm.

By $\sim\Delta$s, $\frac{z+12}{18}=\frac{y}{12}$

$y=\frac{2}{3}(z+12)$



$\therefore 244\pi=\frac{1}{3}y^2(z+12)-256\pi$

$500\pi=\frac{1}{3}\pi\left(\frac{2}{3}(z+12)\right)^2(z+12)$

$500=\frac{4}{27}(z+12)^3$

$(z+12)^3=3375 \Rightarrow z+12=15 \Rightarrow z=3$

Hence, depth of water in vessel is $10+3=13$ (cm).

(ii) Cap of vessel $=640\pi+608\pi=1248\pi=3920$ (cm^3)

Vol of water + metal $=884\pi+1000=3777$ (cm^3)

< Cap of vessel

$\therefore$ NO.

15B.19 HKCEE MA 2011 – I – 13

(a) $\frac{288^\circ}{\cancel{360^\circ}} \times \pi(OX)^2 = 2880\pi \Rightarrow OX = 60$ (mm)

(b) $\widehat{XYZ} = \frac{288^\circ}{\cancel{360^\circ}} \times 2\pi(60) = 96\pi$ (mm)

$\therefore$ Base radius of container $= \frac{96\pi}{2\pi} = 48$ (mm)

$\Rightarrow$ Height of container $= \sqrt{60^2 - 48^2} = 36$ (mm)

(c) Cap of container $= \frac{1}{3}\pi(48)^2(36) = 86859$ mm^3

$= 86.859$ cm$^3 < 150$ cm^3

$\therefore$ YES.

15B.20 HKDSE MA SP – I – 6

(a) $\frac{1}{3}\pi r^2(12) = 2 \times \left(\frac{4}{3}\pi r^3 \times \frac{1}{2}\right)$

$4\pi r^2 = \frac{4}{3}\pi r^3 \Rightarrow r = 3$

(b) Volume $= \frac{2}{3}\pi(3)^3 \times 3 = 54\pi$ (cm^3)

15B.21 HKDSE MA 2012 – I – 9

(a) Base area $= \frac{\text{Volume}}{\text{Height}} = \frac{1020}{10} = 102$ (cm^2)

$\frac{(6+AD) \times 12}{2} = 102 \Rightarrow AD = 11$ (cm)

(b) Perimeter of base $= 11 + 12 + 6 + \sqrt{(11 - 6)^2 + 12^2}$

$= 42$ (cm)

$\therefore$ T.S.A. $= 2 \times 102 + 42 \times 10 = 624$ (cm^2)

15B.22 HKDSE MA 2012 – I – 12

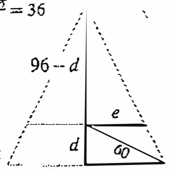
(a) Vol of cone $= \frac{1}{3}\pi(48)^2(96) = 73728\pi$ (cm^3)

(b) (i) Vol of milk $= \frac{4}{3}\pi(60)^3 \div 2 = 144000\pi$ (cm^3)

(ii) In the figure, $d = \sqrt{60^2 - 48^2} = 36$

$\frac{e}{48} = \frac{96-d}{96}$

$e = \frac{60}{96} \times 48 = 30$



Method 1

$\therefore$ Vol of part of cone in milk

$= 73728\pi - \frac{1}{3}\pi(30)^2(60)$

$= 73728\pi - 18000\pi = 55728\pi$ (cm^3)

$\therefore$ Vol of milk remaining $= 144000\pi - 55728\pi$

Method 2

$\because \frac{\text{Height of cone outside milk}}{\text{Height of the whole cone}} = \frac{96-36}{96} = \frac{5}{8}$

$\therefore$ Vol of part of cone in milk

$= 73728\pi \times \left[1 - \left(\frac{5}{8}\right)^3\right] = 55728\pi$ (cm^3)

Hence

$\therefore$ Vol of milk remaining $= 144000\pi - 55728\pi$

$= 88272\pi$ cm^3

$= 277000$ cm^3

$= 0.277$ m$^3 < 0.3$ m^3

$\therefore$ The craftsman is disagreed.

*15B.23 in the end of 15C

15C Similar plane figures and solids

15C.1 HKCEE MA 1981(1/2/3) – I – 1

$\frac{\text{S.A. of bigger tank}}{\text{S.A. of smaller tank}} = \left(\sqrt[3]{\frac{64}{27}}\right)^2$

$\frac{\text{Paint for bigger tank}}{72 \text{ kg}} = \left(\frac{4}{3}\right)^2$

Paint for bigger tank $= \frac{16}{9} \times 72 = 128$ (kg)

15C.2 HKCEE MA 1987(A/B) – I – 9

(a) (i) $108\pi =$ Vol of hemisphere $\times 6$

$108\pi = \left[\frac{4}{3}\pi(r)^3 \div 2\right] \times 6$

$108\pi = 4\pi r^3 \Rightarrow r = 3$

Vol of cylindrical part $= \frac{5}{6}(108\pi)$

$\pi(3)^2(h) = 90\pi \Rightarrow h = 10$

(ii) Vol of water $= 108\pi -$ Vol of empty space

$= 108\pi - \pi(3)^2(4) = 69\pi$ (cm^3)

(b) $\because \frac{\text{Height of vessel}}{\text{Depth of water}} = 2$

$\therefore \frac{\text{Cap of vessel}}{\text{Vol of water}} = 2^3 = 8$

Cap of vessel $= 8 \times 69\pi = 552\pi$ (cm^3)

15C.3 HKCEE MA 1992 – I – 12

(a) (i) Cap of funnel $= \frac{1}{3}\pi(9)^2(10+5+5) = 540\pi$ (cm^3)

(ii) V of water : Total v of water & oil : Cap of funnel

$= (10)^3 : (10+5)^3 : (10+5+5)^3$

$= 8 : 27 : 81$

$\therefore$ V of water : V of oil : Cap of funnel

$= 8 : (27-8) : 81 = 8 : 19 : 81$

(b) V of water $= 540\pi \times \frac{8}{81} = \frac{160}{3}\pi$ (cm^3)

In the tube,

V of water in lower part $= \frac{4}{3}\pi(3)^3 \div 2 = 18\pi$ (cm^3)

$\Rightarrow$ V of water in upper part $= \frac{160}{3}\pi - 18\pi = \frac{106}{3}\pi$ (cm^3)

$\therefore$ Depth of water $= \frac{\frac{106}{3}\pi}{\pi(3)^2} + 3 = \frac{187}{27}$ (cm)

(c) $\because \frac{\text{Vol of oil}}{\text{Cap of funnel}} = \frac{19}{81}$

$\therefore \frac{\text{Depth of oil}}{\text{Height of funnel}} = \sqrt{\frac{19}{81}}$

$\Rightarrow$ Depth of oil $= \sqrt{\frac{19}{81}} \times 20 = 9.69$ (cm, 3 s.f.)

15C.4 HKCEE MA 1994 – I – 2(e)

Ratio of volumes $= \left(\frac{2}{3}\right)^3 = 8 : 27$

15C.5 HKCEE MA 1997 – I – 7

(a) Required ratio $= \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$

15C.6 HKCEE MA 2000 – I – 8

Actual area $= 220$ cm$^2 \times (5000)^2$

$= 5500000000$ cm$^2 = 550000$ m^2

15C.7 HKCEE MA 2002 – I – 6

Method 1

(a) New radius $= 8 \times (1 + 10\%) = 8.8$ (cm)

$\therefore$ New area $= \pi(8.8)^2 = 77.44\pi$ (cm^2)

(b) Original area $= \pi(8)^2 = 64\pi$ (cm^2)

% increase $= \frac{77.44\pi - 64\pi}{64\pi} \times 100\% = 21\%$

Method 2

(a) Original area $= \pi(8)^2 = 64\pi$ (cm^2)

$\therefore$ New area $= 64\pi \times (1 + 10\%)^2 = 77.44\pi$ (cm^2)

(b) % increase $= \frac{(1+10\%)^2 - 1}{1} \times 100\% = 21\%$

15C.8 HKCEE MA 2002 – I – 11

(a) Let $A = hP + kP^2$.

$\begin{cases} 36 = 24h + 576k \\ 9 = 18h + 324k \end{cases} \Rightarrow \begin{cases} h = -\frac{5}{2} \\ k = \frac{1}{6} \end{cases} \Rightarrow A = \frac{-5}{2}P + \frac{1}{6}P^2$

(b) (i) $54 = \frac{-5}{2}P + \frac{1}{6}P^2$

$P^2 - 15P - 324 = 0 \Rightarrow P = 27$ or -12 (rejected)

$\therefore$ The perimeter is 27 cm.

(ii) $\frac{\text{Area of miniature}}{\text{Area of original}} = \frac{8}{54} = \frac{4}{27}$

$\Rightarrow \frac{\text{Perimeter of miniature}}{\text{Perimeter of original}} = \frac{2}{\sqrt{27}}$

$\therefore$ Perimeter of miniature $= \frac{2}{\sqrt{27}} \times 27$

$= 2\sqrt{27}$ (cm) $(= 6\sqrt{3}$ cm)

15C.9 HKCEE MA 2003 – I – 13

(a) (i) $\frac{x^\circ}{\cancel{360^\circ}} \times 2\pi(56 + 24) = 30\pi \Rightarrow x = 67.5$

(ii) Area of $ABCD = \frac{67.5^\circ}{\cancel{360^\circ}} \times [\pi(80)^2 - \pi(56)^2]$

$= 612\pi$ (cm^2)

(b) (i) Area of $EFGH = 612\pi \times \left(\frac{18}{24}\right)^2 = 344.25\pi$ (cm^2)

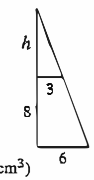
(ii) Base $\odot^{ce} = 30\pi \times \frac{18}{24} = 22.5\pi$ (cm)

$\Rightarrow r = \frac{22.5\pi}{2\pi} = 11.25$

15C.10 HKCEE MA 2006 – I – 13

(a) By $\sim\triangle$s, $\frac{h}{h+8} = \frac{3}{6} = \frac{1}{2}$

$2h = h + 8 \Rightarrow h = 8$



$\therefore$ Vol of frustum

$= \frac{1}{3}\pi(6)^2(8+8) - \frac{1}{3}(3)^2(8)$

$= 192\pi - 24\pi = 168\pi$ (cm^3)

$\Rightarrow$ Vol of $X = 168\pi + \frac{4}{3}\pi(6)^3 \div 2 = 312\pi$ (cm^3)

$\frac{\text{Vol of } Y}{\text{Vol of } X} = \left(\sqrt{\frac{\text{S.A. of } Y}{\text{S.A. of } X}}\right)^3 = \left(\sqrt{\frac{9}{4}}\right)^3 = \frac{27}{8}$

$\Rightarrow$ Vol of $Y = \frac{27}{8}(312\pi) = 1053\pi$ (cm^3)

(b) $\because$ Ratio of S.A. of spheres $= (1:2)^2 = 1:4$

$\neq 4:9$

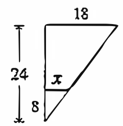
$\therefore$ NO

15C.11 HKCEE MA 2007 – I – 11

(a) *Method 1*

By $\sim\triangle$s, $\frac{x}{18} = \frac{8}{24} = \frac{1}{3}$

$x = 6$



$\therefore$ Vol of water $= \frac{1}{3}\pi(6)^2(8)$

$= 96\pi$ (cm^3)

Method 2

Vol of water $= \left(\frac{8}{24}\right)^3$ Vol of vessel

$= \frac{1}{27} \times \frac{1}{3}\pi(18)^2(24) = 96\pi$ (cm^3

(b) (i) *Method 1*

Area of wet surface $= \pi(6)\sqrt{6^2 + 8^2} = 60\pi$ (cm^2)

Method 2

Area of wet surface $= \left(\frac{8}{24}\right)^2$ C.S.A. of vessel

$= \frac{1}{9} \times \pi(18)\sqrt{18^2 + 24^2}$

$= 60\pi$ (cm^2)

(ii) Ratio of heights $= 24 : 36 = 2 : 3$

Ratio of base radii $= 18 : 27 = 2 : 3$

The two vessels are similar.

$\therefore$ Area of wet surface is also 60π cm^2.

15C.12 HKCEE MA 2008 – I – 13

(a) $\widehat{ABC} = \frac{216^\circ}{\cancel{360^\circ}} \times 2\pi(20) = 24\pi$ (cm)

$\therefore$ Base radius of $X = \frac{24\pi}{2\pi} = 12$ (cm)

Height $= \sqrt{20^2 - 12^2} = 16$ (cm)

(b) Vol of $X = \frac{1}{3}\pi(12)^2(16) = 768\pi$ (cm^3)

(c) *Method 1*

Base radius of $Y = \frac{\frac{108^\circ}{360^\circ} \times 2\pi(10)}{2\pi} = 3$ (cm)

$\because \begin{cases} \frac{\text{Slant height of } X}{\text{Slant height of } Y} = \frac{20}{10} = 2 \\ \frac{\text{Base radius of } X}{\text{Base radius of } Y} = \frac{12}{3} = 4 \neq \frac{\text{Slant height of } X}{\text{Slant height of } Y} \end{cases}$

$\therefore$ NO.

Method 2

Base $\odot^{ce}$ of $X = \frac{216^\circ}{\cancel{360^\circ}} \times 2\pi(20) = 24\pi$ (cm^2)

Base $\odot^{ce}$ of $Y = \frac{108^\circ}{360^\circ} \times 2\pi(10) = 6\pi$ (cm^2)

$\because \begin{cases} \frac{\text{Slant height of } X}{\text{Slant height of } Y} = \frac{20}{10} = 2 \\ \frac{\text{Base } \odot^{ce} \text{ of } X}{\text{Base } \odot^{ce} \text{ of } Y} = \frac{24\pi}{6\pi} = 4 \neq \frac{\text{Slant height of } X}{\text{Slant height of } Y} \end{cases}$

$\therefore$ NO.

Method 3

C.S.A. of $X = \frac{216^\circ}{360^\circ} \times \pi(20)^2 = 240\pi$ (cm^2)

C.S.A. of $Y = \frac{\cancel{108^\circ}}{360^\circ} \times \pi(10)^2 = 30\pi$ (cm^2)

$\because \begin{cases} \frac{\text{Slant height of } X}{\text{Slant height of } Y} = \frac{20}{10} = 2 \\ \frac{\text{C.S.A. of } X}{\text{C.S.A. of } Y} = \frac{240\pi}{30\pi} = 8 \neq \left(\frac{\text{Slant height of } X}{\text{Slant height of } Y}\right)^2 \end{cases}$

$\therefore$ NO.

Method 4

$\because$ Reflex $\angle AOC \neq \angle DPF$

$\therefore$ The sectors are not similar.

$\Rightarrow \dfrac{\text{Area of sector } OABC}{\text{Area of sector } PDEF} \neq \left(\dfrac{OA}{PD}\right)^2$

$\Rightarrow \dfrac{\text{C.S.A. of } X}{\text{C.S.A. of } Y} \neq \left(\dfrac{\text{Slant height of } X}{\text{Slant height of } Y}\right)^2$

$\therefore$ NO.

15C.13 HKCEE MA 2010 – I – 13

(a) Area of $\triangle ABC = \dfrac{16 \times \sqrt{17^2 - (16 \div 2)^2}}{2} = 120\ (\text{cm}^2)$

(b) Vol of $ABCDEF = 120 \times 20 = 2400\ (\text{cm}^3)$

(c) (i) $\dfrac{\text{Area of } \triangle APQ}{\text{Area of } \triangle ABC} = \left(\dfrac{PQ}{BC}\right)^2 = \dfrac{1}{16}$

$\therefore$ Vol of $APQRES = 2400 \times \dfrac{1}{16} = 150\ (\text{cm}^3)$

(ii) *Method 1*

$\because \dfrac{PQ}{BC} = \dfrac{1}{4}$, but $\dfrac{AE}{AE} = 1 \neq \dfrac{PQ}{BC}$

$\therefore$ NO.

Method 2

$\dfrac{PQ}{BC} = \dfrac{1}{4}$

$\because \dfrac{\text{Vol of } APQRES}{\text{Vol of } ABCDEF} = \dfrac{150}{2400} = \dfrac{1}{16} \neq \left(\dfrac{PQ}{BC}\right)^3$

$\therefore$ NO.

15C.14 HKDSE MA 2012 – I – 11

(a) Let $C = h + kA$.

$\begin{cases} 62 = h + 2k \\ 74 = h + 6k \end{cases} \Rightarrow \begin{cases} h = 56 \\ k = 3 \end{cases} \Rightarrow C = 56 + 3A$

$\therefore$ When $A = 13$, cost $= 56 + 3(13) = (\$)95$

(b) Volume is 8 times. $\Rightarrow$ Area is $(\sqrt[3]{8})^2 = 4$ times.

$\therefore$ Cost $= 56 + 3(13 \times 4) = (\$)212$

15C.15 HKDSE MA 2013 – I – 13

(a) (i) $\dfrac{r}{R} = \sqrt{\dfrac{1}{9}} = \dfrac{1}{3}$

(ii) Let H cm be the height of a larger cylinder.

$2 \times \pi R^2 H = 27 \times \pi r^2 (10)$

$2 \times (9\pi r^2) H = 27 \times \pi r^2 (10)$

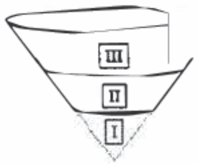
$18H = 270 \Rightarrow H = 15$

Hence, the height is 15 cm.

(b) $\because \dfrac{\text{Height of smaller cylinder}}{\text{Height of larger cylinder}} = \dfrac{10}{15} = \dfrac{2}{3} \neq \dfrac{r}{R}$

$\therefore$ NO.

15C.16 HKDSE MA 2014 – I – 14



(a) *Method 1*

C.S.A. of entire cone $= \pi(72)\sqrt{72^2 + 96^2} = 8640\pi\ (\text{cm}^2)$

With the label in the figure,

C.S.A. of 'I' : C.S.A. of 'I+II' : C.S.A. of 'I+II+III'

$= ((96 - 60) : (96\quad 60 + 28) : 96)^2$

$= (9 : 16 : 24)^2 = 81 : 256 : 576$

$\therefore$ Area of wet curved surface $= 8640\pi \times \dfrac{256 - 81}{576}$

$= 2625\pi\ (\text{cm}^2)$

Method 2

With the label in the figure,

$\dfrac{\text{Base radius of 'I'}}{96 - 60} \quad \dfrac{\text{Base radius of 'I+II'}}{96\quad 60 + 28} = \dfrac{72}{96}$

$\Rightarrow$ Base r of 'I' $= 27$ (cm), Base r of 'I+II' $= 48$ (cm)

$\therefore$ Area of wet curved surface $= \pi(48)\sqrt{48^2 + 64^2}$

$- \pi(27)\sqrt{27^2 + 36^2}$

$= 2625\pi\ (\text{cm}^2)$

(b) *Method 1*

Vol of entire cone $= \dfrac{1}{3}\pi(72)^2(96) = 165888\pi\ (\text{cm}^3)$

$\therefore$ Vol of water $= 165888\pi \times \dfrac{16^3 - 9^3}{24^3}$

$= 40404\pi\ (\text{cm}^3)$

$= 126933\ \text{cm}^3 = 0.127\ \text{m}^3 > 0.1\ \text{m}^3$

$\therefore$ YES.

Method 2

Vol of water $= \dfrac{1}{3}\pi(48)^2(64) \quad \dfrac{1}{3}\pi(27)^2(36)$

$= 40404\pi\ (\text{cm}^3)$

$= 126933\ \text{cm}^3 = 0.127\ \text{m}^3 > 0.1\ \text{m}^3$

$\therefore$ YES.

15C.17 HKDSE MA 2016 – I – 11

(a) Let V cm^3 be the final volume of milk.

$\dfrac{\text{Initial volume of milk}}{\text{Final volume of milk}} = \left(\dfrac{\text{Initial depth of milk}}{\text{Final depth of milk}}\right)^3$

$\Rightarrow \dfrac{V \quad 444\pi}{V} = \left(\dfrac{12}{16}\right)^3 = \dfrac{27}{64} \Rightarrow V = 768\pi$

$\therefore$ The final volume of milk is 768π cm^3.

(b) Let r cm be the final radius of the milk surface.

$\dfrac{1}{3}\pi r^2(16) = 768\pi \Rightarrow r = 12$

$\therefore$ Final area of wet surface $= \pi(12)\sqrt{(12)^2 + 16^2}$

$= 240\pi$

$= 754.0\ (\text{cm}^2) < 800\ \text{cm}^2$

NO.

15C.18 HKDSE MA 2017 – I – 12

(a) Volume of metal $= 84 \times 20 = 1680\ (\text{cm}^3)$

$\dfrac{\text{Vol of smaller pyramid}}{\text{Vol of larger pyramid}} = \left(\sqrt{\dfrac{4}{9}}\right)^3 = \dfrac{8}{27}$

$\therefore$ Vol of larger pyramid $= 1680 \times \dfrac{27}{8 + 27} = 1296\ (\text{cm}^3)$

(b) For the larger pyramid,

Base area $= \dfrac{3 \times \text{Volume}}{\text{Height}} = \dfrac{3 \times 1296}{12} = 324\ (\text{cm}^2)$

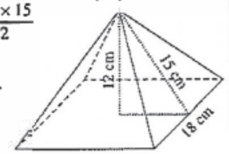
$\Rightarrow$ Length of one side of base $= \sqrt{324} = 18$ (cm)

$\Rightarrow$ Height of each lateral-face $\triangle = \sqrt{12^2 + (18 \div 2)^2}$

$= 15$ (cm)

$\therefore$ T.S.A. $= 324 + 4 \times \dfrac{18 \times 15}{2}$

$= 864\ (\text{cm}^2)$



Hence, for the smaller prd,

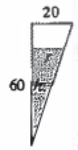
T.S.A. $= 864 \times \dfrac{9}{4}$

$= 384\ (\text{cm}^2)$

15C.19 HKDSE MA 2018 – I – 14

(a) Vol of water $= \pi(8)^2(64) = 4096\pi\ (\text{cm}^3)$

(b) *Method 1*



By $\sim \triangle$s, $\dfrac{h}{60} = \dfrac{r}{20}$

$h = 3r$

$\therefore \dfrac{1}{3}\pi r^2 h = 4096\pi$

$\dfrac{1}{3}\pi r^2(3r) = 4096\pi$

$r^3 = 4096 \Rightarrow r = 16$

Hence, the depth of water is $3(16) = 48$ (cm).

Method 2

Cap of vessel $= \dfrac{1}{3}\pi(20)^2(60) = 8000\pi\ (\text{cm}^3)$

$\therefore$ Depth of water $= \sqrt[3]{\dfrac{4096\pi}{8000\pi}} \times$ Height of vessel

$= \dfrac{4}{5} \times 60 = 48$ (cm)

(c) Vol of sphere $= \dfrac{4}{3}\pi(14)^3 = 3658\dfrac{2}{3}\pi\ (\text{cm}^3)$

Vol of empty space in vessel

$= \dfrac{1}{3}\pi(20)^2(60) - 4096\pi = 3904\pi >$ Vol of sphere

$\therefore$ NO.

15C.20 HKDSE MA 2019 – I – 9

(a) *Method 1*

Let the radii of the smaller and larger spheres be r cm and $2r$ cm respectively.

$\dfrac{4}{3}\pi(r)^3 + \dfrac{4}{3}\pi(2r)^3 = 324\pi$

$r^3 + 8r^3 = 243 \Rightarrow r^3 \quad 27 \Rightarrow r = 3$

$\therefore$ Vol of larger sphere $= \dfrac{4}{3}\pi(2 \times 3)^3 = 288\pi\ (\text{cm}^3)$

Method 2

$\dfrac{\text{V of larger sphere}}{\text{V of smaller sphere}} = \left(\dfrac{\text{R of larger sphere}}{\text{R of smaller sphere}}\right) = 8$

$\therefore$ V of larger sphere $= 324\pi \times \dfrac{8}{1+8} = 288\pi\ (\text{cm}^3)$

(b) R of larger sphere $= \sqrt[3]{288\pi \div \dfrac{4\pi}{3}} = 6$ (cm)

$\therefore$ Sum of S.A. $= 4\pi(6)^2 + 4\pi(6 \div 2)^2 = 180\pi\ (\text{cm}^2)$

****15B.23 HKDSE MA 2020 - I - 12**

12a	The required volume $= \dfrac{\pi}{3}(15)^2(36)\left[\left(\dfrac{2}{3}\right)^3 - \left(\dfrac{1}{3}\right)^3\right]$ $700\pi\ \text{cm}^3$
b	The required curved surface area $\quad \pi(15)\sqrt{15^2 + 36^2}\left[\left(\dfrac{2}{3}\right)^2 - \left(\dfrac{1}{3}\right)^2\right]$ $195\pi\ \text{cm}^2$

16 Coordinate Geometry

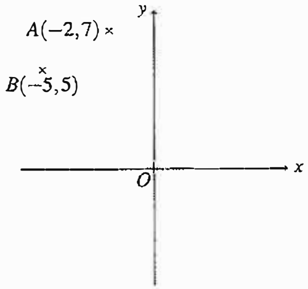
16A Transformation in the rectangular coordinate plane

16A.1 HKCEE MA 2006 – I – 7

In the figure, the coordinates of the points A and B are $(-2,7)$ and $(-5,5)$ respectively. A is rotated clockwise about the origin O through $90°$ to A'. B' is the reflection image of B with respect to the y-axis.

(a) Write down the coordinates of A' and B'.

(b) Are the lengths of AB and $A'B'$ equal? Explain your answer.

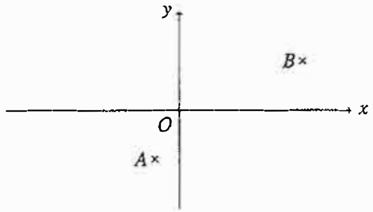


16A.2 HKCEE MA 2009 – I – 9

In the figure, the coordinates of the points A and B are $(-1,-2)$ and $(5,2)$ respectively. A is translated vertically upward by 6 units to A'. B' is the reflection image of B with respect to the y axis.

(a) Write down the coordinates of A' and B'.

(b) Is AB parallel to $A'B'$? Explain your answer.



16A.3 HKCEE MA 2011 I 8

The coordinates of the point A are $(-4,6)$. A is rotated anticlockwise about the origin O through $90°$ to B. M is the mid-point of AB.

(a) Find the coordinates of M.

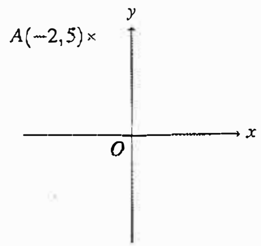
(b) Is OM perpendicular to AB? Explain your answer.

16A.4 HKDSE MA SP – I – 8

In the figure, the coordinates of the point A are $(-2,5)$. A is rotated clockwise about the origin O through $90°$ to A'. A'' is the reflection image of A with respect to the y-axis.

(a) Write down the coordinates of A' and A''.

(b) Is OA'' perpendicular to AA'? Explain your answer.



16A.5 HKDSE MA 2014 I – 8

The coordinates of the points P and Q are $(-3,5)$ and $(2,-7)$ respectively. P is rotated anticlockwise about the origin O through $270°$ to P'. Q is translated leftwards by 21 units to Q'.

(a) Write down the coordinates of P' and Q'.

(b) Prove that PQ is perpendicular to $P'Q'$.

16A.6 HKDSE MA 2017 – I – 6

The coordinates of the points A and B are $(\ 3,4)$ and $(9,\ 9)$ respectively. A is rotated anticlockwise about the origin through $90°$ to A'. B' is the reflection image of B with respect to the x-axis.

(a) Write down the coordinates of A' and B'.

(b) Prove that AB is perpendicular to $A'B'$.

16B Straight lines in the rectangular coordinate plane

16B.1 HKCEE MA 1992 – I – 5

L_1 is the line passing through the point $A(10,5)$ and perpendicular to the line $L_2 : x-2y+5=0$.

(a) Find the equation of L_1.

(b) Find the intersection point of L_1 and L_2.

16B.2 HKCEE MA 1998 I – 8

$A(0,4)$ and $B(-2,1)$ are two points.

(a) Find the slope of AB.

(b) Find the equation of the line passing through $(1,3)$ and perpendicular to AB.

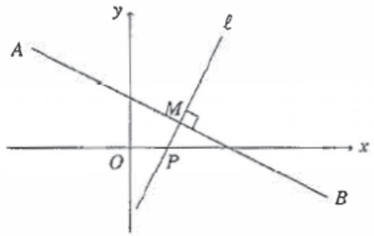
16B.3 HKCEE MA 1999 I – 10

In the figure, $A(-8,8)$ and $B(16,\ \ 4)$ are two points. The perpendicular bisector ℓ of the line segment AB cuts AB at M and the x axis at P.

(a) Find the equation of ℓ.

(b) Find the length of BP.

(c) If N is the mid point of AP, find the length of MN.



16B.4 HKCEE MA 2000 – I – 9

Let L be the straight line passing through $(\ \ 4,4)$ and $(6,0)$.

(a) Find the slope of L.

(b) Find the equation of L.

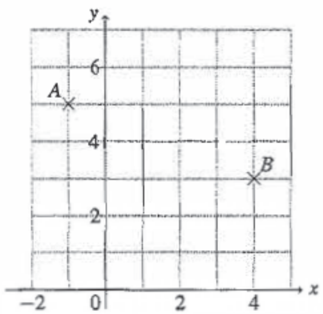
(c) If L intersects the y axis at C, find the coordinates of C.

16B.5 HKCEE MA 2001 – I – 7

Two points A and B are marked in the figure.

(a) Write down the coordinates of A and B.

(b) Find the equation of the straight line joining A and B.

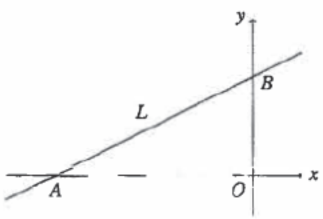


16B.6 HKCEE MA 2002 – I – 8

In the figure, the straight line $L : x\ \ 2y+8=0$ cuts the coordinate axes at A and B.

(a) Find the coordinates of A and B.

(b) Find the coordinates of the mid-point of AB.



16B.7 HKCEE MA 2003 – I – 12

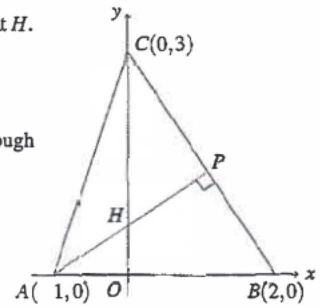
In the figure, AP is an altitude of the triangle ABC. It cuts the y-axis at H.

(a) Find the slope of BC.

(b) Find the equation of AP.

(c) (i) Find the coordinates of H.

(ii) Prove that the three altitudes of the triangle ABC pass through the same point.



16B.8 HKCEE MA 2004 – I – 13

In the figure, $ABCD$ is a rhombus. The diagonals AC and BD cut at E.

(a) Find

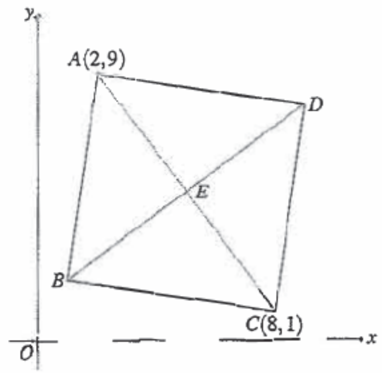
(i) the coordinates of E,

(ii) the equation of BD.

(b) It is given that the equation of AD is $x+7y-65=0$. Find

(i) the equation of BC,

(ii) the length of AB.



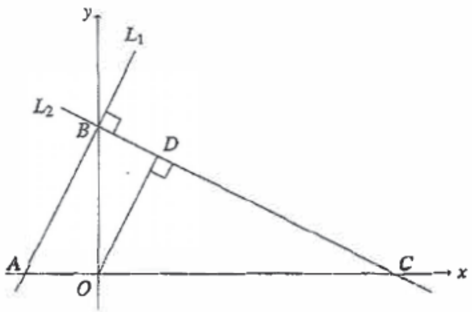
16B.9 HKCEE MA 2005 I – 13

In the figure, the straight line $L_1 : 2x-y+4=0$ cuts the x-axis and the y axis at A and B respectively. The straight line L_2, passing through B and perpendicular to L_1, cuts the x-axis at C. From the origin O, a straight line perpendicular to L_2 is drawn to meet L_2 at D.

(a) Write down the coordinates of A and B.

(b) Find the equation of L_2.

(c) Find the ratio of the area of $\triangle ODC$ to the area of quadrilateral $OABD$.



16B.10 HKCEE MA 2006 I – 12

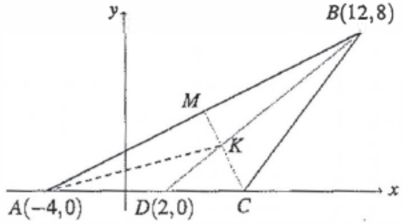
In the figure, CM is the perpendicular bisector of AB, where C and M are points lying on the x axis and AB respectively. BD and CM intersect at K.

(a) Write down the coordinates of M.

(b) Find the equation of CM. Hence, or otherwise, find the coordinates of C.

(c) (i) Find the equation of BD.

(ii) Using the result of (c)(i), find the coordinates of K. Hence find the ratio of the area of $\triangle AMC$ to the area of $\triangle AKC$.



16B.11 HKCEE MA 2007 – I – 13

In the figure, the perpendicular from B to AC meets AC at D.

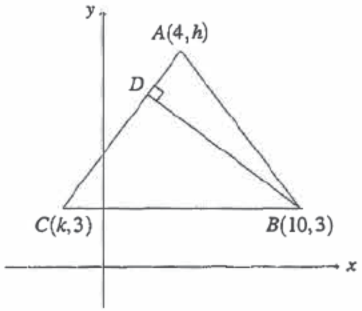
It is given that $AB = AC$ and the slope of AB is $\frac{-4}{3}$.

(a) Find the equation of AB.

(b) Find the value of h.

(c) (i) Write down the value of k

(ii) Find the area of $\triangle ABC$. Hence, or otherwise, find the length of BD.



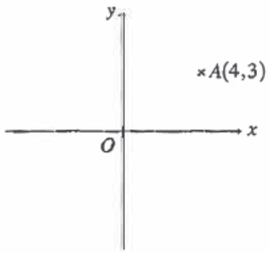
16B.12 HKCEE MA 2008 – I – 12

In the figure, the coordinates of the point A are $(4,3)$. A is rotated anticlockwise about the origin O through $90°$ to B. C is the reflection image of A with respect to the x axis.

(a) Write down the coordinates of B and C.

(b) Are O, B and C collinear? Explain your answer.

(c) A is translated horizontally to D such that $\angle BCD = 90°$. Find the equation of the straight line passing through C and D. Hence, or otherwise, find the coordinates of D.



16B.13 HKCEE MA 2010 I – 12

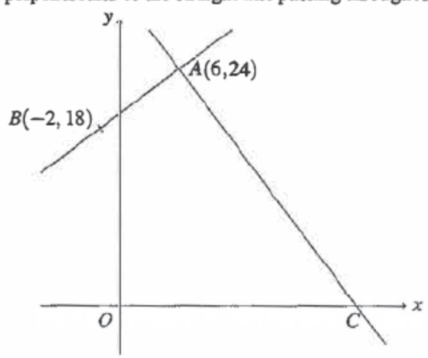
In the figure, the straight line passing through A and B is perpendicular to the straight line passing through A and C, where C is a point lying on the x-axis.

(a) Find the equation of the straight line passing through A and B.

(b) Find the coordinates of C.

(c) Find the area of $\triangle ABC$.

(d) A straight line passing through A cuts the line segment BC at D such that the area of $\triangle ABD$ is 90 square units. Let $BD : DC = r : 1$. Find the value of r.



16B.14 HKCEE AM 1982 II 2

Find the ratio in which the line segment joining $A(3,-1)$ and $B(-1,1)$ is divided by the straight line $x-y-1=0$.

16B.15 HKCEE AM 1982 – II – 10

(a) The lines $3x \quad 2y-8=0$ and $x \quad y-2=0$ meet at a point P. L_1 and L_2 are lines passing through P and having slopes $\frac{1}{2}$ and 2 respectively. Find their equations.

(b) *[Out of syllabus]*

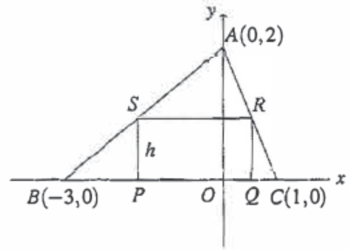
16B.16 (HKCEE AM 1985 II – 10)

$A(0,2)$, $B(-3,0)$ and $C(1,0)$ are the vertices of a triangle. $PQRS$ is a variable rectangle inscribed in the triangle with PQ on the x-axis, R on AC and S on AB, as shown in the figure. Let the length of PS be h.

(a) Find the coordinates of S and R in terms of h.

(b) Let A_1 be the area of $PQRS$ when it is a square, A_2 be the maximum possible area of rectangle $PQRS$, and A_3 be the area of $\triangle ABC$. Find the ratios $A_1 : A_2 : A_3$.

(c) The centre of $PQRS$ is the point $M(x,y)$. Express x and y in terms of h. Hence show that M lies on the line $x-y+1=0$.



16B.17 (HKCEE AM 1984 II – 4)

The area of the triangle bounded by the two lines $L_1 : x+y=4$ and $L_2 : x-y=2p$ and the y-axis is 9.

(a) Find the coordinates of the point of intersection of L_1 and L_2 in terms of p.

(b) Hence, find the possible value(s) of p.

16B.18 HKCEE AM 1988 – II – 2

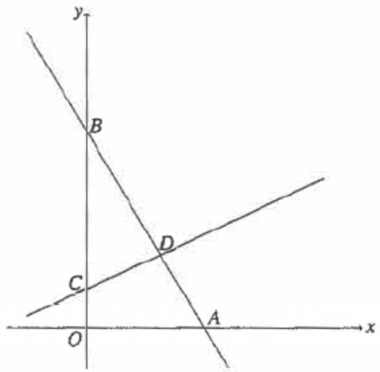
A and B are the points $(1,2)$ and $(7,4)$ respectively. P is a point on the line segment AB such that $\frac{AP}{PB} = k$.

(a) Write down the coordinates of P in terms of k.

(b) Hence find the ratio in which the line $7x-3y \quad 28=0$ divides the line segment AB.

16B.19 HKCEE AM 1990 – II – 7

In the figure, $A(3,0)$, $B(0,5)$ and $C(0,1)$ are three points and O is the origin. D is a point on AB such that the area of $\triangle BCD$ equals half of the area of $\triangle OAB$. Find the equation of the line CD.



16B.20 (HKCEE AM 1996 II 8)

Given two straight lines $L_1 : 2x-y-4=0$ and $L_2 : x-2y+4=0$. Find the equation of the straight line passing through the origin and the point of intersection of L_1 and L_2.

16B.21 (HKCEE AM 1998 – II – 5)

Two lines $L_1 : 2x+y-3=0$ and $L_2 : x-3y+1=0$ intersect at a point P.

(a) Find the coordinates of P.

(b) L is a line passing through P and the origin. Find the equation of L.

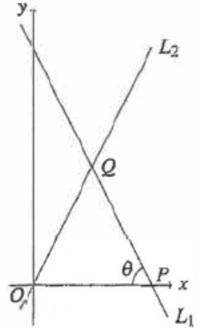
16B.22 HKCEE AM 2005 6

The figure shows the line $L_1 : 2x+y-6=0$ intersecting the x axis at point P.

(a) Let θ be the acute angle between L_1 and the x axis. Find $\tan\theta$.

(b) L_2 is a line with positive slope passing through the origin O. If L_1 intersects L_2 at a point Q such that $OP = OQ$, find the equation of L_2.

(Candidates can use the formula $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.)



16B.23 (HKCEE AM 2009 3)

Given two straight lines $L_1 : x-3y+7=0$ and $L_2 : 3x-y-11=0$. Find the equation of the straight line passing through the point $(2, 1)$ and the point of intersection of L_1 and L_2.

16B.24 HKCEE AM 2010 6

Two straight lines $L_1 : x-2y+3=0$ and $L_2 : 2x-y-1=0$ intersect at a point P. If L is a straight line passing through P and with equal positive intercepts, find the equation of L.

16C Circles in the rectangular coordinate plane

16C.1 HKCEE MA 1980(1/3 I) – B – 15

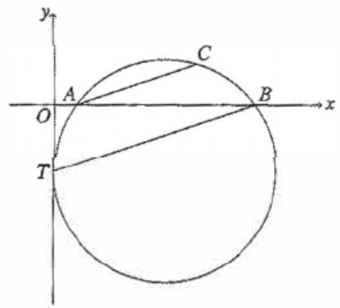
The circle $x^2+y^2-10x+8y+16=0$ cuts the x axis at A and B and touches the y-axis at T as shown in the figure.

(a) Find the coordinates of A, B and T.

(b) C is a point on the circle such that $AC//TB$.

(i) Find the equation of AC.

(ii) Find the coordinates of C by solving simultaneously the equation of AC and the equation of the given circle.



16C.2 HKCEE MA 1981(1/3) I – 13

Figure (1) shows a circle of radius 15 with centre at the origin O. The line TP, of slope $\frac{3}{4}$ ($=\tan\theta$), touches the circle at T and cuts the x axis at P.

(a) Find the equation of the circle.

(b) Calculate the length of OP.

(c) Find the equation of the line TP.

Another circle, with centre C and radius 15, is drawn to touch TP at P (see Figure (2)).

(d) Find the equation of the line OC.

(e) Find the equation of the circle with centre C.

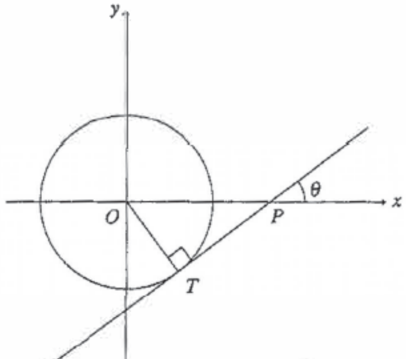


Figure (1)

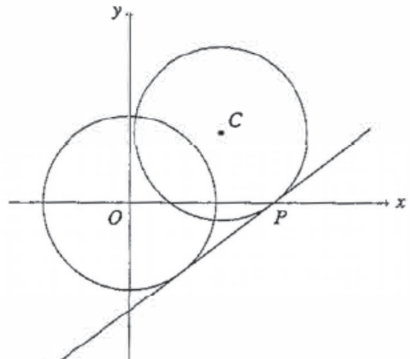


Figure (2)

16C.3 <u>HKCEE MA 1982(1) – I – 13</u>

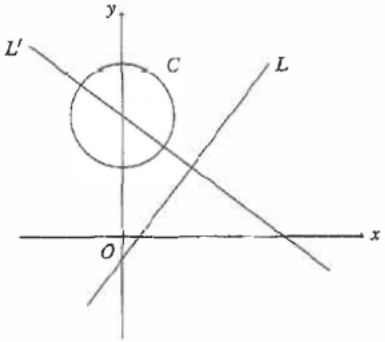
In the figure, C is the circle $x^2+y^2-14y+40=0$ and L is the line $4x-3y-4=0$.

(a) Find the radius and the coordinates of the centre of the circle C.

(b) The line L' passes through the centre of the circle C and is perpendicular to the given line L. Find the equation of the line L'.

(c) Find the coordinates of the point of intersection of the line L and the line L'.

(d) Hence, or otherwise, find the shortest distance between the circle C and the line L.



16C.4 <u>HKCEE MA 1983(A/B) – I – 9</u>

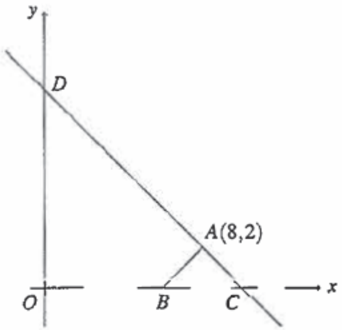
In the figure, O is the origin and A is the point $(8,2)$.

(a) B is a point on the x-axis such that the slope of AB is 1. Find the coordinates of B.

(b) C is another point on the x-axis such that $AB=AC$. Find the coordinates of C.

(c) Find the equation of the straight line AC. If the line AC cuts the y-axis at D, find the coordinates of D.

(d) Find the equation of the circle passing through the points O, B and D. Show that this circle passes through A.



16C.5 <u>HKCEE MA 1984(A/B) – I – 9</u>

Let L be the line $y=k \quad x$ (k being a constant) and C be the circle $x^2+y^2=4$.

(a) If L meets C at exactly one point, find the two values of k.

(b) If L intersects C at the points $A(2,0)$ and B,

 (i) find the value of k and the coordinates of B;

 (ii) find the equation of the circle with AB as diameter.

16C.6 <u>HKCEE MA 1985(A/B) – I – 9</u>

In the figure, $A(2,0)$ and $B(7,5)$ are the end-points of a diameter of the circle. P is a point on AB such that $\dfrac{AP}{PB}=\dfrac{1}{4}$.

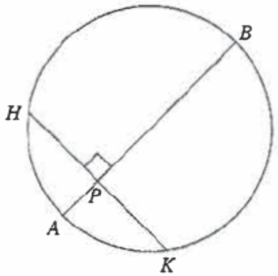
(a) Find the equation of the circle.

(b) Find the coordinates of P.

(c) The chord HPK is perpendicular to AB.

 (i) Find the equation of HPK.

 (ii) Find the coordinates of H and K.



16C.7 <u>HKCEE MA 1986(A/B) – I – 8</u>

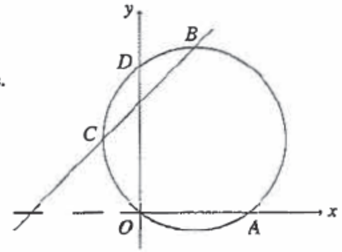
The line $y \quad x-6=0$ cuts the circle $x^2+y^2-6x-8y=0$ at the points B and C as shown in the figure. The circle cuts the x-axis at the origin O and the point A; it also cuts the y axis at D.

(a) Find the coordinates of B and C.

(b) Find the coordinates of A and D.

(c) Find $\angle ADO$, $\angle ABO$ and $\angle ACO$, correct to the nearest degree.

(d) Find the area of $\triangle ACO$.



16C.8 <u>HKCEE MA 1987(A/B) – I – 8</u>

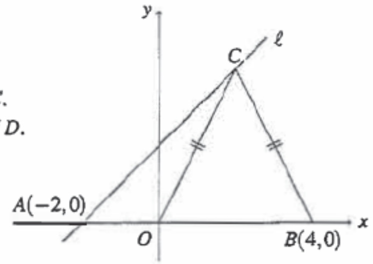
In the figure, O is the origin. A and B are the points $(-2,0)$ and $(4,0)$ respectively. ℓ is a straight line through A with slope 1. C is a point on ℓ such that $CO=CB$.

(a) Find the equation of ℓ.

(b) Find the coordinates of C.

(c) Find the equation of the circle passing through O, B and C.

(d) If the circle OBC cuts ℓ again at D, find the coordinates of D.

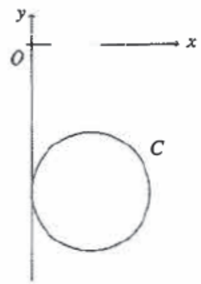


16C.9 <u>HKCEE MA 1988 – I – 7</u>

In the figure, the circle C has equation $x^2+y^2-4x+10y+k=0$, where k is a constant.

(a) Find the coordinates of the centre of C.

(b) If C touches the y axis, find the radius of C and the value of k.



16C.10 <u>HKCEE MA 1989 – I – 8</u>

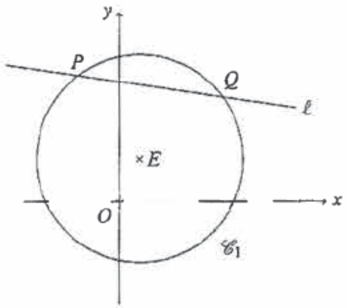
Let E be the centre of the circle $\mathscr{C}_1 : x^2+y^2-2x-4y-20=0$. The line $\ell : x+7y-40=0$ cuts $\mathscr{C}_1$ at the points P and Q as shown in the figure.

(a) Find the coordinates of E.

(b) Find the coordinates of P and Q.

(c) Find the equation of the circle $\mathscr{C}_2$ with PQ as diameter.

(d) Show that $\mathscr{C}_2$ passes through E. Hence, or otherwise, find $\angle EPQ$.



16C.11 <u>HKCEE MA 1990 – I – 8</u>

Let (C_1) be the circle $x^2+y^2-2x+6y+1=0$ and A be the point $(5,0)$.

(a) Find the coordinates of the centre and the radius of (C_1).

(b) Find the distance between the centre of (C_1) and A.
Hence determine whether A lies inside, outside or on (C_1).

(c) Let s be the shortest distance from A to (C_1).
  (i) Find s.
  (ii) Another circle (C_2) has centre A and radius s. Find its equation.

(d) A line touches the above two circles (C_1) and (C_2) at two distinct points E and F respectively.
Draw a rough diagram to show this information.
Find the length of EF.

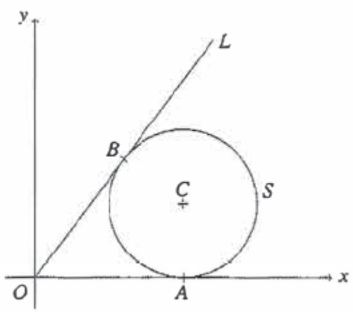
16C.12 <u>HKCEE MA 1991 – I – 9</u>

In the figure, the circle $S : x^2+y^2-4x-2y+4=0$ with centre C touches the x axis at A. The line $L : y=mx$, where m is a non-zero constant, passes through the origin O and touches S at B.

(a) Find the coordinates of C and A.

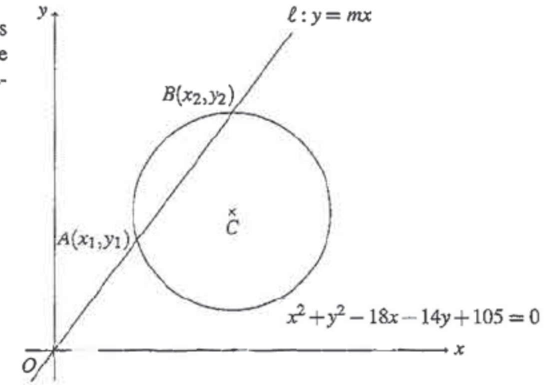
(b) Show that $m=\dfrac{4}{3}$.

(c) (i) Explain why the four points O, A, C, B are concyclic.
  (ii) Find the equation of the circle passing through these four points.



16C.13 <u>HKCEE MA 1992 – I – 13</u>

In the figure, the line $\ell : y=mx$ passes through the origin and intersects the circle $x^2+y^2-18x-14y+105=0$ at two distinct points $A(x_1,y_1)$ and $B(x_2,y_2)$.



(a) Find the coordinates of the centre C and the radius of the circle.

(b) By substituting $y=mx$ into $x^2+y^2-18x-14y+105=0$, show that $x_1x_2=\dfrac{105}{1+m^2}$.

(c) Express the length of OA in terms of m and x_1 and the length of OB in terms of m and x_2.
Hence find the value of the product of OA and OB.

(d) If the perpendicular distance between the line ℓ and the centre C is 3, find the lengths of AB and OA.

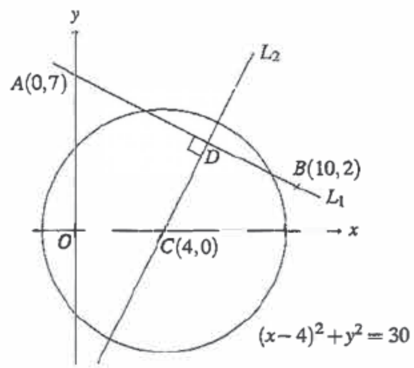
16C.14 <u>HKCEE MA 1993 – I – 8</u>

In the figure, L_1 is the line passing through $A(0,7)$ and $B(10,2)$; L_2 is the line passing through $C(4,0)$ and perpendicular to L_1; L_1 and L_2 meet at D.

(a) Find the equation of L_1.

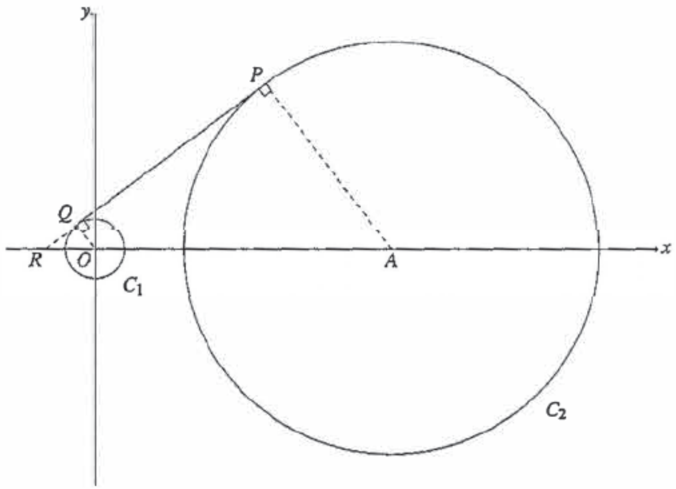
(b) Find the equation of L_2 and the coordinates of D.

(c) P is a point on the line segment AB such that $AP : PB = k : 1$. Find the coordinates of P in terms of k.
If P lies on the circle $(x-4)^2+y^2=30$, show that $2k^2-16k+7=0$ ....(*).
Find the roots of equation (*).
Furthermore, if P lies between A and D, find the value of $\dfrac{AP}{PB}$.



16C.15 HKCEE MA 1994 I – 12

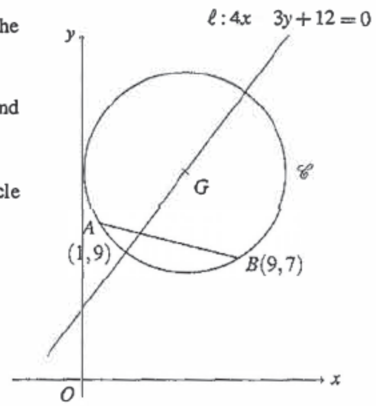
The figure shows two circles $C_1 : x^2+y^2=1$, $C_2 : (x-10)^2+y^2=49$.
O is the origin and A is the centre of C_2. QP is an external common tangent to C_1 and C_2 with points of contact Q and P respectively. The slope of QP is positive.



(a) Write down the coordinates of A and the radius of C_2.
(b) PQ is produced to cut the x axis at R. Find the x-coordinate of R by considering similar triangles.
(c) Using the result in (b), find the slope of QP.
(d) Using the results of (b) and (c), find the equation of the external common tangent QP.
(e) Find the equation of the other external common tangent to C_1 and C_2.

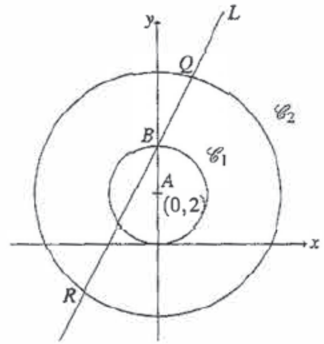
16C.16 HKCEE MA 1995 – I – 10

In the figure, $A(1,9)$ and $B(9,7)$ are points on a circle $\mathscr{C}$. The centre G of the circle lies on the line $\ell : 4x-3y+12=0$.
(a) Find the equation of the line AB.
(b) Find the equation of the perpendicular bisector of AB, and hence the coordinates of G.
(c) Find the equation of the circle $\mathscr{C}$.
(d) If DE (not shown in the figure) is another chord of the circle $\mathscr{C}$ such that AB and DE are equal and parallel, find
  (i) the coordinates of the mid-point of DE, and
  (ii) the equation of the line DE.



16C.17 HKCEE MA 1996 I 11

$\mathscr{C}_1$ is the circle with centre $A(0,2)$ and radius 2. It cuts the y-axis at the origin O and the point B. $\mathscr{C}_2$ is another circle with equation $x^2+(y-2)^2=25$. The line L passing through B with slope 2 cuts $\mathscr{C}_2$ at the points Q and R as shown in the figure.



(a) Find
  (i) the equation of $\mathscr{C}_1$;
  (ii) the equation of L.
(b) Find the coordinates of Q and R.
(c) Find the coordinates of
  (i) the point on L which is nearest to A;
  (ii) the point on $\mathscr{C}_1$ which is nearest to Q.

16C.18 HKCEE MA 1997 – I – 16

(Continued from **12B.14**.)

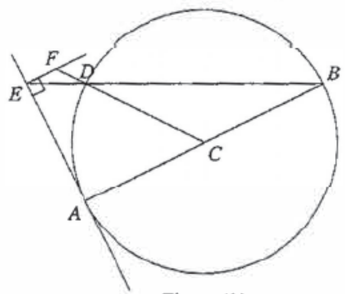


Figure (1)

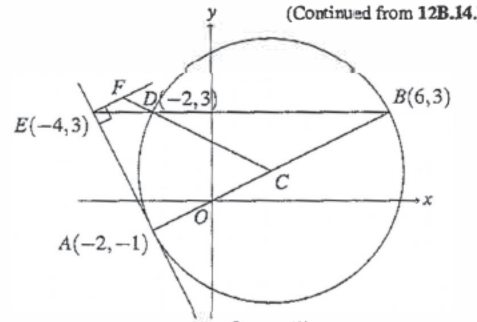
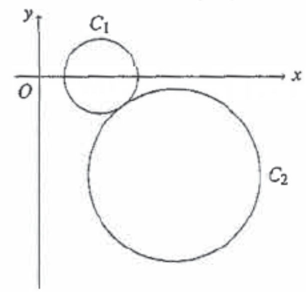


Figure (2)

(a) In Figure (1), D is a point on the circle with AB as diameter and C as the centre. The tangent to the circle at A meets BD produced at E. The perpendicular to this tangent through E meets CD produced at F.
  (i) Prove that $AB // EF$.
  (ii) Prove that $FD = FE$.
  (iii) Explain why F is the centre of the circle passing through D and touching AE at E.
(b) A rectangular coordinate system is introduced in Figure (1) so that the coordinates of A and B are $(\ 2,\ \ 1)$ and $(6,3)$ respectively. It is found that the coordinates of D and E are $(-2,3)$ and $(-4,3)$ respectively as shown in Figure (2). Find the coordinates of F.

16C.19 HKCEE MA 1998 I 15

The figure shows two circles C_1 and C_2 touching each other externally. The centre of C_1 is $(5,0)$ and the equation of C_2 is $(x-11)^2+(y+8)^2=49$.



(a) Find the equation of C_1.
(b) Find the equations of the two tangents to C_1 from the origin.
(c) One of the tangents in (b) cuts C_2 at two distinct points A and B. Find the coordinates of the mid-point of AB.

16C.20 HKCEE MA 1999 – I – 16 (Continued from 12A.17.)

(a) In Figure (1), ABC is a triangle right-angled at B. D is a point on AB. A circle is drawn with DB as a diameter. The line through D and parallel to AC cuts the circle at E. CE is produced to cut the circle at F.

  (i) Prove that A, F, B and C are concyclic.

  (ii) If M is the mid-point of AC, explain why $MB = MF$.

(b) In Figure (2), the equation of circle RST is $x^2 + y^2 + 10x - 6y + 9 = 0$. QST is a straight line. The coordinates of P, Q, R, S are $(-17, 0)$, $(0, 17)$, $(-9, 0)$ and $(-2, 7)$ respectively.

  (i) Prove that $PQ // RS$.

  (ii) Find the coordinates of T.

  (iii) Are the points P, Q, O and T concyclic? Explain your answer.

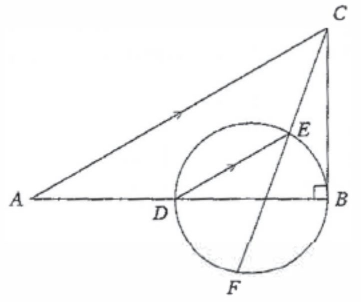


Figure (1)

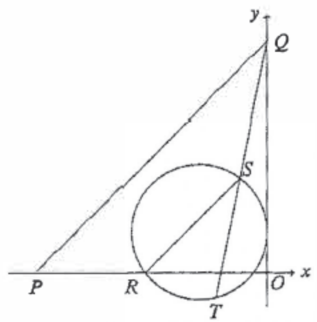


Figure (2)

16C.21 HKCEE MA 2000 – I – 16 (Continued from 12B.15.)

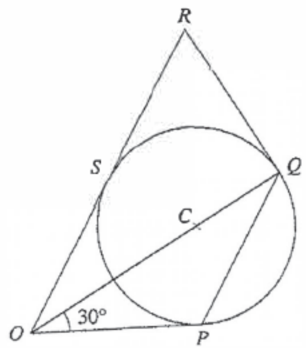
In the figure, C is the centre of the circle PQS. OR and OP are tangent to the circle at S and P respectively. OCQ is a straight line and $\angle QOP = 30°$.

(a) Show that $\angle PQO = 30°$.

(b) Suppose $OPQR$ is a cyclic quadrilateral.

  (i) Show that RQ is tangent to circle PQS at Q.

  (ii) A rectangular coordinate system is introduced in the figure so that the coordinates of O and C are $(0, 0)$ and $(6, 8)$ respectively. Find the equation of QR.



16C.22 HKCEE MA 2001 – I – 17

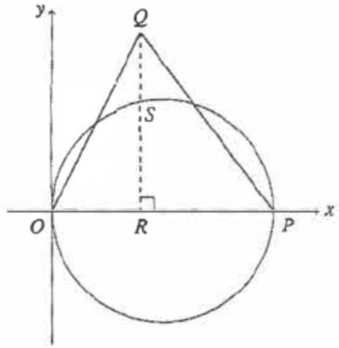


Figure (1)

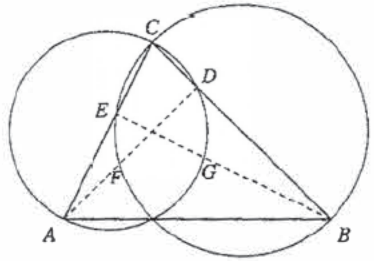


Figure (2)

(a) In Figure (1), OP is a diameter of the circle. The altitude QR of the acute angled triangle OPQ cuts the circle at S. Let the coordinates of P and S be $(p, 0)$ and (a, b) respectively.

  (i) Find the equation of the circle OPS.

  (ii) Using (i) or otherwise, show that $OS^2 = OP \cdot OQ \cos \angle POQ$.

(b) In Figure (2), ABC is an acute angled triangle. AC and BC are diameters of the circles $AGDC$ and $BCEF$ respectively.

  (i) Show that BE is an altitude of $\triangle ABC$.

  (ii) Using (a) or otherwise, compare the length of CF with that of CG. Justify your answer.

16C.23 HKCEE MA 2002 – I – 16 (Continued from 12A.21.)

In the figure, AB is a diameter of the circle $ABEG$ with centre C. The perpendicular from G to AB cuts AB at O. AE cuts OG at D. BE and OG are produced to meet at F.

Mary and John try to prove $OD \cdot OF = OG^2$ by using two different approaches.

(a) Mary tackles the problem by first proving that $\triangle AOD \sim \triangle FOB$ and $\triangle AOG \sim \triangle GOB$. Complete the following tasks for Mary.

  (i) Prove that $\triangle AOD \sim \triangle FOB$.

  (ii) Prove that $\triangle AOG \sim \triangle GOB$.

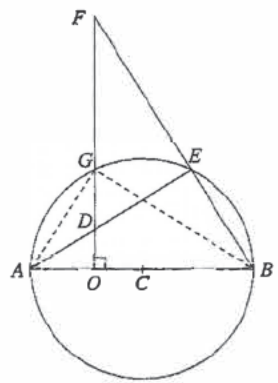
  (iii) Using (a)(i) and (a)(ii), prove that $OD \cdot OF = OG^2$.

(b) John tackles the same problem by introducing a rectangular coordinate system in the figure so that the coordinates of C, D and F are $(c, 0)$, $(0, p)$ and $(0, q)$ respectively, where c, p and q are positive numbers. He denotes the radius of the circle by r.

Complete the following tasks for John.

  (i) Express the slopes of AD and BF in terms of c, p, q and r.

  (ii) Using (b)(i), prove that $OD \cdot OF = OG^2$.



16C.24 HKCEE MA 2003 – I – 17 (Continued from 12B.16.)

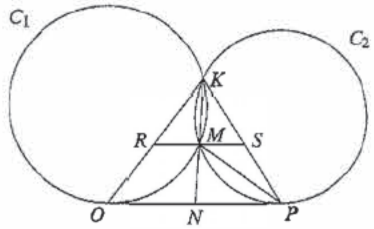


Figure (1)

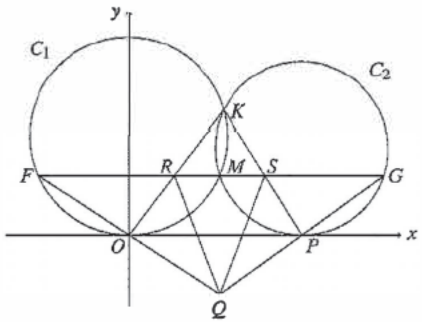


Figure (2)

(a) In Figure (1), OP is a common tangent to the circles C_1 and C_2 at the points O and P respectively. The common chord KM when produced intersects OP at N. R and S are points on KO and KP respectively such that the straight line RMS is parallel to OP.

(i) By considering triangles NPM and NKP, prove that $NP^2 = NK \cdot NM$.

(ii) Prove that $RM = MS$.

(b) A rectangular coordinate system, with O as the origin, is introduced to Figure (1) so that the coordinates of P and M are $(p, 0)$ and (a, b) respectively (see Figure (2)). The straight line RS meets C_1 and C_2 again at F and G respectively while the straight lines FO and GP meet at Q.

(i) Express FG in terms of p.

(ii) Express the coordinates of F and Q in terms of a and b.

(iii) Prove that triangle QRS is isosceles.

16C.25 HKCEE MA 2004 – I – 16 (Continued from 12B.17.)

In the figure, BC is a tangent to the circle OAB with $BC // OA$. OA is produced to D such that $AD = OB$. BD cuts the circle at E.

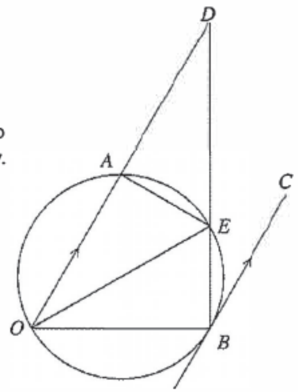
(a) Prove that $\triangle ADE \cong \triangle BOE$.

(b) Prove that $\angle BEO = 2\angle BOE$.

(c) Suppose OE is a diameter of the circle $OAEB$.

(i) Find $\angle BOE$.

(ii) A rectangular coordinate system is introduced in the figure so that the coordinates of O and B are $(0, 0)$ and $(6, 0)$ respectively. Find the equation of the circle $OAEB$.



16C.26 HKCEE MA 2005 – I – 17 (Continued from 12A.22.)

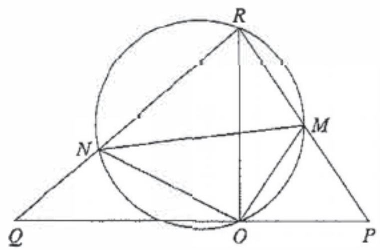


Figure (1)

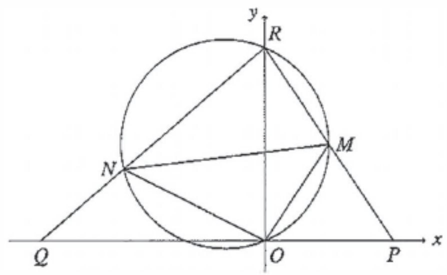


Figure (2)

(a) In Figure (1), MN is a diameter of the circle $MONR$. The chord RO is perpendicular to the straight line POQ. RNQ and RMP are straight lines.

(i) By considering triangles OQR and ORP, prove that $OR^2 = OP \cdot OQ$.

(ii) Prove that $\triangle MON \sim \triangle POR$.

(b) A rectangular coordinate system, with O as the origin, is introduced to Figure (1) so that R lies on the positive y-axis and the coordinates of P and Q are $(4, 0)$ and $(-9, 0)$ respectively (see Figure (2)).

(i) Find the coordinates of R.

(ii) If the centre of the circle $MONR$ lies in the second quadrant and $ON = \frac{3\sqrt{13}}{2}$, find the radius and the coordinates of the centre of the circle $MONR$.

16C.27 HKCEE MA 2006 I 16 (Continued from 12A.23.)

In the figure, G and H are the circumcentre and the orthocentre of $\triangle ABC$ respectively. AH produced meets BC at O. The perpendicular from G to BC meets BC at R. BS is a diameter of the circle which passes through A, B and C.

(a) Prove that

(i) $AHCS$ is a parallelogram,

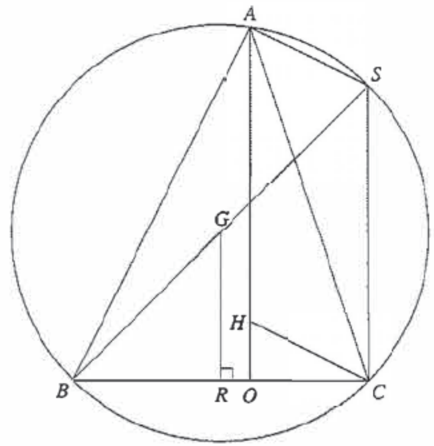
(ii) $AH = 2GR$.

(b) A rectangular coordinate system, with O as the origin, is introduced in the figure so that the coordinates of A, B and C are $(0, 12)$, $(-6, 0)$ and $(4, 0)$ respectively.

(i) Find the equation of the circle which passes through A, B and C.

(ii) Find the coordinates of H.

(iii) Are B, O, H and G concyclic? Explain your answer.



16C.28 HKCEE MA 2007 – I – 17

(Continued from **12A.24.**)

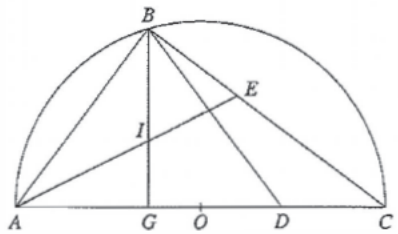


Figure (1)

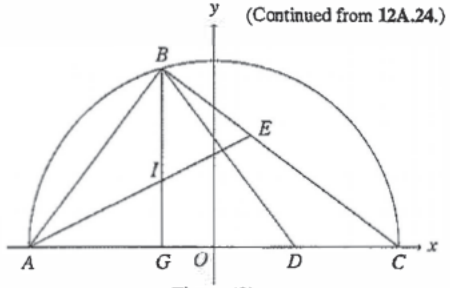


Figure (2)

(a) In Figure (1), AC is the diameter of the semi-circle ABC with centre O. D is a point lying on AC such that $AB = BD$. I is the in centre of $\triangle ABD$. AI is produced to meet BC at E. BI is produced to meet AC at G.

(i) Prove that $\triangle ABG \cong \triangle DBG$.

(ii) By considering the triangles AGI and ABE, prove that $\dfrac{GI}{AG} = \dfrac{BE}{AB}$.

(b) A rectangular coordinate system, with O as the origin, is introduced to Figure (1) so that the coordinates of C and D are $(25,0)$ and $(11,0)$ respectively and B lies in the second quadrant (see Figure (2)). It is found that $BE : AB = 1 : 2$.

(i) Find the coordinates of G.

(ii) Find the equation of the inscribed circle of $\triangle ABD$.

16C.29 HKCEE MA 2008 – I – 17

(Continued from **12A.25.**)

Figure (1) shows a circle passing through A, B and C. I is the in centre of $\triangle ABC$ and AI produced meets the circle at P.

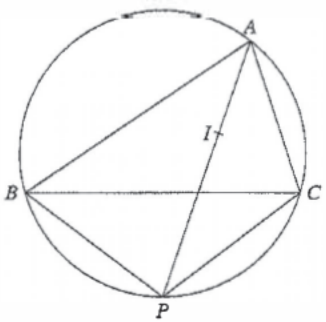


Figure (1)

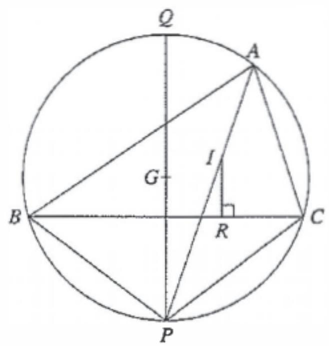


Figure (2)

(a) Prove that $BP = CP = IP$.

(b) Figure (2) is constructed by adding three points G, Q and R to Figure (1), where G is the circumcentre of $\triangle ABC$, PQ is a diameter of the circle and R is the foot of the perpendicular from I to BC. A rectangular coordinate system is then introduced in Figure (2) so that the coordinates of B, C and I are $(-80,0)$, $(64,0)$ and $(0,32)$ respectively.

(i) Find the equation of the circle with centre P and radius BP.

(ii) Find the coordinates of Q.

(iii) Are B, Q, I and R concyclic? Explain your answer.

16C.30 HKCEE MA 2011 – I – 16

In the figure, $\triangle PQR$ is an isosceles triangle with $PQ = PR$. It is given that S is a point lying on QR and the orthocentre of $\triangle PQR$ lies on PS. A rectangular coordinate system is introduced in the figure so that the coordinates of P and Q are $(16,80)$ and $(-32,-48)$ respectively. It is given that QR is parallel to the x axis.

(a) Find the equation of the perpendicular bisector of PR.

(b) Find the coordinates of the circumcentre of $\triangle PQR$.

(c) Let C be the circle which passes through P, Q and R.

(i) Find the equation of C.

(ii) Are the centre C and the in-centre of $\triangle PQR$ the same point? Explain your answer.

16C.31 HKCEE AM 1981 – II – 6

The circles $C_1 : x^2+y^2+7y+11=0$ and $C_2 : x^2+y^2+6x+4y+8=0$ touch each other externally at P.

(a) Find the coordinates of P.

(b) Find the equation of the common tangent at P.

16C.32 (HKCEE AM 1981 – II – 12)

The line $L : y = mx+2$ meets the circle $C : x^2+y^2=1$ at the points $A(x_1,y_1)$ and $B(x_2,y_2)$.

(a) (i) Show that x_1 and x_2 are the roots of the quadratic equation $(m^2+1)x^2+4mx+3=0$.

(ii) Hence, or otherwise, show that the length of the chord AB is $2\sqrt{\dfrac{m^2-3}{m^2+1}}$.

(b) Find the values of m such that

(i) L meets C at two distinct points,

(ii) L is a tangent to C,

(iii) L does not meet C.

(c) For the two tangents in (b)(ii), let the corresponding points of contact be P and Q. Find the equation of PQ.

16C.33 (HKCEE AM 1982 – II – 8)

M is the point $(5,6)$, L is the line $5x+12y=32$ and C is the circle with M as centre and touching L.

(a) (i) Find the equation of the straight line passing through M and perpendicular to L.

(ii) Hence, or otherwise, find the equation of C.

(b) Show that C also touches the y axis.

(c) Find the equation of the tangent (other than the y-axis) to C from the origin.

(d) $P(2,2)$ is a point on C. Q is another point on C such that PQ is a diameter. Find the equation of the circle which passes through P, Q and the origin.

16C.34 HKCEE AM 1984 – II – 6

Given the equation $x^2+y^2-2kx+4ky+6k^2-2=0$.

(a) Find the range of values of k so that the equation represents a circle with radius greater than 1.

(b) *[Out of syllabus]*

16C.35 (HKCEE AM 1985 II – 5)

If the equation $x^2+y^2+kx-(2+k)y=0$ represents a circle with radius $\sqrt{5}$,

(a) find the value(s) of k;

(b) find the equation(s) of the circle(s).

16C.36 HKCEE AM 1986 – II – 10

The circles $C_1: x^2+y^2-4x+2y+1=0$ and $C_2: x^2+y^2-10x-4y+19=0$ have a common chord AB.

(a) (i) Find the equation of the line AB.

(ii) Find the equation of the circle with AB as a chord such that the area of the circle is a minimum.

(b) The circle C_1 and another circle C_3 are concentric. If AB is a tangent to C_3, find the equation of C_3.

(c) *[Out of syllabus]*

16C.37 HKCEE AM 1987 – II – 11

In the figure, A and B are the points $(8,0)$ and $(16,0)$ respectively. The equation of the circle C_1 is $x^2+y^2-16x-4y+64=0$. OH and BH are tangents to C_1.

(a) (i) Show that C_1 touches the x axis at A.

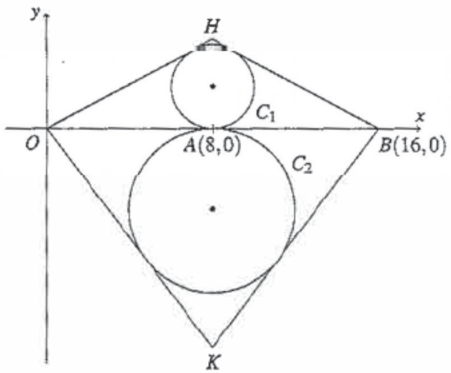
(ii) Find the equation of OH.

(iii) Find the equation of BH.

(b) In the figure, the equation of OK is $4x+3y=0$. The circle $C_2: x^2+y^2-16x+2fy+c=0$ is the inscribed circle of $\triangle OBK$ and touches the x-axis at A.

(i) Find the values of the constants c and f.

(ii) Find area of $\triangle OBH$: area of $\triangle OBK$.

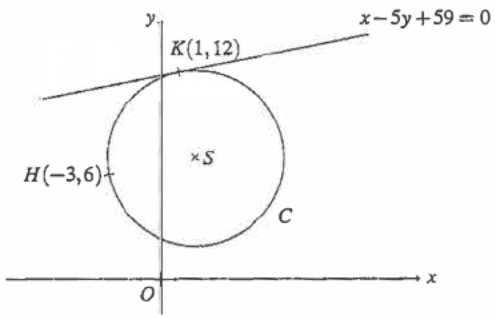


16C.38 (HKCEE AM 1988 II – 11)

In the figure, S is the centre of the circle C which passes through $H(-3,6)$ and touches the line $x-5y+59=0$ at $K(1,12)$.

(a) Find the coordinates of S. Hence, or otherwise, find the equation of the circle C.

(b) The line $L: 3x-2y-5=0$ cuts the circle C at A and B. Find the equation of the circle with AB as diameter.



16C.39 HKCEE AM 1993 – II – 11

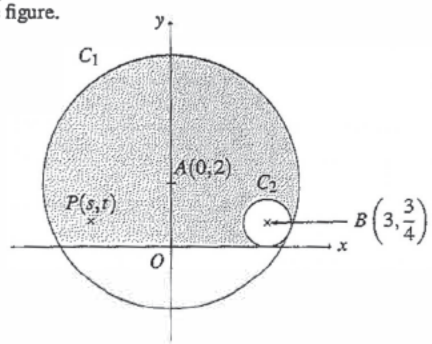
$A(0,2)$ is the centre of circle C_1 with radius 4. $B\left(3,\frac{3}{4}\right)$ is the centre of circle C_2 which touches the x axis. $P(s,t)$ is any point in the shaded region as shown in the figure.

(a) Find AB and the radius of C_2. Hence show that C_1 and C_2 touch each other.

(b) If P is the centre of a circle which touches the x axis and C_1, show that $4t=12-s^2$.

(c) If P is the centre of a circle which touches the x-axis and C_2, show that $3t=(s-3)^2$.

(d) Given that there are two circles in the shaded region, each of which touches the x-axis, C_1 and C_2. Using (b) and (c), find the equations of the two circles, giving your answers in the form $(x-h)^2+(y-k)^2=r^2$.



16C.40 HKCEE AM 1994 – II – 9

Given two points $A(5,5)$ and $B(7,1)$. Let (h,k) be the centre of a circle C which passes through A and B.

(a) Express h in terms of k. Hence show that the equation of C is $x^2+y^2-4kx-2ky+30k-50=0$.

(b) If the tangent to C at B is parallel to the line $y=\frac{1}{2}x$, find the equation of C.

(c) *[Out of syllabus]*

16C.41 HKCEE AM 1995 – II – 10

C_1 is the circle $x^2+y^2-16x-36=0$ and C_2 is a circle centred at the point $A(-7,0)$. C_1 and C_2 touch externally as shown in the figure. $P(h,k)$ is a point in the second quadrant.

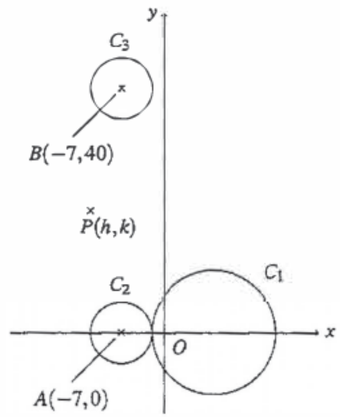
(a) Find the centre and radius of C_1. Hence find the radius of C_2.

(b) If P is the centre of a circle which touches both C_1 and C_2 externally, show that $8h^2-k^2-8h-48=0$.

(c) C_3 is a circle centred at the point $B(-7,40)$ and of the same radius as C_2.

(i) If P is the centre of a circle which touches both C_2 and C_3 externally, write down the equation of the locus of P.

(ii) Find the equation of the circle, with centre P, which touches all the three circles C_1, C_2 and C_3 externally.



16C.42 (HKCEE AM 1996 – II – 10)

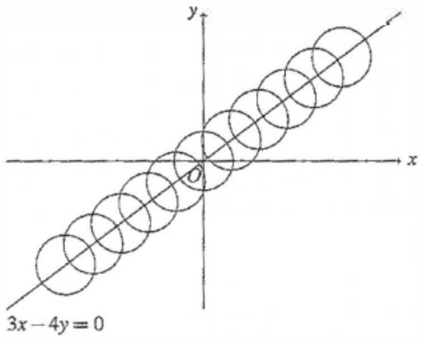
The equation $C_k: x^2+y^2-8kx-6ky+25(k^2-1)=0$, where k is real, represents a circle.

(a) (i) Find the centre of C_k in terms of k. Hence show that the centre of C_k lie on the line $3x-4y=0$ for all values of k.

(ii) Show that C_k has a radius of 5.

(b) The figure shows some C_k's for various values of k. It is given that there are two parallel lines, both of which are common tangents to all C_k's. Write down the slope of these two common tangents. Hence find the equations of these two common tangents.

(c) For a certain value of k, C_k cuts the x-axis at two points A and B. Write down the distance from the centre of the circle to the x axis in terms of k. Hence, or otherwise, find the two possible values of k such that C_k satisfies the condition $AB=8$.



16C.43 (HKCEE AM 1998 – II – 2)

Given a line $L: x-7y+3=0$ and a circle $C: (x-2)^2+(y+5)^2=a$, where a is a positive number. If L is a tangent to C, find the value of a.

16C.44 (HKCEE AM 2000 – II – 9)

A circle has the equation $(F): x^2+y^2+(4k+4)x+(3k+1)y-(8k+8)=0$, where k is real.

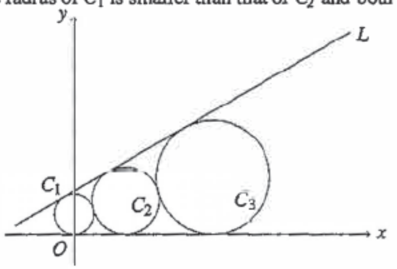
(a) Rewrite the equation (F) in the form $(x-p)^2+(y-q)^2=r^2$.

(b) C_1 and C_2 are two circles described by (F) such that the radius of C_1 is smaller than that of C_2 and both of them touch the x axis.

(i) Find the equations of C_1 and C_2.

(ii) Show that C_1 and C_2 touch each other externally.

(c) The figure shows the circles C_1 and C_2 in (b). L is a common tangent to C_1 and C_2. C_3 is a circle touching C_2, L and the x-axis. Find the equation of C_3. (Hint: The centres of the three circles are collinear.)



16C.45 HKCEE AM 2002 15

(Continued from 12B.18.)

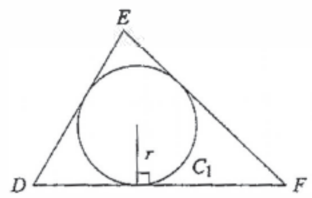


Figure (1)

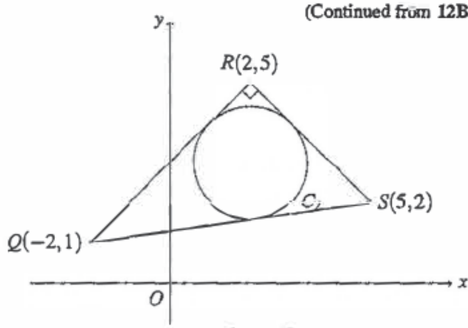


Figure (2)

(a) DEF is a triangle with perimeter p and area A. A circle C_1 of radius r is inscribed in the triangle (see Figure (1)). Show that $A=\frac{1}{2}pr$.

(b) In Figure (2), a circle C_2 is inscribed in a right-angled triangle QRS. The coordinates of Q, R and S are $(-2,1)$, $(2,5)$ and $(5,2)$ respectively.

(i) Using (a), or otherwise, find the radius of C_2.

(ii) Find the equation of C_2.

16C.46 HKCEE AM 2005 – 15

The figure shows a circle $C_1: x^2+y^2-4x-2y+4=0$ centred at point A. L is the straight line $y=kx$.

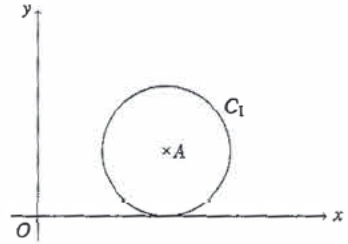
(a) Find the range of k such that C_1 and L intersect.

(b) There are two tangents from the origin O to C_1. Find the equation of the tangent L_1 other than the x-axis.

(c) Suppose that L and C_1 intersect at two distinct points P and Q. Let M be the mid-point of PQ.

(i) Show that the x coordinate of M is $\frac{k+2}{k^2+1}$.

(ii) *[Out of syllabus]*



16C.47 HKCEE AM 2006 – 14

Let J be the circle $x^2+y^2=r^2$, where $r>0$.

(a) Suppose that the straight line $L: y=mx+c$ is a tangent to J.

(i) Show that $c^2=r^2(m^2+1)$.

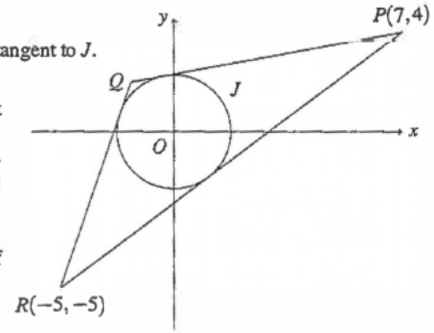
(ii) If L passes through a point (h,k), show that $(k-mh)^2=r^2(m^2+1)$.

(b) J is inscribed in a triangle PQR (see the figure). The coordinates of P and R are $(7,4)$ and $(-5,-5)$ respectively.

(i) Find the radius of J.

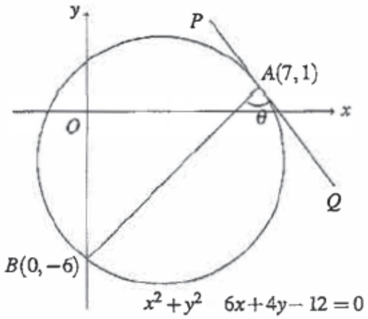
(ii) Using (a)(ii), or otherwise, find the slope of PQ.

(iii) Find the coordinates of Q.



16C.48 HKCEE AM 2010 – 7

In the figure, a tangent PQ is drawn to the circle $x^2+y^2 \quad 6x+4y-12=0$ at the point $A(7,1)$. $B(0,-6)$ is another point lying on the circle. Let θ be the acute angle between AB and PQ. Find the value of $\tan\theta$.



16C.49 HKCEE AM 2010 – 15

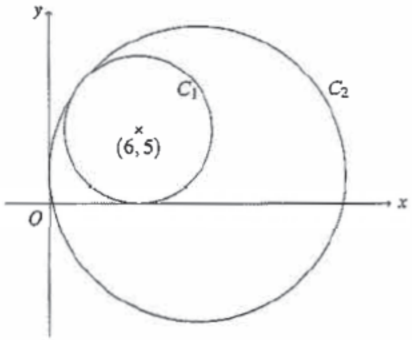
In the figure, C_1 is a circle with centre $(6,5)$ touching the x axis. C_2 is a variable circle which touches the y axis and C_1 internally.

(a) Show that the equation of locus of the centre of C_2 is $x=\frac{1}{2}y^2 \quad 5y+18$.

(b) It is known that the length of the tangent from an external point $P(0,-3)$ to C_2 is 5 and the centre of C_2 is in the first quadrant.

 (i) Find the centre of C_2.

 (ii) Find the equations of the two tangents from P to C_2.



16C.50 HKDSE MA SP – I – 19

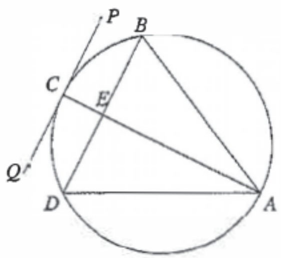
(Continued from 12B.19.)

In the figure, the circle passes through four points A, B, C and D. PQ is the tangent to the circle at C and is parallel to BD. AC and BD intersect at E. It is given that $AB=AD$.

(a) (i) Prove that $\triangle ABE \cong \triangle ADE$.

 (ii) Are the in-centre, the orthocentre, the centroid and the circumcentre of $\triangle ABD$ collinear? Explain your answer.

(b) A rectangular coordinate system is introduced in the figure so that the coordinates of A, B and D are $(14,4)$, $(8,12)$ and $(4,4)$ respectively. Find the equation of the tangent PQ.



16C.51 HKDSE MA PP – I 14

(Continued from 12A.28.)

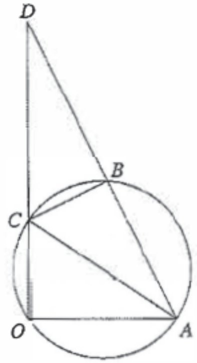
In the figure, $OABC$ is a circle. It is given that AB produced and OC produced meet at D.

(a) Write down a pair of similar triangles in the figure.

(b) Suppose that $\angle AOD = 90°$. A rectangular coordinate system, with O as the origin, is introduced in the figure so that the coordinates of A and D are $(6,0)$ and $(0,12)$ respectively. If the ratio of the area of $\triangle BCD$ to the area of $\triangle OAD$ is $16:45$, find

 (i) the coordinates of C,

 (ii) the equation of the circle $OABC$.



16C.52 HKDSE MA 2012 – I – 17

The coordinates of the centre of the circle C are $(6,10)$. It is given that the x axis is a tangent to C.

(a) Find the equation of C.

(b) The slope and the y intercept of the straight line L is -1 and k respectively. If L cuts C at A and B, express the coordinates of the mid-point of AB in terms of k.

16C.53 HKDSE MA 2015 – I – 14

The coordinates of the points P and Q are $(4,-1)$ and $(-14,23)$ respectively.

(a) Let L be the perpendicular bisector of PQ.

 (i) Find the equation of L.

 (ii) Suppose that G is a point lying on L. Denote the x-coordinate of G by h. Let C be the circle which is centred at G and passes through P and Q.
 Prove that the equation of C is $2x^2+2y^2-4hx-(3h+59)y+13h \quad 93=0$.

(b) The coordinates of the point R are $(26,43)$. Using (a)(ii), or otherwise, find the diameter of the circle which passes through P, Q and R.

16C.54 HKDSE MA 2016 – I – 20

(Continued from 12B.20.)

$\triangle OPQ$ is an obtuse-angled triangle. Denote the in-centre and the circumcentre of $\triangle OPQ$ by I and J respectively. It is given that P, I and J are collinear.

(a) Prove that $OP=PQ$.

(b) A rectangular coordinate system is introduced so that the coordinates of O and Q are $(0,0)$ and $(40,30)$ respectively while the y coordinate of P is 19. Let C be the circle which passes through O, P and Q.

 (i) Find the equation of C.

 (ii) Let L_1 and L_2 be two tangents to C such that the slope of each tangent is $\frac{3}{4}$ and the y-intercept of L_1 is greater than that of L_2. L_1 cuts the x axis and the y-axis at S and T respectively while L_2 cuts the x-axis and y axis at U and V respectively. Someone claims that the area of the trapezium $STUV$ exceeds 17 000. Is the claim correct? Explain your answer.

16C.55 HKDSE MA 2018 – I – 19

The coordinates of the centre of the circle C are $(8,2)$. Denote the radius of C by r. Let L be the straight line $kx-5y-21=0$, where k is a constant. It is given that L is a tangent to C.

(a) Find the equation of C in terms of r. Hence, express r^2 in terms of k.

(b) L passes through the point $D(18,39)$.

  (i) Find r.

  (ii) It is given that L cuts the y-axis at the point E. Let F be a point such that C is the inscribed circle of $\triangle DEF$. Is $\triangle DEF$ an obtuse-angled triangle? Explain your answer.

16C.56 HKDSE MA 2019 I 19 (Continued from 7E.5.)

Let $f(x)=\dfrac{1}{1+k}(x^2+(6k-2)x+(9k+25))$, where k is a positive constant. Denote the point $(4,33)$ by F.

(a) Prove that the graph of $y=f(x)$ passes through F.

(b) The graph of $y=g(x)$ is obtained by reflecting the graph of $y=f(x)$ with respect to the y-axis and then translating the resulting graph upwards by 4 units. Let U be the vertex of the graph of $y=g(x)$. Denote the origin by O.

  (i) Using the method of completing the square, express the coordinates of U in terms of k.

  (ii) Find k such that the area of the circle passing through F, O and U is the least.

  (iii) For any positive constant k, the graph of $y=g(x)$ passes through the same point G. Let V be the vertex of the graph of $y=g(x)$ such that the area of the circle passing through F, O and V is the least. Are F, G, O and V concyclic? Explain your answer.

16C.57 HKDSE MA 2020 I 14

The coordinates of the points A and B are $(\ 10,0)$ and $(30,0)$ respectively. The circle C passes through A and B. Denote the centre of C by G. It is given that the y-coordinate of G is -15.

(a) Find the equation of C. (3 marks)

(b) The straight line L passes through B and G. Another straight line ℓ is parallel to L. Let P be a moving point in the rectangular coordinate plane such that the perpendicular distance from P to L is equal to the perpendicular distance from P to ℓ. Denote the locus of P by Γ. It is given that Γ passes through A.

  (i) Describe the geometric relationship between Γ and L.

  (ii) Find the equation of Γ.

  (iii) Suppose that Γ cuts C at another point H. Someone claims that $\angle GAH<70°$. Do you agree? Explain your answer.

(6 marks)

16D Loci in the rectangular coordinate plane

16D.1 (HKCEE MA 1981(3) I – 7)

The parabola $y^2=4ax$ passes through the points $A(1,4)$ and $B(16,\ \ 16)$. A point P divides AB internally such that $AP:PB\ \ 1:4$.

(a) Find the coordinates of P.

(b) Show that the parabola is the locus of a moving point which is equidistant from P and the line $x=-a$.

16D.2 HKCEE AM 1987 II 10

$P(x,y)$ is a variable point equidistant from the point $S(1,0)$ and the line $x+1=0$.

(a) Show that the equation of the locus of P is $y^2=4x$.

(b) *[Out of syllabus]*

16D.3 (HKCEE AM 1994 II – 4)

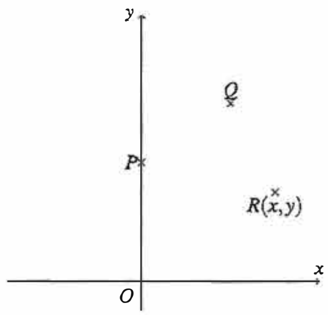
In the figure, $P(0,4)$ and $Q(2,6)$ are two points and $R(x,y)$ is a variable point.

(a) Suppose $R_0=(4,4)$ (not shown in the figure). Find the area of $\triangle PQR_0$.

(b) If the area of $\triangle PQR$ is 4 square units,

  (i) describe the locus of R and sketch it in the figure;

  (ii) find the equation(s) of the locus of R.



16D.4 HKCEE AM 1999 – II 10

$A(-3,0)$ and $B(-1,0)$ are two points and $P(x,y)$ is a variable point such that $PA=\sqrt{3}PB$. Let C be the locus of P.

(a) Show that the equation of C is $x^2+y^2=3$.

(b) $T(a,b)$ is a point on C. Find the equation of the tangent to C at T.

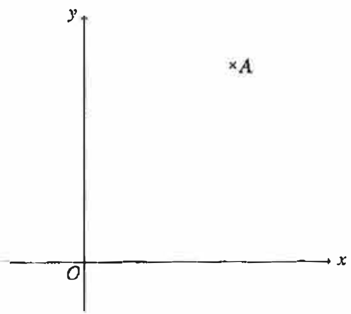
(c) The tangent from A to C touches C at a point S in the second quadrant. Find the coordinates of S.

(d) *[Out of syllabus]*

16D.5 (HKCEE AM 2004 10)

In the figure, O is the origin and A is the point $(3,4)$. P is a variable point (not shown) such that the area of $\triangle OPA$ is always equal to 2.

Describe the locus of P and sketch it in the figure.



16D.6 (HKCEE AM 2011 – 16) ***[Difficult!]***

Figure (1) shows a circle $C_1 : x^2+y^2-10y+16=0$. $Z(x,y)$ is the centre of a circle which touch the x axis and C_1 externally. Let S be the locus of Z.

(a) Show that the equation of S is $y=\frac{1}{16}x^2+1$.

(b) Let C_2 and C_3 be circles touching the x-axis and C_1 externally. It is given that C_2 passes through the point $(20,16)$ and it touches C_3 externally. Suppose that both the centres of C_2 and C_3 lie in the first quadrant (see Figure (2)).
   (i) Find the equation of C_2.
   (ii) Without any algebraic manipulation, determine whether the following sentence is correct:
   "The point of contact of C_2 and C_3 lies on S."

(c) Can we draw a circle satisfying all the following conditions?
   - Its centre lies on S.
   - It touches the x axis.
   - It touches C_1 internally.

   Explain your answer.

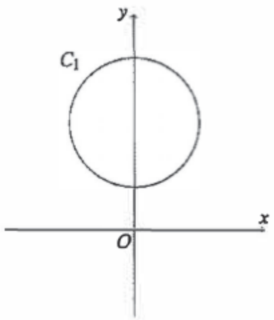


Figure (1)

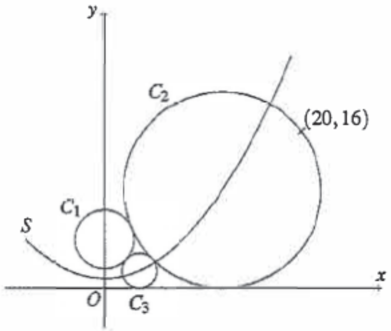


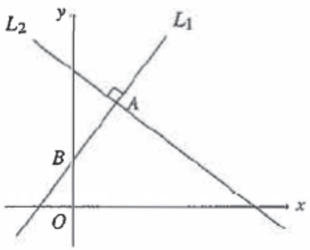
Figure (2)

16D.7 HKDSE MA SP I 13

In the figure, the straight line $L_1 : 4x-3y+12=0$ and the straight line L_2 are perpendicular to each other and intersect at A. It is given that L_1 cuts the y-axis at B and L_2 passes through the point $(4,9)$.

(a) Find the equation of L_2.

(b) Q is a moving point in the coordinate plane such that $AQ=BQ$. Denote the locus of Q by Γ.
   (i) Describe the geometric relationship between Γ and L_2. Explain your answer.
   (ii) Find the equation of Γ.



16D.8 HKDSE MA PP – I – 8

The coordinates of the points A and B are $(-3,4)$ and $(-2,-5)$ respectively. A' is the reflection image of A with respect to the y axis. B is rotated anticlockwise about the origin O through 90° to B'.

(a) Write down the coordinates of A' and B'.

(b) Let P be a moving point in the rectangular coordinate plane such that P is equidistant from A' and B'. Find the equation of the locus of P.

16D.9 HKDSE MA 2012 – I – 14

The y-intercepts of two parallel lines L and ℓ are -1 and 3 respectively and the x intercept of L is 3. P is a moving point in the rectangular coordinate plane such that the perpendicular distance from P to L is equal to the perpendicular distance from P to ℓ. Denote the locus of P by Γ.

(a) (i) Describe the geometric relationship between Γ and L.
   (ii) Find the equation of Γ.

(b) The equation of the circle C is $(x-6)^2+y^2=4$. Denote the centre of C by Q.
   (i) Does Γ pass through Q? Explain your answer.
   (ii) If L cuts C at A and B while Γ cuts C at H and K, find the ratio of the area of $\triangle AQH$ to the area of $\triangle BQK$.

16D.10 HKDSE MA 2013 – I – 14

The equation of the circle C is $x^2+y^2-12x-34y+225=0$. Denote the centre of C by R.

(a) Write down the coordinates of R.

(b) The equation of the straight line L is $4x+3y+50=0$. It is found that C and L do not intersect. Let P be a point lying on L such that P is nearest to R.
   (i) Find the distance between P and R.
   (ii) Let Q be a moving point on C. When Q is nearest to P,
      (1) describe the geometric relationship between P, Q and R;
      (2) find the ratio of the area of $\triangle OPQ$ to the area of $\triangle OQR$, where O is the origin.

16D.11 HKDSE MA 2014 – I – 12

The circle C passes through the point $A(6,11)$ and the centre of C is the point $G(0,3)$.

(a) Find the equation of C.

(b) P is a moving point in the rectangular coordinate plane such that $AP=GP$. Denote the locus of P by Γ.
   (i) Find the equation of Γ.
   (ii) Describe the geometric relationship between Γ and the line segment AG.
   (iii) If Γ cuts C at Q and R, find the perimeter of the quadrilateral $AQGR$.

16D.12 HKDSE MA 2016 – I – 10

The coordinates of the points A and B are $(5,7)$ and $(13,1)$ respectively. Let P be a moving point in the rectangular coordinate plane such that P is equidistant from A and B. Denote the locus of P by Γ.

(a) Find the equation of Γ.

(b) Γ intersects the x-axis and the y axis at H and K respectively. Denote the origin by O. Let C be the circle which passes through O, H and K. Someone claims that the circumference of C exceeds 30. Is the claim correct? Explain your answer.

16D.13 HKDSE MA 2017 – I – 13

The coordinates of the points E, F and G are $(-6,5)$, $(\ 3,11)$ and $(2,-1)$ respectively. The circle C passes through E and the centre of C is G.

(a) Find the equation of C.

(b) Prove that F lies outside C.

(c) Let H be a moving point on C. When H is farthest from F,
   (i) describe the geometric relationship between F, G and H;
   (ii) find the equation of the straight line which passes through F and H.

16D.14 HKDSE MA 2019 – I – 17 (Continued from **12B.21**.)

(a) Let a and p be the area and perimeter of $\triangle CDE$ respectively. Denote the radius of the inscribed circle of $\triangle CDE$ by r. Prove that $pr = 2a$.

(b) The coordinates of the points H and K are $(9,12)$ and $(14,0)$ respectively. Let P be a moving point in the rectangular coordinate plane such that the perpendicular distance from P to OH is equal to the perpendicular distance from P to HK, where O is the origin. Denote the locus of P by Γ.

(i) Describe the geometric relationship between Γ and $\angle OHK$.

(ii) Using (a), find the equation of Γ.

16E Polar coordinates

16E.1 HKCEE MA 2009 – I – 8

In a polar coordinate system, O is the pole. The polar coordinates of the points P and Q are $(k, 123°)$ and $(24, 213°)$ respectively, where k is a positive constant. It is given that $PQ = 25$.

(a) Is $\triangle OPQ$ a right-angled triangle? Explain your answer.

(b) Find the perimeter of $\triangle OPQ$.

16E.2 HKDSE MA PP – I – 6

In a polar coordinate system, the polar coordinates of the points A, B and C are $(13, 157°)$, $(14, 247°)$ and $(15, 337°)$ respectively.

(a) Let O be the pole. Are A, O and C collinear? Explain your answer.

(b) Find the area of $\triangle ABC$.

16E.3 HKDSE MA 2013 – I – 6

In a polar coordinate system, O is the pole. The polar coordinates of the points A and B are $(26, 10°)$ and $(26, 130°)$ respectively. Let L be the axis of reflectional symmetry of $\triangle OAB$.

(a) Describe the geometric relationship between L and $\angle AOB$.

(b) Find the polar coordinates of the point of intersection of L and AB.

16E.4 HKDSE MA 2016 – I – 7

In a polar coordinate system, O is the pole. The polar coordinates of the points A and B are $(12, 75°)$ and $(12, 135°)$ respectively.

(a) Find $\angle AOB$.

(b) Find the perimeter of $\triangle AOB$.

(c) Write down the number of folds of rotational symmetry of $\triangle AOB$.

16 Coordinate Geometry

16A Transformation in the rectangular coordinate plane

16A.1 HKCEE MA 2006 – I – 7

(a) $A' = (7,2),\ B' = (5,5)$

(b) $AB = \sqrt{(-2+5)^2+(7-5)^2} = \sqrt{14}$

$A'B' = \sqrt{(7-5)^2+(2\ \ 5)^2} = \sqrt{14} = AB$

. YES

16A.2 HKCEE MA 2009 – I – 9

(a) $A' = (-1,4),\ B' = (-5,2)$

(b) $m_{AB} = \frac{2+2}{5+1} = \frac{2}{3},\ m_{A'B'} = \frac{4-2}{-1+5} = \frac{1}{2} \neq m_{AB}$

$\therefore$ NO

16A.3 HKCEE MA 2011 – I – 8

(a) $B = (-6,-4),\ M = \left(\frac{-4-6}{2}, \frac{6\ \ 4}{2}\right) = (-5,1)$

(b) $m_{OM} = \frac{1}{-5},\ m_{AB} = 5$

$\because m_{OM} \cdot m_{AB} = -1$

$\therefore OM \perp AB$

16A.4 HKDSE MA SP – I – 8

(a) $A' = (5,2),\ A'' = (2,5)$

(b) $m_{OA''} = \frac{5}{2},\ m_{AA'} = \frac{-3}{7}$

$\because m_{OA''} m_{AA'} = \frac{15}{14} \neq -1$

$\therefore OA''$ is not perpendicular to AA'.

16A.5 HKDSE MA 2014 – I – 8

(a) $P' = (5,3),\ Q' = (-19,\ \ 7)$

(b) $m_{PQ} = \frac{-12}{5},\ m_{P'Q'} = \frac{10}{24} = \frac{5}{12}$

$\because m_{PQ} m_{P'Q'} = -1$

$\therefore PQ \perp P'Q'$

16A.6 HKDSE MA 2017 – I – 6

(a) $A' = (-4,-3),\ B' = (9,9)$

(b) $m_{AB} = \frac{13}{-12},\ m_{A'B'} = \frac{12}{13}$

$\because m_{AB} m_{A'B'} = -1$

$\therefore AB \perp A'B'$

16B Straight lines in the rectangular coordinate plane

16B.1 HKCEE MA 1992 – I – 5

(a) $m_{L_2} = \frac{1}{2} \Rightarrow m_{L_1} = -2$

$\therefore$ Eqn of L_1: $y-5 = -2(x-10) \Rightarrow 2x+y-25 = 0$

(b) $\begin{cases} L_1 : 2x+y-25 = 0 \\ L_2 : x-2y+5 = 0 \end{cases} \Rightarrow (x,y) = (9,7)$

16B.2 HKCEE MA 1998 – I – 8

(a) $m_{AB} = \frac{4-1}{0+2} = \frac{3}{2}$

(b) Required eqn: $y-3 = \frac{1}{\frac{3}{-2}}(x-1) \Rightarrow 2x+3y-11 = 0$

16B.3 HKCEE MA 1999 – I – 10

(a) $M = \left(\frac{-8+16}{2}, \frac{8-4}{2}\right) = (4,2)$

$m_{AB} = \frac{12}{-24} = -\frac{1}{2} \Rightarrow m_\ell = 2$

$\therefore$ Eqn of ℓ: $y-2 = 2(x-4) \Rightarrow 2x-y-6 = 0$

(b) Put $y = 0$ into eqn of $\ell \Rightarrow x = 3 \Rightarrow P = (3,0)$

$BP = \sqrt{(16-3)^2+(-4-0)^2} = \sqrt{185}$

(c) $N = \left(\frac{-8+3}{2}, \frac{8+0}{2}\right) = \left(-\frac{5}{2}, 4\right)$

$\therefore MN = \sqrt{\left(3+\frac{5}{2}\right)^2 + (0-4)^2} = \sqrt{\frac{185}{4}} = \frac{\sqrt{185}}{2}$

16B.4 HKCEE MA 2000 – I – 9

(a) $m_L = \frac{4-0}{-4\ \ 6} = \frac{2}{5}$

(b) Eqn of L: $y-0 = -\frac{2}{5}(x-6) \Rightarrow 2x+5y-12 = 0$

(c) Put $x = 0 \Rightarrow y = \frac{12}{5} \Rightarrow C = \left(0, \frac{12}{5}\right)$

16B.5 HKCEE MA 2001 – I – 7

(a) $A = (-1,5),\ B = (4,3)$

(b) Eqn of AB: $\frac{y-5}{x+1} = \frac{5-3}{1-4} = \frac{2}{-5}$

$-5(y\ \ 5) = -2(x+1) \Rightarrow 2x+5y-23 = 0$

16B.6 HKCEE MA 2002 – I – 8

(a) $x-2y = -8 \Rightarrow \frac{x}{-8}+\frac{y}{4} = 1$

$\therefore A = (\ \ 8,0),\ B = (0,4)$

(b) Mid-pt of $AB = \left(\frac{-8+0}{2}, \frac{0+4}{2}\right) = (\ \ 4,2)$

16B.7 HKCEE MA 2003 – I – 12

(a) $m_{BC} = \frac{3\ \ 0}{0-2} = \frac{3}{2}$

(b) $m_{AP} = -1 \div \frac{3}{2} = \frac{2}{3}$

$\therefore$ Eqn of AP: $y-0 = \frac{2}{3}(x+1) \Rightarrow 2x-3y+2 = 0$

(c) (i) Put $x = 0 \Rightarrow y = \frac{2}{3} \Rightarrow H = \left(0, \frac{2}{3}\right)$

(ii) $m_{HB} = \frac{\frac{2}{3}-0}{0-2} = \frac{-1}{3},\ m_{AC} = \frac{3-0}{0+1} = 3 = \frac{-1}{m_{HB}}$

$\therefore HB \perp AC$

Hence the 3 altitudes of $\triangle ABC$ are CO, AP and HB, all passing through H.

16B.8 HKCEE MA 2004 – I – 13

(a) (i) $E =$ mid-pt of $AC = \left(\frac{2+8}{2}, \frac{9+1}{2}\right) = (5,5)$

(ii) $m_{AC} = \frac{9-1}{2\ \ 8} = \ \frac{4}{3} \Rightarrow m_{BD} = \frac{3}{4}$

$\therefore$ Eqn of BD: $y-5 = \frac{3}{4}(x-5) \Rightarrow 3x-4y+5 = 0$

(b) (i) *Method 1*

$m_{AD} = \frac{1}{7}$

$\Rightarrow BC$: $y-1 = \frac{-1}{7}(x-8) \Rightarrow x+7y-15 = 0$

Method 2

Let BC be $x+7y+K\ \ 0$.

Put C: $(8)+7(1)+K = 0 \Rightarrow K = -15$

$\therefore$ Eqn of BC is $x+7y-15 = 0$.

(ii) $\begin{cases} BD : 3x-4y+5 = 0 \\ BC : x+7y\ \ 15 = 0 \end{cases} \Rightarrow B = (1,2)$

$\therefore AB = \sqrt{(2-1)^2+(9-2)^2} = \sqrt{50}$

16B.9 HKCEE MA 2005 – I – 13

(a) $A = (-2,0),\ B = (0,4)$

(b) $m_{L_1} = 2 \Rightarrow m_{L_2} = \frac{-1}{2}$

$\therefore$ Eqn of L_2: $y = \frac{-1}{2}x+4$

(c) $C = (8,0)$

$OC : AC = 8 : (8+2) = 4 : 5$

$\therefore$ Area of $\triangle ODC$: Area of $\triangle ABC = 16 : 25$

$\Rightarrow$ Area of $\triangle ODC$: Area of $OABD = 16 : (25-16)$

$= 16 : 9$

16B.10 HKCEE MA 2006 – I – 12

(a) $M = (4,4)$

(b) $m_{AB} = \frac{1}{2} \Rightarrow m_{CM} = \ \ 2$

$\therefore$ Eqn of CM: $y-4 = -2(x-4) \Rightarrow 2x+y-12 = 0$

Hence, put $y = 0 \Rightarrow C = (6,0)$

(c) (i) Eqn of BD: $\frac{y-0}{x-2} = \frac{8-0}{12-2} = \frac{4}{5} \Rightarrow 4x\ \ 5y\ \ 8 = 0$

(ii) $\begin{cases} CM : 2x+y-12 = 0 \\ BD : 4x\ \ 5y-8 = 0 \end{cases} \Rightarrow K = \left(\frac{34}{7}, \frac{16}{7}\right)$

Method 1

$\frac{\text{Area of } \triangle AMC}{\text{Area of } \triangle AKC}\ \ \frac{y\text{-coor of } M}{y\text{-coor of } K} = \frac{4}{\frac{16}{7}} = \frac{7}{4}$

Method 2

$\frac{\text{Area of } \triangle AMC}{\text{Area of } \triangle AKC} = \frac{MC}{KC} = \frac{\sqrt{(4-6)^2+(4-0)^2}}{\sqrt{(6-\frac{34}{7})^2+(0-\frac{16}{7})^2}}$

$= \frac{\sqrt{20}}{\sqrt{\frac{320}{49}}} = \frac{7}{4}$

Method 3

Let $MK : KC = r : s \Rightarrow \frac{16}{7} = \frac{s(4)+r(0)}{r+s}$

$16r+16s = 28s$

$r : s = 12 : 16 = 3 : 4$

$\therefore \frac{\text{Area of } \triangle AMC}{\text{Area of } \triangle AKC} = \frac{MC}{KC} = \frac{7}{4}$

16B.11 HKCEE MA 2007 – I – 13

(a) Eqn of AB: $y\ \ 3 = \frac{-4}{3}(x-10) \Rightarrow 4x+3y-49 = 0$

(b) Put $x = 4 \Rightarrow y = 11 \Rightarrow h = 11$

(c) (i) (Since $\triangle ABC$ is isosceles, A should lie 'above' the mid-point fo BC.)

$\frac{k+10}{2} = 4 \Rightarrow k = -2$

(ii) Area of $\triangle ABC = \frac{(10+2)(11-3)}{2} = 48$

$AC = \sqrt{(4+2)^2+(11-3)^2} = 10$

$\therefore BD = \frac{2 \times \text{Area of } \triangle ABC}{AC} = \frac{48}{5}$

16B.12 HKCEE MA 2008 – I – 12

(a) $B = (-3,4),\ C = (4,-3)$

(b) $m_{OB} = \frac{4}{-3},\ m_{OC} = \frac{-3}{4} \neq m_{OB}$

.. NO

(c) $m_{CD} = \frac{-1}{m_{BC}} = 1$

$\therefore$ Eqn of CD: $y+3 = 1(x-4) \Rightarrow x-y\ \ 7 = 0$

$\because$ D is translated horizontally from A,

$\therefore$ y-coordinate of $D = y$-coordinat eof $A = 3$

Put into eqn of $CD \Rightarrow x = 10 \Rightarrow D = (10,3)$

16B.13 HKCEE MA 2010 – I – 12

(a) Eqn of AB: $\frac{y-24}{x-6} = \frac{18-24}{-2-6} = \frac{3}{4} \Rightarrow 3x-4y+78 = 0$

(b) Let $C = (x,0)$.

$m_{AC} = \frac{-1}{m_{AB}} = \frac{-4}{3}$

$\frac{24\ \ 0}{6-x} = \frac{-4}{3} \Rightarrow x = 24 \Rightarrow C = (24,0)$

(c) $AB = \sqrt{(24-18)^2+(6+2)^2} = 10$

$AC = \sqrt{(24-6)^2+(0-24)^2} = 30$

$\therefore$ Area of $\triangle ABC = \frac{10 \times 30}{2} = 150$

(d) $\frac{BD}{DC} = \frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ADC} \Rightarrow \frac{r}{1} = \frac{90}{150-90} \Rightarrow r = 1.5$

16B.14 HKCEE AM 1982 – II – 2

Method 1

Eqn of AB: $\frac{y-1}{x+1} = \frac{-1-1}{3+1} = \frac{-1}{2} \Rightarrow x+2y-1 = 0$

Let P be the pt of division. $\begin{cases} x+2y-1 = 0 \\ x-y-1 = 0 \end{cases} \Rightarrow P = (1,0)$

Let $AP : PB = r : 1 \Rightarrow 0 = \frac{-1+(1)r}{r+1} = \frac{r-1}{r+1} \Rightarrow r = 1$

$\therefore$ The required ratio is 1 : 1.

Method 2

Let the point of division be P, and $AP : PB = r : 1$.

$P = \left(\frac{3+(-1)r}{r+1}, \frac{-1+(1)r}{r+1}\right) = \left(\frac{3-r}{r+1}, \frac{r-1}{r+1}\right)$

If P lies on $x\ \ y-1 = 0$,

$\left(\frac{3-r}{r+1}\right) - \left(\frac{r-1}{r+1}\right) - 1 = 0 \Rightarrow r = 1$

$\therefore$ The required ratio is 1 : 1.

16B.15 <u>HKCEE AM 1982 – II – 10</u>

(a) $\begin{cases}3x\ \ 2y\ \ 8=0\\ x-y-2=0\end{cases} \Rightarrow P=(4,2)$

Eqn of L_1: $y\ \ 2=\frac{1}{2}(x\ \ 4) \Rightarrow x+2y-8=0$

Eqn of L_2: $y\ \ 2=2(x-4) \Rightarrow 2x-y-6=0$

16B.16 <u>(HKCEE AM 1985 – II – 10)</u>

(a) <u>*Method 1* – Use collinearity of points</u>

Let $R=(r,h)$ and $S=(s,h)$

$m_{RC}=m_{AC} \Rightarrow \frac{h}{r-1}=\frac{2-0}{0-1} \Rightarrow r=1-\frac{h}{2}$

$m_{SB}=m_{AB} \Rightarrow \frac{h}{s+3}=\frac{2-0}{0+3} \Rightarrow s=\frac{3}{2}h-3$

$\therefore S=\left(\frac{3}{2}h-3,h\right),\ R=\left(1\ \ \frac{h}{2},h\right)$

<u>*Method 2* – Use eqns of straight lines</u>

Eqn of AB: $\frac{y-0}{x+3}=\frac{2\ \ 0}{0+3} \Rightarrow 2x-3y+6=0$

Put $y=h \Rightarrow x=\frac{3}{2}h-3 \Rightarrow S=\left(\frac{3}{2}h-3,h\right)$

Eqn of AC: $\frac{y-0}{x-1}=\frac{2\ \ 0}{0-1} \Rightarrow 2x+y-2=0$

Put $y=h \Rightarrow x=1-\frac{h}{2} \Rightarrow R=\left(1-\frac{h}{2},h\right)$

<u>*Method 3* – Use similar triangles</u>

$\Delta BSP \sim \Delta BAO \Rightarrow \frac{h}{2}=\frac{BP}{3} \Rightarrow BP=\frac{3}{2}h$

$\therefore$ x-coordinate of $S=-3+\frac{3}{2}h \Rightarrow S=\left(\frac{3}{2}h-3,h\right)$

$\Delta AOC \sim \Delta RQC \Rightarrow \frac{2}{h}=\frac{1}{QC} \Rightarrow QC=\frac{h}{2}$

$\therefore$ x-coordinate of $R=1-\frac{h}{2} \Rightarrow R=\left(1-\frac{h}{2},h\right)$

(b) $RS=\left(1-\frac{h}{2}\right)-\left(\frac{3}{2}h-3\right)=4-2h$

When $PQRS$ is a square,

$PS=RS \Rightarrow h=4-2h \Rightarrow h=\frac{4}{3} \Rightarrow A_1=h^2=\frac{16}{9}$

Area of $PQRS=h(4-2h)=\ \ 2(h^2-2h)$
$=-2(h-1)^2+2 \Rightarrow A_2=2$

$A_3=\frac{2\times 4}{2}=4$

$\therefore A_1:A_2:A_3=\frac{16}{9}:2:4=8:9:18$

(c) $M=$ mid-pt of $PR=\left(\frac{h}{2}-1,\frac{h}{2}\right)$

i.e. $x=\frac{h}{2}-1,\ y=\frac{h}{2}$

Put into $x\ \ y+1=0$:

$\text{LHS}=\left(\frac{h}{2}-1\right)\ \ \left(\frac{h}{2}\right)+1=0=\text{RHS}$

$\therefore M$ lies on $x-y+1=0$

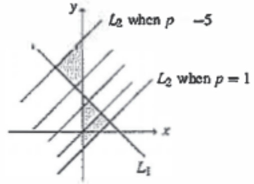
16B.17 <u>(HKCEE AM 1984 – II – 4)</u>

(a) $\begin{cases}L_1: x+y=4\\ L_2: x-y=2p\end{cases} \Rightarrow (x,y)=(2+p,2\ \ p)$

(b) y-intercept of $L_1=4$, y-intercept of $L_2=-2p$

$\therefore$ Area of $\Delta=\frac{[4-(-2p)](2+p)}{2}$

$9=(2+p)^2 \Rightarrow p=-5$ or 1



16B.18 <u>HKCEE AM 1988 – II – 2</u>

(a) $P\ \ \left(\frac{7k+1}{k+1},\frac{4k+2}{k+1}\right)$

(b) When P lies on $7x-3y-28=0$,

$7\left(\frac{7k+1}{k+1}\right)-3\left(\frac{4k+2}{k+1}\right)\ \ 28=0$

$7(7k+1)-3(4k+2)\ \ 28(k+1)=0$

$9k\ \ 27=0 \Rightarrow k=3$

$\therefore$ The ratio is 3 : 1.

16B.19 <u>HKCEE AM 1990 – II – 7</u>

<u>*Method 1* – Use algebra to find D</u>

Eqn of AB: $\frac{x}{3}+\frac{y}{5}=1 \Rightarrow 5x+3y-15=0$

Area of $\Delta OAB=\frac{5\times 3}{2}=\frac{15}{2} \Rightarrow$ Area of $\Delta BCD=\frac{15}{4}$

Let $D=(h,k)$. Then

$\begin{cases}5h+3k\ \ 15=0\\ \frac{15}{4}=\frac{(5-1)h}{2}=2h\end{cases} \Rightarrow D=\left(\frac{15}{8},\frac{15}{8}\right)$

<u>*Method 2* – Use ratios of areas to find D</u>

Area of $\Delta OAB=\frac{15}{2}$, $\Delta OAC=\frac{3}{2}$, $\Delta BCD=\frac{15}{4}$

$\Rightarrow$ Area of $\Delta ACD=\frac{15}{2}-\frac{15}{4}-\frac{3}{2}=\frac{9}{4}$

$\Rightarrow \frac{BD}{DA}=\frac{\text{Area of }\Delta BCD}{\text{Area of }\Delta ACD}=\frac{\frac{15}{4}}{\frac{9}{4}}=\frac{5}{3}$

$\therefore D=\left(\frac{3(0)+5(3)}{5+3},\frac{3(5)+5(0)}{5+3}\right)=\left(\frac{15}{8},\frac{15}{8}\right)$

<u>*Hence*,</u>

Eqn of CD: $\frac{y\ \ 1}{x-0}=\frac{\frac{15}{8}-1}{\frac{15}{8}-0}-\frac{7}{15} \Rightarrow 7x-15y+15=0$

16B.20 <u>(HKCEE AM 1996 – II – 8)</u>

$\begin{cases}L_1: 2x-y-4=0\\ L_2: x-2y+4=0\end{cases} \Rightarrow (x,y)=(4,4)$

$\therefore$ Eqn of required line: $\frac{y\ \ 0}{x-0}=\frac{4\ \ 0}{4-0} \Rightarrow y=x$

16B.21 <u>(HKCEE AM 1998 – II – 5)</u>

(a) $\begin{cases}L_1: 2x+y-3=0\\ L_2: x-3y+1=0\end{cases} \Rightarrow P=\left(\frac{8}{7},\frac{5}{7}\right)$

(b) Eqn of L: $\frac{y\ \ 0}{x\ \ 0}=\frac{\frac{5}{7}-0}{\frac{8}{7}-0} \Rightarrow y=\frac{5}{8}x$

16B.22 <u>HKCEE AM 2005 – 6</u>

(a) $\tan\theta=m_{L_1}=\ \ 2$

(b) $\angle OQP=\theta \Rightarrow \angle QOP=180°\ \ 2\theta$

$\therefore$ Eqn of L_2: $y=x\tan\angle QOP=x\tan(180°\ \ 2\theta)$
$=\ \ x\tan 2\theta$
$=-x\cdot\frac{2\tan\theta}{1-\tan^2\theta}$
$=-x\cdot\frac{2(\ \ 2)}{1-(-2)^2}$
$\Rightarrow y=\frac{4}{5}x$

16B.23 <u>(HKCEE AM 2009 – 3)</u>

$\begin{cases}L_1: x-2y+3=0\\ L_2: 2x-y-1=0\end{cases} \Rightarrow P=\left(\frac{5}{3},\frac{7}{3}\right)$

<u>*Method 1*</u>

Let the eqn of L be $\frac{x}{a}+\frac{y}{a}=1$, where $a>0$.

$\because P$ lies on L

$\therefore \left(\frac{5}{3}\right)+\left(\frac{7}{3}\right)\ \ a \Rightarrow a=4$

$\therefore$ Required line: $\frac{x}{4}+\frac{y}{4}=1 \Rightarrow x+y-4=0$

<u>*Method 2*</u>

Let L be $y-\frac{7}{3}=m\left(x-\frac{5}{3}\right) \Rightarrow 3mx\ \ 3y+7\ \ 5m=0$

$\Rightarrow$ x-intercept $=\frac{5m-7}{3m}$, y-intercept $=\frac{7\ \ 5m}{3}$

$\Rightarrow \frac{5m\ \ 7}{3m}=\frac{7\ \ 5m}{3} \Rightarrow 5m-7=-m(5m-7)$

$m=\frac{7}{5}$ or -1

However, when $m=\frac{7}{5}$, L becomes $7x-5y=0$, which has zero x- and y-intercepts. Rejected.

$\therefore$ Eqn of L is: $3(\ \ 1)x\ \ 3y+7\ \ 5(-1)=0 \Rightarrow x+y-4=0$

16B.24 <u>HKCEE AM 2010 – 6</u>

$\begin{cases}L_1: x-3y+7=0\\ L_2: 3x-y-11=0\end{cases} \Rightarrow (x,y)=(5,4)$

$\therefore$ Eqn of required line: $\frac{y-1}{x-2}=\frac{4-1}{5-2}=1 \Rightarrow x-y-1=0$

16C Circles in the rectangular coordinate plane

16C.1 <u>HKCEE MA 1980(1/3 I) – B – 15</u>

(a) Put $y=0 \Rightarrow x^2-10x+16=0 \Rightarrow x=2$ or 8

$\therefore A=(2,0)$, $B=(8,0)$

Put $x=0 \Rightarrow y^2+8y+16=0 \Rightarrow y=-4$

$\therefore T=(0,\ \ 4)$

(b) (i) $m_{TB}=\frac{0+4}{8-0}=\frac{1}{2}$

$\therefore$ Eqn of AC: $y\ \ 0=\frac{1}{2}(x\ \ 2) \Rightarrow x-2y-2=0$

(ii) $\begin{cases}x^2+y^2-10x+8y+16=0\\ x-2y\ \ 2=0\end{cases}$

$(2y+2)^2+y^2-10(2y+2)+8y+16=0$
$5y^2-8y=0$
$y=0$ or $\frac{8}{5}$

Put $y=\frac{8}{5} \Rightarrow x=\frac{26}{5} \Rightarrow C=\left(\frac{26}{5},\frac{8}{5}\right)$

16C.2 <u>HKCEE MA 1981(1/3) – I – 13</u>

(a) $x^2+y^2=15^2 \Rightarrow x^2+y^2-225=0$

(b) $OP=\frac{OT}{\sin\angle OPT}=\frac{OT}{\sin\theta}=\frac{15}{\frac{3}{\sqrt{3^2+4^2}}}=25$

(c) $P=(25,0)$

$\therefore$ Eqn of TP: $y\ \ 0=\frac{3}{4}(x\ \ 25) \Rightarrow 3x-4y-75=0$

(d) By geometry, $OCPT$ is a rectangle.

i.e. Eqn of OC: $y=\frac{3}{4}x$

(e) Let $C=(h,k)$. Then $k=\frac{3}{4}h$

$15=CP=\sqrt{(h-25)^2+(\frac{3}{4}h)^2}$

$225=\frac{25}{16}h^2-50h+625$

$h^2-32h+256=0 \Rightarrow h=16 \Rightarrow C=(16,12)$

Hence, eqn of circle is $(x-16)^2+(y-12)^2=15^2$
$x^2+y^2-32x-24y+175=0$

16C.3 <u>HKCEE MA 1982(I) – I – 13</u>

(a) $C: x^2+y^2-14y+40=0 \Rightarrow x^2+(y-7)^2=3^2$

$\therefore$ Centre $=(0,7)$, Radius $=3$

(b) $m_L=\frac{4}{3} \Rightarrow m_{L'}=\frac{3}{4}$

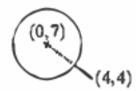
$\therefore$ Eqn of L': $y=\frac{-3}{4}x+7$

(c) $\begin{cases}L: 4x-3y-4=0\\ L': y=\frac{-3}{4}x+7\end{cases} \Rightarrow (x,y)=(4,4)$

(d) <u>Distance between centre</u> of C and $(4,4)$

$=\sqrt{(0-4)^2+(7-4)^2}=5$

$\Rightarrow$ Shortest dist $=5-$ radius $=2$



16C.4 HKCEE MA 1983(A/B) – I – 9

(a) Let $B=(b,0)$.
$1=m_{AB}=\frac{2\ \ 0}{8-6} \Rightarrow b=6 \Rightarrow B=(6,0)$

(b) Let $C=(c,0)$. Since $\triangle ABC$ is isosceles, A lies 'above' the mid-point of BC.
$\therefore \frac{c+6}{2}=8 \Rightarrow c=10 \Rightarrow C=(10,0)$

(c) Eqn of AC: $\frac{y-0}{x-10}=\frac{2\ \ 0}{8-10} \Rightarrow y=-x+10$
$\therefore D=(0,10)$

(d) $BD=\sqrt{6+10^2}=\sqrt{136}$
Mid-pt of $BD=\left(\frac{6+0}{2},\frac{0+10}{2}\right)=(3,5)$
Eqn of circle OBD is $(x-3)^2+(y-5)^2=\left(\frac{\sqrt{136}}{2}\right)^2$
$\Rightarrow x^2+y^2\ \ 6x\ \ 10y=0$
Put A int othe equation:
LHS $=(8)^2+(2)^2\ \ 6(8)\ \ 10(2)=0=$ RHS
$\therefore$ A lies on the circle.

16C.5 HKCEE MA 1984(A/B) – I – 9

(a) $\begin{cases}x^2+y^2=4\\ y=k\ \ x\end{cases} \Rightarrow x^2+(k\ \ x)^2=4$
$2x^2-2kx+k^2\ \ 4=0\ldots(*)$
$\Delta=4k^2\ \ 8(k^2-4)=0 \Rightarrow k=\pm\sqrt{8}$

(b) (i) If $A(2,0)$ is one fo the intersections of C and L, 2 is a root of the equation $(*)$
$2(2)^2\ \ 2k(2)+(2)^2\ \ 4=0 \Rightarrow k=2$
Then $(*)$ becomes $2x^2\ \ 4x=0 \Rightarrow x=2$ or 0
$\therefore B=(0,k\ \ 0)=(0,2)$

(ii) $AB=\sqrt{(2\ \ 0)^2+(0-2)^2}=\sqrt{8}$
Mid-pt of $AB=\left(\frac{2+0}{2},\frac{0+2}{2}\right)=(1,1)$
. Eqn of circle is $(x-1)^2+(y-1)^2=\left(\frac{\sqrt{8}}{2}\right)^2$
$\Rightarrow x^2+y^2\ \ 2x\ \ 2y=0$

16C.6 HKCEE MA 1985(A/B) I – 9

(a) $AB=\sqrt{(2-7)^2+(0\ \ 5)^2}=\sqrt{50}$
Mid-pt of $AB=\left(\frac{2+7}{2},\frac{0+5}{2}\right)=\left(\frac{9}{2},\frac{5}{2}\right)$
$\therefore$ Eqn of circle is $\left(x\ \ \frac{9}{2}\right)^2+\left(y\ \ \frac{5}{2}\right)^2=\left(\frac{\sqrt{50}}{2}\right)^2$
$\Rightarrow x^2+y^2-9x\ \ 5y+14=0$

(b) $P=\left(\frac{4(2)+1(7)}{1+4},\frac{4(0)+1(5)}{1+4}\right)=(3,1)$

(c) (i) $m_{AB}=\frac{0-5}{2\ \ 7}=1 \Rightarrow m_{HPK}=\ \ 1$
$\therefore$ Eqn of HPK: $y\ \ 1=-1(x\ \ 3) \Rightarrow x+y-4=0$

(ii) $\begin{cases}x^2+y^2-9x\ \ 5y+14=0\\ x+y-4=0\end{cases}$
$\Rightarrow x^2+(4\ \ x)^2\ \ 9x\ \ 5(4\ \ x)+14=0$
$2x^2-12x+10=0$
$x=1$ or 5
$\Rightarrow y=3$ or $\ \ 1$
$\therefore H=(1,3),\ K=(5,\ \ 1)$

16C.7 HKCEE MA 1986(A/B) – I – 8

(a) $\begin{cases}x^2+y^2-6x-8y=0\\ y-x\ \ 6=0\end{cases}$
$\Rightarrow x^2+(x+6)^2-6x-8(x+6)=0$
$2x^2\ \ 2x-12=0$
$x=3$ or $\ \ 2$
$y=9$ or 4
$\therefore B=(3,9),\ C=(\ \ 2,4)$

(b) Put $y=0 \Rightarrow x=0$ or $6 \Rightarrow A=(6,0)$
Put $x=0 \Rightarrow y=0$ or $8 \Rightarrow D=(0,8)$

(c) $\angle ADO=\tan^{-1}\frac{AO}{DO}=\tan^{-1}\frac{6}{8}=37°$ (nearest degree)
$\therefore \angle ABO=\angle ACO=\angle ADO=37°$

(d) Area of $\triangle ACO=\frac{6\times 4}{2}=12$

16C.8 HKCEE MA 1987(A/B) – I – 8

(a) Eqn of ℓ: $y\ \ 0=1(x+2) \Rightarrow x\ \ y+2=0$

(b) x-coordinate of $C=x$-coordinate of mid-pt of $OB=2$
Put $x=2$ into $\ell \Rightarrow y=4 \Rightarrow C=(2,4)$

(c) Let the centre of the circle be $(2,k)$.
$k^2+4=(4\ \ k)^2$
$k^2+4=16-8k+k^2 \Rightarrow k=\frac{3}{2}$
$\therefore$ Eqn of circle: $(x\ \ 2)^2+\left(y-\frac{3}{2}\right)^2=\left(4\ \ \frac{3}{2}\right)^2$
$\Rightarrow x^2+y^2\ \ 4x-3y=0$

(d) $\begin{cases}x^2+y^2-4x-3y=0\\ x-y+2=0\end{cases}$
$\Rightarrow x^2+(x+2)^2-4x-3(x+2)=0$
$2x^2\ \ 3x-2=0 \Rightarrow x=2$ or $\frac{1}{2}$
$\therefore D=\left(\frac{1}{2},\frac{1}{2}+2\right)=\left(\frac{1}{2},\frac{5}{2}\right)$

16C.9 HKCEE MA 1988 – I – 7

(a) $(2,\ \ 5)$

(b) Radius of $C=x$-coordinate of centre $=2$
$\therefore \sqrt{2^2+5^2-k}=2 \Rightarrow k=5$

16C.10 HKCEE MA 1989 – I – 8

(a) $E=(1,2)$

(b) $\begin{cases}x^2+y^2-2x\ \ 4y\ \ 20=0\\ x+7y-40=0\end{cases}$
$\Rightarrow (40\ \ 7y)^2+y^2-2(40\ \ 7y)\ \ 4y\ \ 20=0$
$50y^2\ \ 55y+1500=0$
$y=5$ or 6
$x=5$ or -2
$\therefore P=(\ \ 2,6),\ Q=(5,5)$

(c) $PQ=\sqrt{(\ \ 2\ \ 5)^2+(6\ \ 5)^2}=\sqrt{50}$
Mid-pt of $PQ=\left(\frac{-2+5}{2},\frac{6+5}{2}\right)=\left(\frac{3}{2},\frac{11}{2}\right)$
$\therefore$ Eqn of $\mathscr{C}_2$: $\left(x\ \ \frac{3}{2}\right)^2+\left(y\ \ \frac{11}{2}\right)^2=\left(\frac{\sqrt{50}}{2}\right)^2$
$\Rightarrow x^2+y^2-3x-11y+20=0$

(d) Put $E(1,2)$ into $\mathscr{C}_2$:
LHS $=(1)^2+(2)^2-3(1)-11(2)+20=0=$ RHS
$\therefore E$ lies on $\mathscr{C}_2 \Rightarrow \angle EPQ=90°$

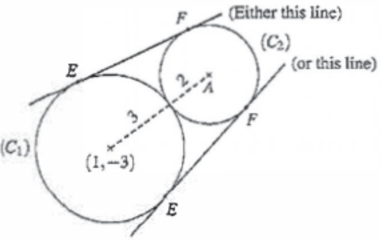
16C.11 HKCEE MA 1990 – I – 8

(a) $(C_1):(x\ \ 1)^2+(y+3)^2=3^2$
$\therefore$ Centre $=(1,\ \ 3)$, Radius $=3$

(b) Required distance $=\sqrt{(1-5)^2+(\ \ 3-0)^2}=5>3$
$\therefore$ Outside

(c) (i) $s=5-3=2$
(ii) Eqn of C_2: $(x\ \ 5)^2+(y-0)^2=2^2$
$\Rightarrow x^2+y^2-10x+21=0$

(d)



16C.12 HKCEE MA 1991 – I – 9

(a) $S:(x\ \ 2)^2+(y-1)^2=1^2$
$\therefore C=(2,1),\ A=(2,0)$

(b) $\begin{cases}y=mx\\ x^2+y^2\ \ 4x-2y+4=0\end{cases}$
$x^2+(mx)^2\ \ 4x\ \ 2(mx)+4=0$
$(1+m^2)x^2-2(2+m)x+4=0$
$\Delta=4(2+m)^2-16(1+m^2)=0$
$(2+m)^2\ \ 4(1+m^2)=0$
$3m^2-4m=0 \Rightarrow m=0$ (rej.) or $\frac{4}{3}$

(c) (i) $\because \angle OBC=\angle OAC=90°$ (tangent properties)
$\therefore \angle OBC+\angle OAC=180°$
$\therefore O, A, C$ and B are concyclic. (opp $\angle$s supp.)

(ii) $OC=\sqrt{2^2+1^2}=\sqrt{5}$
Mid-pt of $OC=\left(\frac{2+0}{2},\frac{1+0}{2}\right)=\left(1,\frac{1}{2}\right)$
$\therefore$ Eqn of circle: $(x-1)^2+\left(y-\frac{1}{2}\right)^2=\left(\frac{\sqrt{5}}{2}\right)^2$
$\Rightarrow x^2+y^2-2x-y=0$

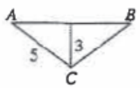
16C.13 HKCEE MA 1992 – I – 13

(a) $x^2+y^2\ \ 18x\ \ 14y+105=0 \Rightarrow (x-9)^2+(y-7)^2=5^2$
$\therefore C=(9,7)$, Radius $=5$

(b) $x^2+(mx)^2-18x\ \ 14(mx)+105=0$
$(1+m^2)x^2\ \ 2(9+7m)x+105=0$
$\therefore x_1x_2=$ product of roots $=\frac{105}{1+m^2}$

(c) $OA=\sqrt{x_1^2+y_1^2}=\sqrt{x_1^2+(mx_1)^2}=x_1\sqrt{1+m^2}$
Similarly, $OB=x_2\sqrt{1+m^2}$
$\therefore OA\cdot OB=(1+m^2)x_1x_2=(1+m^2)\cdot\frac{105}{1+m^2}=105$

(d) $AB=2\sqrt{5^2-3^2}=8$
$OA\cdot(OA+AB)=105$
$OA^2+8OA-105=0$
$\Rightarrow OA=\ \ 15$ (rej.) or 7



16C.14 HKCEE MA 1993 – I – 8

(a) $L_1:\frac{y\ \ 7}{x\ \ 0}=\frac{2\ \ 7}{10-0}=\frac{\ \ 1}{2} \Rightarrow x+2y-14=0$

(b) $m_{L_2}=\frac{1}{\frac{-1}{2}}=2$
$\therefore$ Eqn of L_2: $y-0=2(x-4) \Rightarrow 2x\ \ y-8\ \ 0$
$\begin{cases}x+2y-14=0\\ 2x-y-8=0\end{cases} \Rightarrow D=(x,y)=(6,4)$

(c) $P=\left(\frac{1(0)+k(10)}{k+1},\frac{1(7)+k(2)}{k+1}\right)=\left(\frac{10k}{k+1},\frac{7+2k}{k+1}\right)$
If P lies on the circle,
$\left[\left(\frac{10k}{k+1}\right)-4\right]^2+\left(\frac{7+2k}{k+1}\right)^2=30$
$(6k\ \ 4)^2+(7+2k)^2=30(k+1)^2$
$10k^2\ \ 80k+35=0$
$k=\frac{16\pm\sqrt{200}}{4}=4\pm\frac{5\sqrt{2}}{2}$
$\because \frac{AD}{DB}=\frac{6-0}{10-6}=\frac{3}{2}$
$\therefore k<\frac{3}{2}$ if P lies bet ween$A$ and D.
i.e. $\frac{AP}{PB}=k=4-\frac{5\sqrt{2}}{2}$

16C.15 HKCEE MA 1994 – I – 12

(a) $A=(10,0)$, Radius of $C_2=7$

(b) $\frac{RO}{RA}=\frac{OQ}{AP} \Rightarrow \frac{RO}{RO+10}=\frac{1}{7} \Rightarrow RO=\frac{5}{3}$
$\therefore$ x-coordinate of $R=\ \ \frac{5}{3}$

(c) $m_{QP}=\tan\angle QRO=\frac{OQ}{QR}=\frac{1}{\sqrt{(\frac{5}{3})^2-1^2}}=\frac{3}{4}$

(d) Eqn of QP: $y\ \ 0=\frac{3}{4}\left(x+\frac{5}{3}\right) \Rightarrow 3x-4y+5=0$

(e) By symmetry, the other tangent is:
$y-0=\frac{-3}{4}\left(x+\frac{5}{3}\right) \Rightarrow 3x+4y+5=0$

16C.16 HKCEE MA 1995 – I – 10

(a) Eqn of AB: $\frac{y\ \ 7}{x\ \ 9}=\frac{9-7}{1-9}=\frac{\ \ 1}{4} \Rightarrow x+4y-37=0$

(b) Mid-pt of $AB=\left(\frac{1+9}{2},\frac{9+7}{2}\right)=(5,8)$
Slope of $\perp$ bisector of $AB=4$
$\therefore$ Eqn of $\perp$ bisector is: $y\ \ 8=4(x\ \ 5) \Rightarrow y=4x\ \ 12$
$\begin{cases}4x-3y+12=0\\ y=4x-12\end{cases} \Rightarrow G=(6,12)$

(c) Radius $=\sqrt{(6\ \ 1)^2+(12\ \ 9)^2}=\sqrt{34}$
$\therefore$ Eqn of $\mathscr{C}$: $(x-6)^2+(y\ \ 12)^2=34$
$x^2+y^2\ \ 12x\ \ 24y+146=0$

(d) (i) Let the mid-pt of DE be (m,n). Then G is the mid-pt of $(5,8)$ and (m,n).
$\therefore \left(\frac{5+m}{2},\frac{8+n}{2}\right)=(6,12) \Rightarrow G=(m,n)=(7,16)$

(ii) $m_{DE}=m_{AB}=\frac{\ \ 1}{4}$
$\therefore$ Eqn of DE: $y-16=\frac{\ \ 1}{4}(x-7)$
$\Rightarrow x+4y\ \ 57=0$

16C.17 HKCEE MA 1996 – I – 11

(a) (i) $\mathscr{C}_1 : (x-0)^2+(y-2)^2=2^2 \Rightarrow x^2+y^2\quad 4y=0$

(ii) $B=(0,4) \Rightarrow$ Eqn of L: $y=2x+4$

(b) $\begin{cases} L: y=2x+4 \\ \mathscr{C}_2: x^2+(y-2)^2=25 \end{cases}$

$x^2+(2x+2)^2=25$

$5x^2+8x-21=0 \Rightarrow x=-3 \text{ or } \frac{7}{5} \Rightarrow y=-2 \text{ or } \frac{34}{5}$

$\therefore Q=\left(\frac{7}{5},\frac{34}{5}\right),\ R=(-3,-2)$

(c) (i) Req. pt = mid-pt of $QR=\left(\frac{-4}{5},\frac{12}{5}\right)$

(ii) Req. pt = Intersection of AQ and $\mathscr{C}_1$

= the pt 'P' with $AP:PQ=2:(5-2)$

$=\left(\frac{3(0)+2(\frac{7}{5})}{2+3},\frac{3(2)+2(\frac{34}{5})}{2+3}\right)=\left(\frac{14}{25},\frac{98}{25}\right)$

16C.18 HKCEE MA 1997 – I – 16

(a) (i) $\angle EAB=90°$ (tangent $\perp$ radius)

$\therefore \angle FEA+\angle EAB=90°+90°=180°$

$\therefore AB//EF$ (int. $\angle$s supp.)

(ii) $\angle FDE=\angle BDC$ (vert. opp. $\angle$s)

$=\angle DBC$ (base $\angle$s, isos. $\triangle$)

$=\angle FED$ (alt. $\angle$s, $AB//EF$)

$\therefore FD=FE$ (sides opp. equal $\angle$s)

(iii) If the circle touches AE at E, then its centre lies on EF.

If ED is a chord, the centre lies on the $\perp$ bisector of ED.

$\therefore$ The intersection of these two lines, F, is the centre of the circle described.

(b) $C=\left(\frac{6-2}{2},\frac{3-1}{2}\right)=(2,1)$

$\because FD=FE$,

$\therefore$ Let $F=\left(\frac{\ \ 4\ \ 2}{2},k\right)=(-3,k)$

F, D, C collinear $\Rightarrow m_{FD}\quad m_{CD}$

$\frac{k-3}{-3+2}=\frac{3-1}{-2-2} \Rightarrow k=\frac{7}{2}$

$\therefore F=\left(-3,\frac{7}{2}\right)$

16C.19 HKCEE MA 1998 – I – 15

(a) Centre of $C_2=(11,-8)$, Radius of $C_2=7$

Dist btwn the 2 centres $=\sqrt{(11-5)^2+(-8-0)^2}=10$

$\therefore$ Radius of $C_1=10-7=3$

$\therefore$ Eqn of C_1: $(x-5)^2+(y-0)^2=3^2$

$\Rightarrow x^2+y^2-10x+16=0$

(b) Let the tangent be $y=mx$.

$\begin{cases} y=mx \\ x^2+y^2-10x+16=0 \end{cases} \Rightarrow (1+m^2)x^2-10x+16=0$

$\Delta=100-64(1+m^2)=0 \Rightarrow m=\pm\frac{1}{2}$

$\therefore$ The tangents are $y=\pm\frac{1}{2}x$

(c) $\begin{cases} y=\frac{-1}{2}x \\ (x-11)^2+(y+8)^2=49 \end{cases} \Rightarrow \frac{5}{4}x^2-30x+136=0$

Sum of rts $=\frac{30}{\frac{5}{4}}=24 \Rightarrow$ x-coor of mid-pt of $AB=12$

$\Rightarrow$ y-coor $=\frac{-1}{2}(12)=-6 \Rightarrow$ The mid-pt $=(12,-6)$

16C.20 HKCEE MA 1999 – I – 16

(a) (i) $\angle BFE=\angle BDE$ ($\angle$s in the same segment)

$=\angle BAC$ (corr. $\angle$s, $AC//DE$)

$\therefore A, F, B$ and C are concyclic.

(converse of $\angle$s in the same segment)

(ii) $\because \angle ABC=90°$ (given)

$\therefore AC$ is a diameter of circle $AFBC$.

(converse of $\angle$ in semi-circle)

$\Rightarrow M$ is the centre of circle $AFBC \Rightarrow MB=MF$

(b) (i) $m_{PQ}=\frac{17-0}{0+17}=1$

$m_{RS}=\frac{7-0}{-2+9}=1=m_{PQ}$

$\therefore PQ//RS$

(ii) Eqn of QS: $\frac{y-17}{x-0}=\frac{17-7}{0+2} \Rightarrow y=5x+17$

$\begin{cases} y=5x+17 \\ x^2+y^2+10x-6y+9=0 \end{cases}$

$x^2+(5x+17)^2+10x\quad 6(5x+17)+9=0$

$26x^2+150x+196=0$

$x=-2 \text{ or } -\frac{49}{13}$

$\therefore T=\left(-\frac{49}{13},5\left(-\frac{49}{13}\right)+17\right)=\left(-\frac{49}{13},-\frac{24}{13}\right)$

(iii) *Method 1*

Let the mid-pt of PQ be $N=\left(\frac{-17}{2},\frac{17}{2}\right)$

$NO\quad\sqrt{\left(\frac{-17}{2}\right)^2+\left(\frac{17}{2}\right)^2}=\sqrt{\frac{289}{2}}$

$NT=\sqrt{\left(\frac{-49}{13}+\frac{17}{2}\right)^2+\left(\frac{-24}{13}-\frac{17}{2}\right)^2}=\sqrt{\frac{3365}{26}}$

Hence, $NT\neq NO$.

If P, Q, O and T are concyclic, the result of (a)(ii) should apply, i.e. $NO=NT$. Thus they are not concyclic.

Method 2

$\because m_{PT}\quad m_{QT}=\frac{0+\frac{24}{13}}{-17+\frac{49}{13}}\cdot\frac{17+\frac{24}{13}}{0+\frac{49}{13}}=\frac{-30}{43}\neq-1$

$\therefore \angle PTQ\neq 90°$

Thus, $\angle PTQ+\angle POQ\neq 90°+90°=180°$, and P, Q, O and T are not concyclic.

16C.21 HKCEE MA 2000 – I – 16

(a) In $\triangle OCP$, $\angle CPO=90°$ (tangent $\perp$ radius)

$\angle PCO=180°-30°-90°$ ($\angle$ sum of $\triangle$)

$\therefore \angle PQO=60°\div 2=30°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

(b) (i) $\angle SOC=\angle POC=30°$ (tangent properties)

$\angle PQR=180°-\angle POS$ (opp. $\angle$s, cyclic quad.)

$=120°$

$\Rightarrow \angle RQO=120°-30°=90°$

$\therefore RQ$ is tangent to the circle at Q.

(converse of tangent $\perp$ radius)

(ii) $OC=\sqrt{6^2+8^2}=10$

$CQ=CP=OC\sin 30°=5$

$\because OC:CQ=10:5=2:1$

$\therefore Q=(9,12)$

$m_{OC}=\frac{4}{3} \Rightarrow m_{QR}=\frac{-3}{4}$

$\therefore$ Eqn of QR: $y\quad 12=\frac{-3}{4}(x-9)$

$\Rightarrow 3x+4y-21=0$

16C.22 HKCEE MA 2001 – I – 17

(a) (i) Centre $=\left(\frac{p}{2},0\right)$, Radius $=\frac{p}{2}$

$\therefore$ Eqn of OPS: $\left(x-\frac{p}{2}\right)^2+y^2=\left(\frac{p}{2}\right)^2$

$\Rightarrow x^2+y^2-px=0$

(ii) *'Hence'*

$S(a,b)$ lies on the circle

$\Rightarrow a^2+b^2-pa=0 \Rightarrow a^2+b^2=pa$

$\therefore OS^2=(a-0)^2+(b-0)^2=a^2+b^2$

$=pa$

$=OP\cdot OQ\cos\angle POQ$

'Otherwise'

$\angle OSP=90°$ ($\angle$ in semi-circle)

In $\triangle OPS$ and $\triangle OSR$,

$\angle POS=\angle SOR$ (common)

$\angle ORS=\angle OSP=90°$ (proved)

$\therefore \triangle OPS\sim\triangle OSR$ (AA)

$\Rightarrow \frac{OS}{OR}=\frac{OP}{OS}$ (corr. sides, $\sim\triangle$s)

$OS^2=OP\cdot OR$

$=OP\cdot OQ\cos\angle POQ$

(b) (i) In circle BCE, $\angle CEB=90°$ ($\angle$ in semi-circle)

i.e. BE is an altitude of $\triangle ABC$.

(ii) By (a), $CG^2=AC\cdot BC\cos\angle ACB$

Similarly, AD is an altitude of $\triangle ABC$ by considering circle ACD.

$\Rightarrow CF^2=BC\cdot AC\cos\angle ACB=CG^2$

$\therefore CF=CG$

16C.23 HKCEE MA 2002 – I – 16

(b) (i) $A=(c\quad r,0)$, $B=(c+r,0)$

$m_{AD}=\frac{p\quad 0}{0-(c-r)}=\frac{p}{r-c}$

$m_{BF}=\frac{q-0}{0-(c+r)}=\frac{q}{r+c}$

(ii) $AD\perp BF \Rightarrow \frac{p}{r\quad c}\cdot\frac{-q}{r+c}=-1$

$pq=r^2-c^2$

i.e. $OD\cdot OF=CG^2-OC^2$

$=OG^2$

16C.24 HKCEE MA 2003 – I – 17

(a) (i) In $\triangle NPM$ and $\triangle NKP$,

$\angle PNM=\angle KNP$ (common)

$\angle NPM=\angle NKP$ ($\angle$ in alt. segment)

$\angle PMN=\angle KPN$ ($\angle$ sum of $\triangle$)

$\therefore \triangle NPM\sim\triangle NKP$ (AAA)

$\Rightarrow \frac{NP}{NM}=\frac{NK}{NP}$ (corr. sides, $\sim\triangle$s)

$NP^2=NK\cdot NM$

(ii) $\because RS//OP$ (given)

$\therefore \triangle KRM\sim\triangle KON$ and $\triangle KSM\sim\triangle KPN$

$\Rightarrow \frac{RM}{ON}=\frac{KM}{KN}$ and $\frac{SM}{PN}=\frac{KM}{KN}$

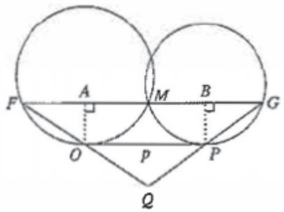
$\Rightarrow \frac{RM}{ON}=\frac{SM}{PN}$

Similar to (a), we have $NO^2=NK\cdot NM$

$\therefore NP=NO$

Hence, $RM=MS$.

(b) (i)



With the notation above, note that OA (extended) and PB (extended) are diameters of C_1 and C_2 respectively.

$\therefore FA=AM$ and $MB=BG$

($\perp$ from centre to chord bisects chord)

Hence, $FG=2AM+2MB=2AB=2p$

(ii) $\because M=(a,b)$ and $FA=AM$,

$\therefore F=(\quad a,b)$

Since $\triangle QOP\sim\triangle QFG$ and $FG=2OP$, we have

$FQ=2OQ \Rightarrow O$ is the mid-pt of FQ

$\Rightarrow Q=(a,\quad b)$

(iii) Note that QM is vertical. Thus $QM\perp RS$.

In $\triangle QMR$ and $\triangle QMS$,

$QM=QM$ (common)

$RM=SM$ (proved)

$\angle QMR=\angle QMS=90°$ (proved)

$\therefore \triangle QMR\cong\triangle QMS$ (SAS)

$\Rightarrow QR=QS$ (corr. sides, $\cong\triangle$s)

i.e. $\triangle QRS$ is isosceles.

16C 25 HKCEE MA 2004 – I – 16

(a) In $\triangle ADE$ and $\triangle BOE$,

$\angle ADE=\angle EBC$ (alt. $\angle$s, $OD//BC$)

$=\angle BOE$ ($\angle$ in alt. segment)

$\angle DAE=\angle OBE$ (ext. $\angle$, cyclic quad.)

$AD=BO$ (given)

$\therefore \triangle ADE\cong\triangle BOE$ (ASA)

(b) $DE=OE$ (corr. sides, $\cong\triangle$s)

$\angle BOE=\angle ADE$ (proved)

$=\angle AOE$ (base $\angle$s, isos. $\triangle$)

i.e. $\angle AOB=2\angle BOE$

$\therefore \angle BEO=\angle AED$ (corr. $\angle$s, $\cong\triangle$s)

$=\angle AOB$ (ext. $\angle$, cyclic quad.)

$=2\angle BOE$ (proved)

(c) Suppose OE is a diameter of the circle $OAEB$.

(i) $\angle OBE=90°$ ($\angle$ in semi-circle)

In $\triangle OBE$, $\angle BOE=180°-90°-(2\angle BOE)$

($\angle$ sum of $\triangle$)

$3\angle BOE=90° \Rightarrow \angle BOE=30°$

(ii) $OB=6 \Rightarrow BE=OB\tan\angle BOE \Rightarrow E=(6,2\sqrt{3})$

$OE=\frac{OB}{\cos 30°}=4\sqrt{3}$

Mid-pt of $OE=(3,\sqrt{3})$

$\therefore$ Eqn of circle: $(x-3)^2+(y-\sqrt{3})^2=\left(\frac{4\sqrt{3}}{2}\right)^2$

$\Rightarrow x^2+y^2-6x\quad 2\sqrt{3}y=0$

16C.26 HKCEE MA 2005 – I – 17

(a) (i) $\because$ MN is a diameter (given)
$\therefore$ $\angle NOM = \angle QRP = 90^\circ$ ($\angle$ in semi-circle)
In $\triangle OQR$ and $\triangle ORP$,
$\angle ROQ = \angle POR = 90^\circ$ (given)
$\angle QRO = \angle QRP\ \ \angle PRO$
$= 90^\circ\ \ \angle PRO$
$\angle POR = 180^\circ - \angle ROP - \angle PRO$ ($\angle$ sum of $\triangle$)
$= 90^\circ\ \ \angle PRO$
$\Rightarrow \angle QPO = \angle PRO$
$\angle RQO = \angle PRO$ ($\angle$ sum of $\triangle$)
$\therefore \triangle OQR \sim \triangle ORP$ (AAA)
$\Rightarrow \dfrac{OR}{OQ} = \dfrac{OP}{OR}$ (corr. sides, $\sim\triangle$s)
$OR^2 = OP \cdot OQ$

(ii) In $\triangle MON$ and $\triangle POR$,
$\angle NMO = \angle QRO$ ($\angle$s in the same segment)
$= \angle RPO$ (proved)
$\angle MON = \angle POR$ (proved)
$\angle MNO = \angle RQO$ ($\angle$ sum of $\triangle$)
$\therefore \triangle MON \sim \triangle RQO$ (AAA)

(b) (i) $OR = \sqrt{OP \cdot OQ} = \sqrt{4 \cdot 9} = 6 \Rightarrow R = (0, 6)$

(ii) In $\triangle POR$, $PR = \sqrt{4^2 + 6^2} = \sqrt{52}$
$\dfrac{MN}{ON} = \dfrac{PR}{OR} = \dfrac{\sqrt{52}}{6} \Rightarrow MN = \dfrac{\sqrt{13}}{3} \cdot \dfrac{3\sqrt{13}}{2} = \dfrac{13}{2}$
$\therefore$ Radius $= \dfrac{13}{2} \div 2 = \dfrac{13}{4}$
Let the centre be $(h, 6 \div 2) = (h, 3)$
(since it lies on the $\perp$ bisector of OR).
$\Rightarrow \sqrt{(h\ \ 0)^2 + (3\ \ 0)^2} = \dfrac{13}{4} \Rightarrow h = -\dfrac{5}{2}$ $(h < 0)$
$\therefore$ The centre is $\left(-\dfrac{5}{2}, 3\right)$

16C.27 HKCEE MA 2006 – I – 16

(a) (i) $\because$ G is the circumcentre (given)
$\therefore$ $SC \perp BC$ and $SA \perp AB$ ($\angle$ in semi-circle)
$\because$ H is the orthocentre (given)
$\therefore$ $AH \perp BC$ and $CH \perp AB$
Thus, $SC // AH$ and $SA // CH \Rightarrow AHCS$ is a //gram.

(ii) *Method 1*
$\because$ $\angle GRB = \angle SCB = 90^\circ$ (proved)
$\therefore$ $GR // SC$ (corr. $\angle$s equal)
$\because$ $BG = GS =$ radius
$\therefore$ $BR = RC$ (intercept thm)
$\Rightarrow SC = 2GR$ (mid-pt thm)
Hence, $AH = SC = 2GR$ (property of //gram)

Method 2
$\because$ $BG = GS =$ radius
and $BR = RC$ ($\perp$ from centre to chord bisects chord)
$\Rightarrow SC = 2GR$ (mid-pt thm)
Hence, $AH = SC = 2GR$ (property of //gram)

(b) (i) Let the circle be $x^2 + y^2 + Dx + Ey + F = 0$
$$\begin{cases} 0^2 + 12^2 + 0D + 12E + F = 0 \\ (-6)^2 + 0^2 - 6D + 0E + F = 0 \\ 4^2 + 0^2 + 4D + 0E + F = 0 \end{cases} \Rightarrow \begin{cases} D = 2 \\ E = -10 \\ F = -24 \end{cases}$$
$\therefore$ The circle is $x^2 + y^2 + 2x - 10y\ \ 24 = 0$.

(ii) $G = (\ \ 1, 5) \Rightarrow GR = 5$
$\therefore$ $H = (0, 12\ \ 2 \times 5) = (0, 2)$ (by (a)(ii))

(iii) $m_{BG} \cdot m_{GH} = \dfrac{5 - 0}{1 + 6} \cdot \dfrac{5\ \ 2}{-1 - 0} =\ \ 3 \neq -1$
$\therefore$ $\angle BGH \neq 90^\circ \Rightarrow \angle BOH + \angle BGH \neq 180^\circ$
Hence, B, O, H and G are not concyclic.

16C.28 HKCEE MA 2007 – I – 17

(a) (i) $\because$ I is the incentre of $\triangle ABD$ (given)
$\therefore$ $\angle ABG = \angle DBG$ and $\angle BAE = \angle CAE$
In $\triangle ABG$ and $\triangle DBG$,
$\angle ABG = \angle DBG$ (proved)
$AB = DB$ (given)
$BG = BG$ (common)
$\therefore \triangle ABG \cong \triangle DBG$ (SAS)

(ii) $\because$ $\triangle ABD$ is isosceles and $\angle ABG = \angle DBG$
$\therefore$ $\angle BGA = 90^\circ$ (property of isos. $\triangle$)
In $\triangle AGI$ and $\triangle ABE$,
$\angle AGI = 90^\circ = \angle ABE$ ($\angle$ in semi-circle)
$\angle IAG = \angle EAB$ (proved)
$\angle AIG = \angle AEB$ ($\angle$ sum of $\triangle$)
$\therefore \triangle AGI \sim \triangle ABE$ (AAA)
$\Rightarrow \dfrac{GI}{AG} = \dfrac{BE}{AB}$ (corr. sides, $\sim\triangle$s)

(b) (i) $\because$ $AG = DG$
$\therefore$ $AG = (\text{Diameter}\ \ CD) \div 2$
$= (25 \times 2 - (25 - 11)) \div 2 = 18$
$\therefore$ $G = (\ \ 25 + 18, 0) = (\ \ 7, 0)$

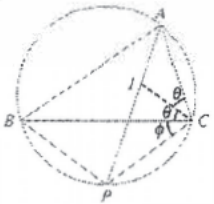
(ii) By (a)(ii), $GI = \dfrac{1}{2} \times AG = 9 \Rightarrow I = (\ \ 7, 9)$
Radius of inscribed circle $= GI = 9$
$\therefore$ Eqn of circle is $(x + 7)^2 + (y - 9)^2 = 9^2$
$\Rightarrow x^2 + y^2 + 14x\ \ 18y + 49 = 0$

16C.29 HKCEE MA 2008 – I – 17

(a) *Method 1*
$\because$ I is the incentre of $\triangle ABC$ (given)
$\therefore$ $\angle BAP = \angle CAP$
$\therefore$ $BP = CP$ (equal $\angle$s, equal chords)

Method 2
$\because$ I is the incentre of $\triangle ABC$ (given)
$\therefore$ $\angle BAP = \angle CAP$
$\angle BCP = \angle BAP$ ($\angle$s in the same segment)
$= \angle CAP$ (proved)
$= \angle CBP$ ($\angle$s in the same segment)
$\Rightarrow BP = CP$ (sides opp. equal $\angle$s)

Both methods



Join CI. Let $\angle ACI = \angle BCI = \theta$ and $\angle BCP = \phi$.
$\angle PAC = \phi$ (equal chords, equal $\angle$s)
$\Rightarrow \angle PIC = \angle PAC + \angle ACI = \theta + \phi$ (ext. $\angle$ of $\triangle$)
$= \angle PCI$
$\therefore$ $IP = CP$ (sides opp. equal $\angle$s)
i.e. $BP = CP = IP$

(b) (i) Let $P = \left(\ \ \dfrac{80 + 64}{2}, k\right) = (\ \ 8, k)$
$\because$ $BP = IP$
$\therefore$ $(-8 + 380)^2 + (k\ \ 0)^2 = (\ \ 8\ \ 0)^2 + (k - 32)^2$
$5184 + k^2 = 64 + k^2\ \ 64k + 1024$
$k = -64 \Rightarrow P = (-8, -64)$
$\therefore$ $P = (\ \ 8, -64)$
Radius of circle $BIC = \sqrt{5184 + (-64)^2} = \sqrt{9280}$
$\therefore$ Eqn of circle: $(x + 8)^2 + (y + 64)^2 = 9280$
$\Rightarrow x^2 + y^2 + 16y + 128y\ \ 5120 = 0$

(ii) *Method 1*
$\because$ $GB = GP$
$\therefore$ $(-8 + 80)^2 + (g - 0)^2 = (g + 64)^2$
$72^2 + g^2 = g^2 + 128g + 64^2$
$g = 8.5$
$\therefore$ $Q = (-8,\ \ 64 + 2GP)$
$= (-8,\ \ 64 + 2(8.5 + 64)) = (\ \ 8, 81)$

Method 2
Let the equation of circle be $x^2 + y^2 + Dx + Ey + F = 0$
$$\begin{cases} (-80)^2 + 0^2\ \ 80D + 0E + F = 0 \\ 64^2 + 0^2 + 64D + 0E + F = 0 \\ (\ \ 8)^2 + (-64)^2 - 8D\ \ 64E + F = 0 \end{cases} \Rightarrow \begin{cases} D = 16 \\ E = -17 \\ F = -5120 \end{cases}$$
$\therefore$ Eqn of circle is $x^2 + y^2 + 16x\ \ 17y\ \ 5120 = 0$
Put $x =\ \ 8 \Rightarrow y^2 - 17y - 5184 = 0$
$\Rightarrow y = 81$ or $\ \ 64 \Rightarrow Q = (-8, 81)$

(iii) *Method 1*
$m_{BQ} \cdot m_{IQ} = \dfrac{81\ \ 0}{-8 + 80} \cdot \dfrac{81 - 32}{-8 - 0} = -\dfrac{441}{64} \neq -1$
$\Rightarrow \angle BQI \neq 90^\circ \Rightarrow \angle BQI + \angle BRI \neq 180^\circ$
$\therefore$ They are not concyclic.

Method 2
Mid-pt of $BI = \left(\ \ \dfrac{80 + 0}{2}, \dfrac{0 + 32}{2}\right) = (\ \ 40, 16)$
$BI = \sqrt{80^2 + 32^2} = \sqrt{7424}$
$\therefore$ Eqn of circle BRI:
$(x + 40)^2 + (y\ \ 16)^2 = (\sqrt{7424} \div 2)^2$
$x^2 + y^2 + 80x\ \ 32y = 0$
Put $Q(-8, 81)$ into the equation:
LHS $= (-8)^2 + (81)^2 + 80(\ \ 8)\ \ 32(81)$
$= 3393 \neq$ RHS
Thus, Q does not lie on the circle through B, R and I.
The 4 points are not concyclic.

16C.30 HKCEE MA 2011 – I – 16

(a) $S = (16, -48)$
$R = (\ \ 32 + 2 \times (16 + 32), -48) = (64,\ \ 48)$

Method 1
Mid-pt of $PR = \left(\dfrac{16 + 64}{2}, \dfrac{80\ \ 48}{2}\right) = (40, 16)$
$m_{PR} = \dfrac{48\ \ 80}{64 - 16} = \dfrac{-8}{3}$
$\therefore$ Eqn of $\perp$ bisector: $y - 16 = \dfrac{-1}{\frac{-8}{3}}(x - 40)$
$\Rightarrow 3x - 8y + 8 = 0$

Method 2
$\sqrt{(x\ \ 16)^2 + (y - 80)^2} = \sqrt{(x\ \ 64)^2 + (y + 48)^2}$
$x^2 + y^2 - 32x\ \ 160y + 6656 = x^2 + y^2\ \ 128x + 96y + 6400$
$96x\ \ 256y + 256 = 0 \Rightarrow 3x\ \ 8y + 8 = 0$

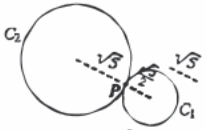
(b) Since $PQ = PR$ and $PS \perp QR$, PS is the $\perp$ bisector of QR. (property of isos. $\triangle$)
Thus the circumcentre of $\triangle PQR$ is the intersection of the line in (a) and PS.
Put $x = 16$ into the eqn in (a) $\Rightarrow y = 7 \Rightarrow (16, 7)$

(c) (i) Radius $= 80\ \ 7 = 73$
$\therefore$ Eqn of C: $(x - 16)^2 + (y\ \ 7)^2 = 73^2$
$\Rightarrow x^2 + y^2\ \ 32x\ \ 14y - 5024 = 0$

(ii) If the centre of C is the in-centre of $\triangle PQR$, its distances to each of PR, QR and PQ would also be the same (the radii of the inscribed circle).
From (a), the foot of $\perp$ from centre to $PR = (40, 16)$
$\Rightarrow$ Dist from centre to $PR = \sqrt{(16 - 40)^2 + (7\ \ 16)^2}$
$\sqrt{657}$
Dist from centre to $QR = 7 - (\ \ 48)\ \ 56 \neq \sqrt{657}$
Therefore, the centre of C cannot be the in-centre of $\triangle PQR$. The claim is disagreed.

16C.31 HKCEE AM 1981 – II – 6

(a) C_1: Centre $= \left(0, -\dfrac{7}{2}\right)$, Radius $= \sqrt{\left(\dfrac{7}{2}\right)^2 - 11} = \dfrac{\sqrt{5}}{2}$
C_2: Centre $= (-3,\ \ 2)$, Radius $= \sqrt{3^2 + 2^2 - 8} = \sqrt{5}$
$\therefore$ $P\left(\dfrac{2(0) + 1(\ \ 3)}{1 + 2}, \dfrac{2(\ \ \frac{7}{2}) + 1(-2)}{1 + 2}\right) = (-1, -3)$



(b) Slope of line joining centres $= \dfrac{\frac{-7}{2} + 2}{0 + 3} = \dfrac{-1}{2}$
$\therefore$ Eqn of tg: $y + 3 = \dfrac{-1}{\frac{-1}{2}}(x + 1) \Rightarrow 2x - y\ \ 1 = 0$

16C.32 (HKCEE AM 1981 – II – 12)

(a) (i) $\begin{cases} L: y = mx + 2 \\ C: x^2 + y^2 = 1 \end{cases} \Rightarrow x^2 + (mx + 2)^2 = 1$
$\Rightarrow (1 + m^2)x^2 + 4mx + 3 = 0$
$\therefore$ x_1 and x_2 are the roots of this equation.

(ii) $x_1 + x_2 = \dfrac{-4m}{1 + m^2}$, $x_1 x_2 = \dfrac{3}{1 + m^2}$
$\Rightarrow AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
$= \sqrt{(x_1 - x_2)^2 + (mx_1 + 2 - mx_2\ \ 2)^2}$
$= \sqrt{(x_1\ \ x_2)^2 + m^2(x_1 - x_2)^2}$
$= \sqrt{(1 + m^2)[(x_1 + x_2)^2 - 4x_1x_2]}$
$= \sqrt{(1 + m^2)\left[\dfrac{16m^2}{(1 + m^2)^2} - \dfrac{12}{1 + m^2}\right]}$
$= \sqrt{\dfrac{16m^2 - 12(1 + m^2)}{1 + m^2}} = 2\sqrt{\dfrac{m^2\ \ 3}{m^2 + 1}}$

(b) (i) 2 distinct pts $\Rightarrow 2\sqrt{\dfrac{m^2 - 3}{m^2 + 1}} > 0 \Rightarrow m^2 - 3 > 0$
$\Rightarrow m <\ \ \sqrt{3}$ or $m > \sqrt{3}$

(ii) Tg to $C \Rightarrow 2\sqrt{\dfrac{m^2 - 3}{m^2 + 1}} = 0 \Rightarrow m = \pm\sqrt{3}$

(iii) No intsn $\Rightarrow \dfrac{m^2 - 3}{m^2 + 1} < 0 \Rightarrow -\sqrt{3} < m < \sqrt{3}$

(c) For $m = \pm\sqrt{3}$, the eqn in (a)(i) becomes
$10x^2 \pm 4\sqrt{3}x + 3 = 0 \Rightarrow x = \dfrac{\mp 4\sqrt{3} \pm \sqrt{0}}{20} = \mp\dfrac{\sqrt{3}}{5}$
$\Rightarrow y = \pm\sqrt{3}\left(\mp\dfrac{\sqrt{3}}{5}\right) + 2 = \dfrac{8}{5}$
$\therefore$ Eqn of PQ is $y = \dfrac{8}{5}$ (since it is horizontal)

16C.33 (HKCEE AM 1982 – II – 8)

(a) (i) $m_L = \frac{-5}{12}$

$\therefore$ Req eqn: $y-6 = \frac{-1}{\frac{-5}{12}}(x-5) \Rightarrow y = \frac{12}{5}x-6$

(ii) *'Hence'*

$\begin{cases} 5x+12y=32 \\ y=\frac{12}{5}x-6 \end{cases} \Rightarrow (x,y) = \left(\frac{40}{13}, \frac{18}{13}\right)$

Radius of circle $= \sqrt{\left(5-\frac{40}{13}\right)^2 + \left(6-\frac{18}{13}\right)^2} = 5$

. Eqn of C: $(x-5)^2+(y-6)^2 = 5^2$

$\Rightarrow x^2+y^2-10x-12y+36=0$

'Otherwise'

Let C be $(x-5)^2+(y-6)^2=r^2$.

$\begin{cases} 5x+12y=32 \\ (x-5)^2+(y-6)^2=r^2 \end{cases}$

$\Rightarrow (x-5)^2+\left(\frac{32-5x}{12}-6\right)^2 = r^2$

$\frac{169}{144}x^2 - \frac{65}{9}x + \frac{325}{9} - r^2 = 0$

$\Delta = \left(\frac{65}{9}\right)^2 - 4\cdot\frac{169}{144}\left(\frac{325}{9}-r^2\right) = 0 \Rightarrow r^2 = 25$

$\therefore$ Eqn of C: $(x-5)^2+(y-6)^2=5^2$

$\Rightarrow x^2+y^2-10x-12y+36=0$

(b) *Method 1*

x-coordinate of centre $= 5 =$ radius

$\therefore$ C touches the y-axis.

Method 2

Put $x=0 \Rightarrow y^2-12y+36=0 \Rightarrow y=6$ (repeated)

$\therefore$ y-axis is tangent to C.

(c) Let the tangent be $y=mx$.

$\begin{cases} y=mx \\ x^2+y^2-10x-12y+36=0 \end{cases}$

$\Rightarrow (1+m^2)x^2-2(5+6m)x+36=0$

$\Delta = 4(5+6m)^2-4\cdot 36(1+m^2)=0 \Rightarrow m=\frac{5}{12}$

$\therefore$ The required tangent is $y=\frac{5}{12}x$.

(d) Let $Q=(m,n)$ Since M is the mid-pt of PQ,

$\left(\frac{2+m}{2}, \frac{2+n}{2}\right) = (5,6) \Rightarrow (m,n)=(8,10)$

Let $x^2+y^2+Dx+Ey+F=0$ be the circle through P, Q and O.

$\begin{cases} 0^2+0^2+0D+0E+F=0 \\ 2^2+2^2+2D+2E+F=0 \\ 8^2+10^2+8D+10E+F=0 \end{cases} \Rightarrow \begin{cases} D=62 \\ E=-66 \\ F=0 \end{cases}$

$\therefore$ The circle is $x^2+y^2+62x-66y=0$.

16C.34 HKCEE AM 1984 – II – 6

(a) $x^2+y^2-2kx+4ky+6k^2-2=0$

Radius $= \sqrt{(-k)^2+(2k)^2-(6k^2-2)} > 1$

$k^2+4k^2-6k^2+2 > 1^2$

$k^2 < 1$

$-1 < k < 1$

16C.35 (HKCEE AM 1985 – II – 5)

(a) Radius $= \sqrt{\left(\frac{k}{2}\right)^2+\left(\frac{2+k}{2}\right)^2} = \sqrt{5}$

$\frac{k^2}{4}+1+k+\frac{k^2}{4}=5$

$k^2+2k-8=0 \Rightarrow k=-4$ or 2

(b) $k=-4 \Rightarrow x^2+y^2-4x+2y=0$

$k=2 \Rightarrow x^2+y^2+2x-4y=0$

16C.36 HKCEE AM 1986 – II – 10

(a) (i) $\begin{cases} C_1: x^2+y^2-4x+2y+1=0 \\ C_2: x^2+y^2-10x-4y+19=0 \end{cases}$

$\Rightarrow 6x+6y\ \ 18=0 \Rightarrow y=3-x$

$\Rightarrow x^2+(3-x)^2-4x+2(3-x)+1=0$

$2x^2-12x+16=0$

$x=2$ or 4

$y=1$ or -1

Hence, A and B are $(2,1)$ and $(4,-1)$.

$\therefore$ Eqn of AB: $\frac{y-1}{x-2} = \frac{-1-1}{4-2} = \frac{-1}{2}$

$\Rightarrow x+2y\ \ 4=0$

(ii) The required circle has AB as a diameter.

Mid-pt of $AB = \left(\frac{2+4}{2}, \frac{1-1}{2}\right) = (3,0)$

$AB = \sqrt{(4-2)^2+(-1-1)^2} = \sqrt{8}$

$\therefore$ Req. circle is: $(x-3)^2+(y-0)^2 = \left(\frac{\sqrt{8}}{2}\right)^2$

$\Rightarrow x^2+y^2-6x+7=0$

(b) Centre of C_3 = Centre of $C_1 = (2,-1)$

Radius of C_3 = Dist. from $(2,-1)$ to AB

$= \sqrt{(\text{Radius of } C_1)^2-(\frac{1}{2}AB)^2}$

$= \sqrt{(2)^2+(1)^2-1-2} = \sqrt{2}$

$\therefore$ Eqn of C_3: $(x-2)^2+(y+1)^2=2$

$\Rightarrow x^2+y^2-4x+2y+3=0$

16C.37 HKCEE AM 1987 – II – 11

(a) (i) *Method 1*

$C_1: (x-8)^2+(y\ \ 2)^2=2^2$

$\Rightarrow$ Radius $=2=y$-coordinate of centre

$\therefore$ C_1 touches the x-axis, and the point of contact is (x-coordinate of centre, 0) $= (8,0) = A$.

Method 2

Put $y=0 \Rightarrow x^2-16x+64=0 \Rightarrow x=8$ (repeated)

$\therefore$ $A(8,0)$ is the only pt of contact of C_1 and x-axis.

(ii) Let OH be $y=mx$.

$\begin{cases} y=mx \\ x^2+y^2-16x-4y+64=0 \end{cases}$

$\Rightarrow x^2+(mx)^2-16x-4(mx)+64=0$

$(1+m^2)x^2-4(4+m)x+64=0$

$\Delta = 16(4+m)^2-4\cdot 64(1+m^2)=0$

$m^2+8m+16-16-16m^2=0$

$15m^2-8m=0$

$m=0$ or $\frac{8}{15}$

.. Eqn of OH is $y=\frac{8}{15}x$.

(iii) By symmetry, $m_{BH} = \frac{-8}{15}$

$\therefore$ Eqn of BH: $y-0 = \frac{-8}{15}(x-16)$

$\Rightarrow y = \frac{-8}{15}+\frac{128}{15}$

(b) (i) Sub $A \Rightarrow 8^2+0^2-16(8)+0+c=0 \Rightarrow c=64$

Method 1

$\begin{cases} 4x+3y=0 \\ x^2+y^2-16x+2fy+64=0 \end{cases}$

$x^2+\left(\frac{-4}{3}x\right)^2-16x+2f\left(\frac{-4}{3}x\right)+64=0$

$\frac{25}{9}x^2-8\left(2+\frac{f}{3}\right)x+64=0$

$\Delta = 64\left(2+\frac{f}{3}\right)^2-4\ \frac{25}{9}\cdot 64=0$

$\left(2+\frac{f}{3}\right)^2 = \frac{100}{9}$

$2+\frac{f}{3} = \pm\frac{10}{3}$

$f=4$ or -16

Since the centre is in Quad IV, $f>0$. $\therefore f=4$

Method 2

Suppose the point of contact of OK and C_2 is P. Then $OP=OA=8$. Let $P=\left(p, \frac{-4}{3}p\right)$.

$\sqrt{(p)^2+\left(\frac{-4}{3}p\right)^2}=8$

$\frac{25}{9}p^2=64 \Rightarrow p=\pm\frac{24}{5}$

As P is in Quad IV, $p=\frac{24}{5} \Rightarrow P=\left(\frac{24}{5}, \frac{-32}{5}\right)$

Put into C_2:

$\left(\frac{24}{5}\right)^2+\left(\frac{-32}{5}\right)^2-16\left(\frac{24}{5}\right)+2f\left(-\frac{32}{5}\right)+64=0$

$\frac{256}{5}-\frac{64}{5}f=0$

$f=4$

(ii) Put $x=8$ into OH and OK respectively.

$OH \Rightarrow y=\frac{8}{15}(8)=\frac{64}{15} \Rightarrow H=\left(8, \frac{64}{15}\right)$

$OK \Rightarrow y=\frac{-4}{3}(8)=\frac{-32}{3} \Rightarrow K=\left(8, \frac{-32}{3}\right)$

$\therefore \frac{\text{Area of } \triangle OBH}{\text{Area of } \triangle OBK} \frac{y\text{-coor of } H}{-(y\text{-coor of } K)} = \frac{\frac{64}{15}}{\frac{32}{3}} = \frac{2}{5}$

16C.38 (HKCEE AM 1988 – II – 11)

(a) *Method 1*

Let $S=(h,k)$.

$\because KS \perp (x-5y+59=0)$

$\therefore \frac{k-12}{\text{[illegible]}} = m_{KS} = \frac{-1}{\frac{1}{5}} = -5 \Rightarrow k=-5h+17$

$\because SK=SH$

$\therefore (h-1)^2+(k-12)^2=(h+3)^2+(k-6)^2$

$-2h-24k+145=6h-12k+45 \Rightarrow 2h+3k=25$

Solving, $h=2$, $k=7 \Rightarrow S=(2,7)$

Method 2

Eqn of KS: $y-12=\frac{-1}{\frac{1}{5}}(x-1) \Rightarrow y=-5x+17$

Eqn of $\perp$ bisector of HK:

$(x-1)^2+(y-12)^2=(x+3)^2+(y-6)^2$

$\Rightarrow 2x+3y=25$

Solving, $(x,y)=(2,7) \Rightarrow S=(2,7)$

(Note how different concepts gave similar calculations.)

Hence,

Radius of $C=\sqrt{(1-2)^2+(12-7)^2}=\sqrt{26}$

$\Rightarrow$ Eqn of C: $(x-2)^2+(y-7)^2=26$

$\Rightarrow x^2+y^2-4x-14y+27=0$

(b) $\begin{cases} L: 3x-2y-5=0 \\ C: x^2+y^2-4x-14y+27=0 \end{cases}$

$\Rightarrow x^2+\left(\frac{3x-5}{2}\right)^2-4x-14\left(\frac{3x-5}{2}\right)+27=0$

$\frac{13}{4}x^2-\frac{65}{2}x+\frac{273}{4}=0$

$x=3$ or 7

$\Rightarrow y=2$ or 8

$\therefore$ A and B are $(3,2)$ and $(7,8)$.

$\Rightarrow$ Centre of circle $=\left(\frac{7+3}{2}, \frac{8+2}{2}\right)=(5,5)$

Radius $\frac{1}{2}\sqrt{(7-3)^2+(8-2)^2}=\frac{1}{2}\sqrt{52}=\sqrt{13}$

$\therefore$ Eqn of circle: $(x-5)^2+(y-5)^2=13$

$\Rightarrow x^2+y^2-10x-10y+37=0$

16C.39 HKCEE AM 1993 – II – 11

(a) $AB=\sqrt{(3-0)^2+\left(\frac{3}{4}-2\right)^2}=\frac{13}{4}$

Radius of $C_2 = y$-coordinate of $B = \frac{3}{4}$

$\because$ Radius of C_1 − Radius of $C_2 = 4-\frac{3}{4}=\frac{13}{4}=AB$

$\therefore$ C_1 and C_2 touch internally.

(b) $AP = 4-$ Radius of circle

$s^2+(t-2)^2=(4-t)^2$

$s^2+t^2-4t+4=16-8t+t^2 \Rightarrow 4t=12-s^2$

(c) $BP=\frac{13}{4}+$ Radius of circle

$(s-3)^2+\left(t-\frac{3}{4}\right)^2=\left(\frac{3}{4}+t\right)^2$

$(s-3)^2=\left(t+\frac{3}{4}\right)^2-\left(t-\frac{3}{4}\right)^2=3t$

(d) $\begin{cases} 4t=12-s^2 \\ 3t=(s-3)^2 \end{cases}$

$\Rightarrow 3(12-s^2)=4(s-3)^2$

$36-3s^2=4s^2-24s+36$

$7s^2-24s=0$

$s=0$ or $\frac{24}{7} \Rightarrow t=3$ or $\frac{3}{49}$

$\therefore$ The required circles are $(x-0)^2+(y-3)^2=3^2$ and $\left(x-\frac{24}{7}\right)^2+\left(y-\frac{3}{49}\right)^2=\left(\frac{3}{49}\right)^2$.

16C.40 HKCEE AM 1994 – II – 9

(a) $(h-5)^2+(k-5)^2=(h-7)^2+(k-1)^2$

$-10h+25-10k+25=-14h+49-2k+1$

$4h=8k \Rightarrow h=2k$

Hence, the equation of C is

$(x-h)^2+(y-k)^2=(h-5)^2+(k-4)^2$

$x^2+y^2-2hx-2ky=-10h+25-10k+25$

$x^2+y^2-2(2k)x-2ky+10(2k)+10k-50=0$

$x^2+y^2-4kx-2ky+30k-50=0$

(b) Denote the centre of C by G.

$m_{BG}=\frac{-1}{\frac{1}{2}}=-2$

$\frac{k-1}{h-7}=-2 \Rightarrow k-1=-2(2k\ \ 7) \Rightarrow k=3$

$\therefore$ Eqn of C is $x^2+y^2-4(3)x-2(3)y+30(3)-50=0$

$\Rightarrow x^2+y^2-12x\ \ 6y+40=0$

16C.41 HKCEE AM 1995 – II 10

(a) $C_1:(x-8)^2+(y-0)^2=10^2$
$\therefore$ Centre $=(8,0)$, Radius $=10$
$\therefore$ Radius of $C_2=$ (Dist. btwn centres of C_1 and C_2) -10
$=15-10=5$

(b) $\sqrt{(h-8)^2+(k-0)^2}-10=\sqrt{(h+7)^2+(k-0)^2}-5$
$h^2+14h+49+k^2=\left(\sqrt{h^2-16h+64+k^2}-5\right)^2$
$30h-40=10\sqrt{h^2-16h+64+k^2}$
$(3h-4)^2=h^2-16h+64+k^2$
$9h^2-24h+16=h^2-16h+64+k^2$
$8h^2-k^2-8h-48=0$

(c) (i) $y=\dfrac{40+0}{2}=20$
(The centre lies on the $\perp$ bisector of the segment joining the two centres. This is true because the radii of C_2 and C_3 are the same.)

(ii) From (c)(i), $k=20$
Put into the result of (b):
$8h^2-(20)^2-8h-48=0$
$h^2-h-56=0 \Rightarrow h=8$ (rej.) or -7
Centre $=(-7,20)$, Radius $=20-5-15$
$\therefore$ Eqn of req. circle: $(x+7)^2+(y-20)^2=15^2$
$\Rightarrow x^2+y^2+14x-40y+224=0$

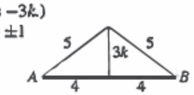
16C.42 (HKCEE AM 1996 – II 10)

(a) (i) Centre $=(4k,3k)$
Put into the line: LHS $=3(4k)-4(3k)=0=$ RHS
$\therefore$ The centre lies on $3x-4y=0$.

(ii) Radius $=\sqrt{(4k)^2+(3k)^2-25(k^2-1)}=\sqrt{25}=5$

(b) Slope $=\dfrac{3}{4}$
Pick a value of k for C_k, e.g. $C_0:x^2+y^2-25=0$.
Let the equation of tangent be $y=\dfrac{3}{4}x+b$.
$\begin{cases}y=\frac{3}{4}x+b\\ x^2+y^2-25=0\end{cases} \Rightarrow x^2+\left(\dfrac{3}{4}+b\right)^2-25=0$
$\dfrac{25}{16}x^2+\dfrac{3}{2}bx+b^2-25=0$
$\Delta=\left(\dfrac{3}{2}b\right)^2-4\cdot\dfrac{25}{16}(b^2-25)=0 \Rightarrow b=\pm\dfrac{25}{4}$
$\therefore$ The tangents are $y=\dfrac{3}{4}x\pm\dfrac{25}{4}$.

(c) Distance $=y$-coordinate of centre $=3k$
(If k is negative, the distance is $-3k$.)
$\therefore 5^2=(3k)^2+(4)^2 \Rightarrow k=\pm1$



16C.43 (HKCEE AM 1998 – II 2)

$\begin{cases}L:x-7y+3=0\\ C:(x-2)^2+(y+5)^2=a\end{cases}$
$\Rightarrow (7y-3-2)^2+(y+5)^2=a \Rightarrow 50y^2-60y+50-a=0$
$\therefore \Delta=3600-4\cdot50(50-a)=0 \Rightarrow 18-(50-a)=0$
$\Rightarrow a=32$

16C.44 (HKCEE AM 2000 – II 9)

(a) $(x+2k+2)^2+\left(y+\dfrac{3k+1}{2}\right)^2=(8k+8)+(2k+2)^2+\left(\dfrac{3k+1}{2}\right)^2$
$(x+2k+2)^2+\left(y+\dfrac{3k+1}{2}\right)^2=\dfrac{25}{4}k^2+\dfrac{35}{2}k+\dfrac{49}{4}$
$(x+2k+2)^2+\left(y+\dfrac{3k+1}{2}\right)^2=\left(\dfrac{5k+7}{2}\right)^2$

(b) (i) $\because$ Touches x-axis
$\therefore \dfrac{3k+1}{2}=\pm\left(\dfrac{5k+7}{2}\right) \Rightarrow k=-3$ or -1
$\therefore$ The circles are $x^2+(y-1)^2=1$ (C_1) and $(x-4)^2+(y-4)^2=16$ (C_2)

(ii) Dist. between centres $=\sqrt{(4-0)^2+(4-1)^2}$
$=5=1+4$
$\therefore$ Touch externally

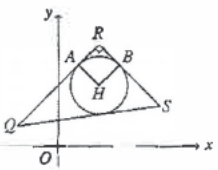
(c) Let the centre of C_3 be (a,b).
$\because$ Collinear with centres of C_1 and C_2
$\therefore \dfrac{b-1}{a-0}=\dfrac{4-1}{4-0}=\dfrac{3}{4} \Rightarrow b=\dfrac{3}{4}a+1$
$\because$ Touches x-axis
$\therefore$ Radius $=b$
$\because$ Touches C_2 externally
$\therefore \sqrt{(a-4)^2+(b-4)^2}=4+b$
$a^2-8a+16+b^2-8b+16=(4+b)^2$
$a^2-8a+16-8b=+8b$
$a^2-8a+16=16b$
$=16\left(\dfrac{3}{4}a+1\right)$
$a^2-20a=0$
$\Rightarrow a=0$ or $20 \Rightarrow b=1$ or 16
$\because (0,1)$ is the centre of C_1
$\therefore C_3$ is $(x-20)^2+(y-16)^2=16^2$

16C.45 HKCEE AM 2002 – 15

(a) Suppose the centre is G. Then
$A=$ Area of $\triangle GDE+$ Area of $\triangle GEF+$ Area of $\triangle GFD$
$=\dfrac{1}{2}DE\cdot r+\dfrac{1}{2}EF\cdot r+\dfrac{1}{2}FD\cdot r$
$=\dfrac{1}{2}(DE+EF+FD)r=\dfrac{1}{2}pr$

(b) (i) Perimeter of $\triangle QRS$
$=\sqrt{4^2+4^2}+\sqrt{3^2+3^2}+\sqrt{7^2+1^2}$
$=4\sqrt{2}+3\sqrt{2}+5\sqrt{2}=12\sqrt{2}$
$\therefore$ Radius of $C_2=\dfrac{\frac{1}{2}\cdot4\sqrt{2}\cdot3\sqrt{2}}{\frac{1}{2}\cdot12\sqrt{2}}=\sqrt{2}$

(ii) Denote the points where C_2 touches QR and RS by A and B respectively. Also let H be the centre of C_2. Then $RAHB$ is a square.



i.e. $RA=AH=HB=BR=\sqrt{2}$
$RH=\sqrt{(\sqrt{2})^2+(\sqrt{2})^2}=2$
$\because m_{RA}=\dfrac{5-1}{2+2}=1$ and $m_{RS}=\dfrac{5-2}{2-5}=-1$
$\therefore RH$ is vertical.
Thus, $H=(2,5-2)=(2,3)$.
$\therefore$ Eqn of C_2 is $(x-2)^2+(y-3)^2=2$

16C.46 HKCEE AM 2005 – 15

(a) $\begin{cases}L:y=kx\\ C:x^2+y^2-4x-2y+4=0\end{cases}$
$\Rightarrow x^2+(kx)^2-4x-2(kx)+4=0$
$(1+k^2)x^2-2(2+k)x+4=0\ldots(*)$
$\Delta=4(2+k)^2-4(4)(1+k^2)>0$
$k^2+4k+4-4-4k^2>0$
$3k^2-4k<0 \Rightarrow 0<k<\dfrac{4}{3}$

(b) From (a), equation of the tangent is $y=\dfrac{4}{3}x$.

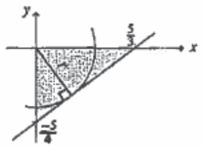
(c) (i) The x-coordinates of P and Q are the roots of $(*)$.
$\Rightarrow$ Sum of roots $=\dfrac{2(2+k)}{1+k^2}$
$\therefore$ x-coordinate of $M=\dfrac{\text{Sum of roots}}{2}=\dfrac{2+k}{1+k^2}$

16C.47 HKCEE AM 2006 – 14

(a) (i) $\begin{cases}L:y=mx+c\\ J:x^2+y^2=r^2\end{cases}$
$\Rightarrow x^2+(mx+c)^2=r^2$
$(1+m^2)x^2+2mcx+c^2-r^2=0$
$\Delta=4m^2c^2-4(1+m^2)(c^2-r^2)=0$
$m^2c^2-c^2-m^2c^2+r^2+r^2m^2=0$
$c^2=r^2(m^2+1)$

(ii) Put (h,k) into L: $k=mh+c$
$\therefore (k-mh)^2=c^2=r^2(m^2+1)$

(b) (i) PR: $\dfrac{y-4}{x-7}=\dfrac{-5-4}{-5-7}=\dfrac{3}{4} \Rightarrow 3x-4y-5=0$
$\Rightarrow$ x-intercept $=\dfrac{5}{3}$
y-intercept $=\dfrac{-5}{4}$



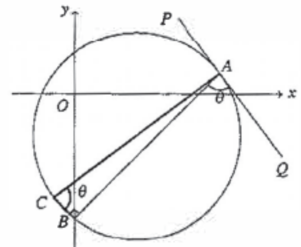
In the shaded triangle,
$\dfrac{1}{2}r\sqrt{\left(\dfrac{5}{3}\right)^2+\left(\dfrac{5}{4}\right)^2}=\dfrac{1}{2}\cdot\dfrac{5}{3}\cdot\dfrac{5}{4}=$ Area
$\Rightarrow r=\dfrac{25}{12}\div\dfrac{25}{12}=1$

(ii) Use (a)(ii) with $(h,k)=(7,4)$ and $r=1$.
$(4-7m)^2=m^2+1$
$48m^2-56m+15=0 \Rightarrow m=\dfrac{3}{4}$ or $\dfrac{5}{12}$
$\therefore m_{PQ}=\dfrac{5}{12}$

(iii) Use (a)(ii) with $(h,k)=R=(-5,-5)$ and $r=1$.
$(-5+5m)^2=m^2+1$
$24m^2-50m+24=0 \Rightarrow m=\dfrac{3}{4}$ or $\dfrac{4}{3}$
$\therefore m_{QR}=\dfrac{4}{3}$
Let $Q=(a,b)$. Then
$\begin{cases}\dfrac{b-4}{a-7}=\dfrac{5}{12} \Rightarrow 5a-12b=-13\\ \dfrac{b+5}{a+5}=\dfrac{4}{3} \Rightarrow 4a-3b=-5\end{cases}$
$\Rightarrow Q=(a,b)=\left(\dfrac{-7}{11},\dfrac{9}{11}\right)$

16C.48 HKCEE AM 2010 – 7

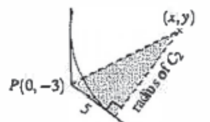
Centre $=(3,-2)$, Radius $=5$
Let $C(m,n)$ be the diametrically opposite pt of A on the circle.
Then $\left(\dfrac{m+7}{2},\dfrac{n+1}{2}\right)=(3,-2) \Rightarrow C=(m,n)=(-1,-5)$
$\because \angle ACB=\theta$ ($\angle$ in alt. segment)
and $\angle ABC=90°$ ($\angle$ in semi-circle)
$\therefore \tan\theta=\dfrac{AB}{BC}=\dfrac{\sqrt{(7-0)^2+(1+6)^2}}{\sqrt{(0+1)^2+(-6+5)^2}}=7$



16C.49 HKCEE AM 2010 – 15

(a) Let the centre of C_2 be (x,y).
Dist. between centres $=$ Radius of C_2- Radius of C_1
$(x-6)^2+(y-5)^2=(x-5)^2$
$-12x+36+y^2-10y=-10x$
$y^2-10y+36=2x \Rightarrow x=\dfrac{1}{2}y^2-5y+18$

(b) (i) By Pyth. thm, $(x-0)^2+(y+3)^2=5^2+x^2$
$(y+3)^2=5^2$
$y=2$ or -8 (rej)
$\Rightarrow x=\frac{1}{2}(2)^2-5(2)+18$
$=10$
$\therefore$ Centre of $C_2=(10,2)$



(ii) Eqn of C_2: $(x-10)^2+(y-2)^2=10^2$
Let the eqns of tangents be $y=mx-3$.
$\begin{cases}y=mx-3\\ (x-10)^2+(y-2)^2=100\end{cases}$
$\Rightarrow (x-10)^2+(mx-5)^2=100$
$(1+m^2)x^2-10(m+2)x+25=0$
$\Delta=100(m+2)^2-100(1+m^2)=0$
$m^2+4m+m-1-m^2=0 \Rightarrow m=\dfrac{-3}{4}$
$\therefore$ Eqns of tgs are $y=\dfrac{-3}{4}x-3$ and $x=0$ (y-axis).

16C.50 HKDSE MA SP – I – 19

(a) (i) Join B and C.

$\angle DAE = \angle DBC$ ($\angle$s in the same segment)
$= \angle PCB$ (alt. $\angle$s, $PQ//BD$)
$= \angle BAE$ ($\angle$ in alt. segment)

In $\triangle ABE$ and $\triangle ADE$,
$AB = AD$ (given)
$\angle BAE = \angle DAE$ (proved)
$AE = AE$ (common)
$\therefore \triangle ABE \cong \triangle ADE$ (SAS)

(ii) $\because \angle BAE = \angle DAE$ (corr. $\angle$s, $\cong\triangle$s)
$\therefore AE$ is an $\angle$ bisector of $\triangle ABD$.
Hence, $AE \perp BD$ (property of isos. $\triangle$)
$\Rightarrow AE$ is an altitude of $\triangle ABD$.
$BE = DE$ (property of isos. $\triangle$)
$\Rightarrow AE$ is a median of $\triangle ABD$.
$\Rightarrow AE$ is a $\perp$ bisector of $\triangle ABD$.
Thus, the in centre, orthocentre, centroid and circumcentre of $\triangle ABE$ all lie on AE. They are collinear.

(b) $m_{PQ} = m_{BD} = \dfrac{12-4}{8-4} = 2$
From (a)(ii), AC is a diameter of the circle.

Method 1
Let the circle be $x^2+y^2+Dx+Ey+F=0$.
$$\begin{cases}14^2+4^2+14D+4E+F=0\\8^2+12^2+8D+12E+F=0\\4^2+4^2+4D+4E+F=0\end{cases} \Rightarrow \begin{cases}D=-18\\E=-13\\F=92\end{cases}$$
$\therefore$ The circle is $x^2+y^2-18x-13y+92=0$.
$\Rightarrow$ Centre $=(9,6.5)$

Method 2
Eqn of $\perp$ bisector of BD (i.e. AC):
$$\sqrt{(x-8)^2+(y-12)^2} = \sqrt{(x-4)^2+(y-4)^2}$$
$$-16x+64-24y+144 = -8x+16-8y+16$$
$$x+2y-22=0$$
Eqn of $\perp$ bisector of AD: $x = \dfrac{14+4}{2} = 9$
($\because$ AD is parallel to the x-axis.)
Solving $\begin{cases}x+2y-22=0\\x=9\end{cases} \Rightarrow$ Circumcentre $=(9,6.5)$

Method 3
Let the centre be $\left(\dfrac{14+4}{2},k\right)=(9,k)$.
Radius $=\sqrt{(9-8)^2+(k-12)^2}=\sqrt{(9-4)^2+(k-4)^2}$
$$k^2-24k+145=k^2-8k+41$$
$$k=6.5$$
$\therefore$ Centre $=(9,6.5)$

Hence,
Let $C=(m,n)$. Then
$\left(\dfrac{m+14}{2},\dfrac{n+4}{2}\right)=(9,6.5) \Rightarrow C=(m,n)=(4,9)$
$\therefore$ Eqn of PQ: $y-9=2(x-4) \Rightarrow 2x-y+1=0$

16C.51 HKDSE MA PP – I – 14

(a) $\triangle BCD \sim \triangle OAD$

(b) (i) $AD=\sqrt{6^2+12^2}=\sqrt{180}$
$\dfrac{CD}{AD}=\sqrt{\dfrac{16}{45}} \Rightarrow CD=\sqrt{\dfrac{16}{45}\times 180}=8$
$\therefore C=(0,12-8)=(0,4)$

(ii) AC is a diameter of the circle.
Mid-pt of $AC=\left(\dfrac{6+0}{2},\dfrac{0+4}{2}\right)=(3,2)$
$AC=\sqrt{6^2+4^2}=\sqrt{52}$
$\therefore$ Eqn of circle $OABC$: $(x-3)^2+(y-2)^2=\left(\dfrac{\sqrt{52}}{2}\right)^2$
$\Rightarrow x^2+y^2-6x-4y=0$

16C.52 HKDSE MA 2012 – I – 17

(a) Radius $=y$-coordinate of centre $=10$
$\therefore$ Eqn of C: $(x-6)^2+(y-10)^2=100$

(b) Eqn of L: $y=-x+k$
$$\begin{cases}y=-x+k\\(x-6)^2+(y-10)^2=100\end{cases}$$
$\Rightarrow (x-6)^2+(-x+k-10)^2=100$
$2x^2+(8-2k)x+(k^2-20k+36)=0$
Sum of roots $=-\dfrac{8-2k}{2}=k-4$
$\Rightarrow$ x-coordinate of mid-pt of $AB=\dfrac{k-4}{2}$
y-coordinate of mid-pt of $AB=-\left(\dfrac{k-4}{2}\right)+k$
$\therefore$ Mid-point of $AB=\left(\dfrac{k-4}{2},\dfrac{k+4}{2}\right)$

16C.53 HKDSE MA 2015 – I – 14

(a) (i) *Method 1*
Mid-pt of $PQ=\left(\dfrac{4-14}{2},\dfrac{-1+23}{2}\right)=(-5,11)$
$m_{PQ}=\dfrac{-1-23}{4+14}=\dfrac{-4}{3}$
$\therefore$ Eqn of L: $y-11=\dfrac{-1}{\frac{-4}{3}}(x+5) \Rightarrow y=\dfrac{3}{4}x+\dfrac{59}{4}$

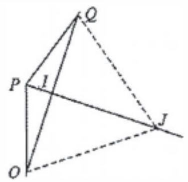
Method 2
$$\sqrt{(x-4)^2+(y+1)^2}=\sqrt{(x+14)^2+(y-23)^2}$$
$$-8x+16+2y+1=28x+196-46y+529$$
$$3x-4y+59=0$$

(ii) Centre $=\left(h,\dfrac{3h+59}{4}\right)$
Radius $=\sqrt{(4-h)^2+\left(-1-\dfrac{3h+59}{4}\right)^2}$
Eqn of C:
$$(x-h)^2+\left(y-\tfrac{3h+59}{4}\right)^2=(4-h)^2+\left(-1-\tfrac{3h+59}{4}\right)^2$$
$$x^2-2hx+y^2-\tfrac{3h+59}{2}y=16-8h+1+\tfrac{3h+59}{2}$$
$$x^2+y^2-2hx-\tfrac{3h+59}{2}y+\tfrac{13h-93}{2}=0$$
$$2x^2+2y^2-4hx-(3h+59)y+13h-93=0$$

(b) If C passes through R,
$$2(26)^2+2(43)^2-4h(26)-(3h+59)(43)+13h-93=0$$
$$2420-220h=0$$
$$h=11$$
$\therefore$ Diameter $=2\sqrt{(4-11)^2+\left(-1-\frac{3(11)+59}{4}\right)^2}=50$

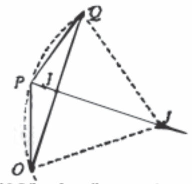
16C.54 HKDSE MA 2016 – I – 20

(a) *Method 1*



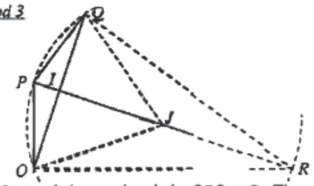
Let $\angle OPJ=\angle QPJ=\theta$. (in-centre)
$OJ=PJ=QJ$ (radii)
In $\triangle POJ$, $\angle POJ=\angle OPJ=\theta$ (base $\angle$s, isos. $\triangle$)
In $\triangle PQJ$, $\angle PQJ=\angle QPJ=\theta$ (base $\angle$s, isos. $\triangle$)
In $\triangle POJ$ and $\triangle PQJ$,
$\angle OPJ=\angle QPJ=\theta$ (in-centre)
$\angle POJ=\angle PQJ=\theta$ (proved)
$PJ=PJ$ (common)
$\therefore \triangle POJ\cong\triangle PQJ$ (AAS)
$\therefore PO=PQ$ (corr. sides, $\cong\triangle$s)

Method 2



Let $\angle OPJ=\angle QPJ=\theta$. (in-centre)
$OJ=PJ=QJ$ (radii)
In $\triangle POJ$, $\angle POJ=\angle OPJ=\theta$ (base $\angle$s, isos. $\triangle$)
$\Rightarrow \angle PJO=180^\circ-2\theta$ ($\angle$ sum of $\triangle$)
$\Rightarrow \angle PQO=(180^\circ-2\theta)\div 2=90^\circ-\theta$
($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
In $\triangle PQJ$, $\angle PQJ=\angle QPJ=\theta$ (base $\angle$s, isos. $\triangle$)
$\Rightarrow \angle PJQ=180^\circ-2\theta$ ($\angle$ sum of $\triangle$)
$\Rightarrow \angle POQ=(180^\circ-2\theta)\div 2=90^\circ-\theta$
($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
$\because \angle PQO=\angle POQ=90^\circ-\theta$ (proved)
$\therefore PO=PQ$ (sides opp. equal $\angle$s)

Method 3



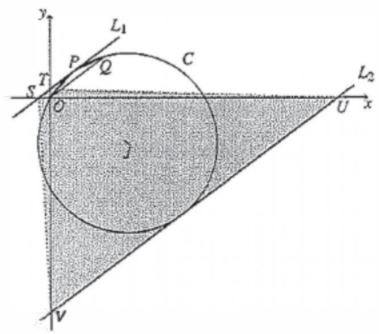
Let PJ extended meet the circle OPQ at R. Then PR is a diameter of the circle.
$\therefore \angle POR=\angle PQR=90^\circ$ ($\angle$ in semi-circle)
Let $\angle OPR=\angle QPR=\theta$. (in-centre)
In $\triangle OPR$, $PO=PR\cos\theta$
In $\triangle QPR$, $PQ=PR\cos\theta$
$\therefore PO=PQ$

(b) (i) Let $P=(x,19)$. By (a),
$$OP=PQ$$
$$\sqrt{x^2+19^2}=\sqrt{(x-40)^2+(19-30)^2}$$
$$x^2+361=x^2-80x+1600+121$$
$$x=17 \Rightarrow P=(17,19)$$

Method 1
Let C be $x^2+y^2+Dx+Ey+F=0$.
$$\begin{cases}0^2+0^2+0+0+F=0\\17^2+19^2+17D+19E+F=0\\40^2+30^2+40D+30E+F=0\end{cases} \Rightarrow \begin{cases}D=-112\\E=66\\F=0\end{cases}$$
$\therefore$ Eqn of C is $x^2+y^2-112x+66y=0$.

Method 2
The centre J lies on the $\perp$ bisector of OQ.
$$\begin{cases}\text{Mid-pt of } OQ=\left(\dfrac{40}{2},\dfrac{30}{2}\right)=(20,15)\\m_{OQ}=\dfrac{30}{40}=\dfrac{3}{4} \Rightarrow m_{\perp\text{ bisector}}=\dfrac{-4}{3}\end{cases}$$
$\therefore$ Eqn of $\perp$ bisector: $y-15=\dfrac{-4}{3}(x-20)$
$\Rightarrow y=\dfrac{125-4x}{3}$
Let $J=(h,k)$. Then
$$\begin{cases}k=\dfrac{125-4h}{3}\\(h-17)^2+(k-19)^2=(h-0)^2+(k-0)^2\end{cases}$$
$$h^2-34h+289+k^2-38k+361=h^2+k^2$$
$$-34h-38\left(\frac{125-4h}{3}\right)+650=0$$
$$\frac{50}{3}h-\frac{2800}{3}=0$$
$$h=56 \Rightarrow k=-33$$
$\therefore$ Eqn of C is
$(x-56)^2+(y+33)^2=(0-56)^2+(0+33)^2$
$\Rightarrow x^2+y^2-112x+66y=0$

(ii)



$\boxed{\textit{Approach One} - \text{Find } L_1 \text{ and } L_2}$

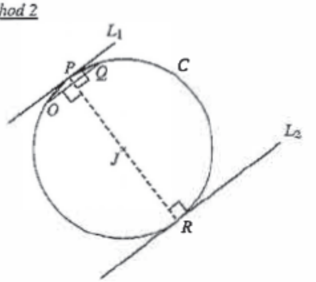
Method 1
Let L_1 and L_2 be $y=\dfrac{3}{4}x+c$.
$$\begin{cases}y=\dfrac{3}{4}x+c\\x^2+y^2-112x+66y=0\end{cases}$$
$$x^2+\left(\frac{3}{4}x+c\right)^2-112x+66\left(\frac{3}{4}x+c\right)=0$$
$$\frac{25}{16}x^2+\left(\frac{3c-125}{2}\right)x+(c^2+66c)=0$$

$$\Delta = \frac{(3c-125)^2}{4} - 4\cdot\frac{25}{16}\cdot(c^2+66c) = 0$$
$$-16c^2 - 2400c + 15625 = 0$$
$$c = -\frac{625}{4} \text{ or } \frac{25}{4}$$

$\therefore$ L_1 is $y = \frac{3}{4}x + \frac{25}{4} \Rightarrow \begin{cases} S = \left(\frac{-25}{3}, 0\right) \\ T = \left(0, \frac{25}{4}\right) \end{cases}$

L_2 is $y = \frac{3}{4}x - \frac{625}{4} \Rightarrow \begin{cases} U = \left(\frac{625}{3}, 0\right) \\ V = \left(0, \frac{-625}{4}\right) \end{cases}$

Method 2



$\because$ $OP = PQ$ and $\angle OPJ = \angle QPJ$ (proved)
$\therefore$ $OQ \perp PJ$ (property of isos. $\triangle$)
$\Rightarrow$ $L_1 \perp PJ$ ($OQ // L_1$)
$\Rightarrow$ L_1 is tangent to C at P.
(converse of $\angle$s in the same segment)

$\therefore$ Eqn of L_1: $y - 19 = \frac{3}{4}(x-17) \Rightarrow y = \frac{3}{4}x + \frac{25}{4}$
$\Rightarrow S = \left(\frac{-25}{3}, 0\right),\ T = \left(0, \frac{25}{4}\right)$

Let the diameter of C through P meet C again at $R(r,s)$. Then $\left(\frac{17+r}{2}, \frac{19+s}{2}\right) = J = (56, -33)$
$\Rightarrow R = (95, -85)$

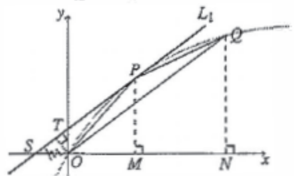
$\because$ L_2 is tangent to C at R
$\therefore$ Eqn of L_2: $y + 85 = \frac{3}{4}(x - 95) \Rightarrow y = \frac{3}{4}x - \frac{625}{4}$
$\Rightarrow U = \left(\frac{625}{3}, 0\right),\ V = \left(0, \frac{-625}{4}\right)$

Therefore, (Method 1)
Area of trapezium $STUV$
$= $ Area of $\triangle STU +$ Area of $\triangle SVU$
$= \frac{(\frac{625}{3}+\frac{25}{3})(\frac{25}{4})}{2} + \frac{(\frac{625}{3}+\frac{25}{3})(\frac{625}{4})}{2}$
$= \frac{105625}{6} = 17604.2 > 17000 \Rightarrow$ YES

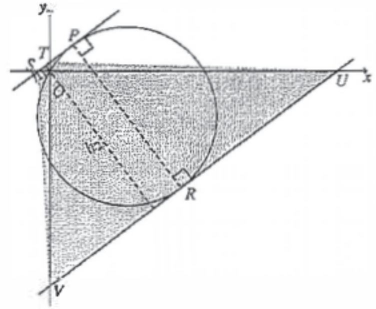
Therefore, (Method 2)
$ST = \sqrt{\left(0+\frac{25}{3}\right)^2 + \left(\frac{25}{4}-0\right)^2} = \frac{125}{12}$
$UV = \sqrt{\left(\frac{625}{3} - 0\right)^2 + \left(-\frac{625}{4} - 0\right)^2} = \frac{3125}{12}$
Height of $STUV =$ Diameter of $C = 130$
$\therefore$ Area of $STUV = \frac{(\frac{125}{12}+\frac{3125}{12})(130)}{2}$
$= \frac{105625}{6} > 17000 \Rightarrow$ YES

Approach Two – Find S, T, U, V without L_1 and L_2

Method 3
Let the foots of perpendiculars from P and Q to the x-axis be M and N respectively. Note that $OQ // L_1 // L_2$.
$\because$ $\triangle SPM \sim \triangle OQN$
$\therefore$ $\frac{PM}{SM} = \frac{QN}{ON} = \frac{3}{4} \Rightarrow SM = \frac{4}{3}(19) = \frac{76}{3}$
$\Rightarrow S = \left(17 - \frac{76}{3}, 0\right) = \left(-\frac{25}{3}, 0\right)$
In $\triangle OST$, $OT = \frac{3}{4}OS = \frac{25}{4} \Rightarrow T = \left(0, \frac{25}{4}\right)$
Area of $\triangle OST = \frac{1}{2} \times \frac{25}{3} \times \frac{25}{4} = \frac{625}{24}$
$ST = \sqrt{\left(\frac{25}{3}\right)^2 + \left(\frac{25}{4}\right)^2} = \frac{125}{12}$
$\Rightarrow$ Height of $\triangle OST$ from O to ST ('h_1')
$= \frac{2 \times \frac{625}{24}}{\frac{125}{12}} = 5$



Referring to *Method 2*, PR is the height of trapezium $STUV$ as $PR \perp L_1$.
$\therefore$ Height of $\triangle OUV$ from O to UV ('h_2')
$=$ Diameter of $C - h_1 = 2\sqrt{56^2+33^2} - 5 = 125$



$\because$ $\triangle OST \sim \triangle OUV$
$\therefore$ $\frac{OV}{OT} = \frac{OU}{OS} = \frac{h_2}{h_1} = 25$
Area of $\triangle OUV = \left(\frac{h_2}{h_1}\right)^2$ (Area of $\triangle OST$)
$= 625$(Area of $\triangle OST$)
Area of $\triangle OTU = \left(\frac{OU}{OS}\right)$ (Area of $\triangle OST$)
$= 25$(Area of $\triangle OST$)
Area of $\triangle OSV = \left(\frac{OV}{OT}\right)$ (Area of $\triangle OST$)
$= 25$(Area of $\triangle OST$)
$\therefore$ Area of $STUV = (1+625+25+25)$(A of $\triangle OST$)
$= \frac{105625}{6} > 17000 \Rightarrow$ YES

Approach Three – A hybrid of Methods 1 and 3

Method 4
Let L_1 and L_2 be $y = \frac{3}{4}x + c$
$$\begin{cases} y = \frac{3}{4}x + c \\ x^2 + y^2 - 112x + 66y = 0 \end{cases}$$
$$x^2 + \left(\frac{3}{4}x + c\right)^2 - 112x + 66\left(\frac{3}{4}x + c\right) = 0$$
$$\frac{25}{16}x^2 + \left(\frac{3c-125}{2}\right)x + (c^2+66c) = 0$$
$$\Delta = \frac{(3c-125)^2}{4} - 4\cdot\frac{25}{16}\cdot(c^2+66c) = 0$$
$$-16c^2 - 2400c + 15625 = 0$$
$$c = -\frac{625}{4} \text{ or } \frac{25}{4}$$
$\therefore$ $OT = \frac{25}{4},\ OV = \frac{625}{4}$
$\Rightarrow \frac{OV}{OT} = 25 \Rightarrow \frac{OU}{OS} = 25$ ($\because \triangle OST \sim \triangle OUV$)
Thus,
Area of $\triangle OUV = (25)^2$(Area of $\triangle OST$)
Area of $\triangle OTU = \left(\frac{OU}{OS}\right)$ (Area of $\triangle OST$)
$= 25$(Area of $\triangle OST$)
Area of $\triangle OSV = \left(\frac{OV}{OT}\right)$ (Area of $\triangle OST$)
$= 25$(Area of $\triangle OST$)
Besides, for $\triangle OST$, $\frac{OT}{OS} = $ slope $= \frac{3}{4} \Rightarrow OS = \frac{25}{3}$
$\Rightarrow$ Area $= \frac{1}{2} \times \frac{25}{3} \times \frac{25}{4} = \frac{625}{24}$
$\therefore$ Area of $STUV = (1+625+25+25)$(A of $\triangle OST$)
$= \frac{105625}{6} > 17000 \Rightarrow$ YES

16C.55 HKDSE MA 2018 – I – 19

(a) Eqn of C: $(x-8)^2 + (y-2)^2 = r^2$
$$\begin{cases} L: kx - 5y - 21 = 0 \\ C: (x-8)^2 + (y-2) = r^2 \end{cases}$$
$$\Rightarrow (x-8)^2 + \left(\frac{kx-21}{5} - 2\right)^2 = r^2$$
$$(x-8)^2 + \left(\frac{k}{5}x - \frac{31}{5}\right)^2 - r^2 = 0$$
$$\left(1+\frac{k^2}{25}\right)x^2 - 2\left(\frac{31}{25}k + 8\right)x + \frac{2561}{25} - r^2 = 0$$
$$\Delta = 0 = 4\left(\frac{31}{25}k+8\right)^2 - 4\left(1+\frac{k^2}{25}\right)\left(\frac{2561}{25} - r^2\right)$$
$$\left(\frac{31}{25}k+8\right)^2 = \left(1+\frac{k^2}{25}\right)\left(\frac{2561}{25} - r^2\right)$$
$$\frac{2561}{25} - r^2 = \frac{(31k+200)^2}{25(25+k^2)}$$
$$r^2 = \frac{2561(25+k^2) - (31k+200)^2}{25(25+k^2)}$$
$$= \frac{961 - 496k + 64k^2}{25+k^2}$$

(b) (i) Put D into L:
$k(18) - 5(39) - 21 = 0 \Rightarrow k = 12$
$r = \sqrt{\frac{961 - 496(12) + 64(12)^2}{25+(12)^2}} = 5$

(ii) $E = \left(0, \frac{-21}{5}\right)$
Denote the centre of C be G, which is the in-centre of $\triangle DEF$.
$DG = \sqrt{(18-8)^2 + (39-2)^2} = \sqrt{1469}$
$\Rightarrow \angle GDE = \sin^{-1}\frac{r}{DG} = 7.49586°$
$\Rightarrow \angle FDE = 2\angle GDE = 14.99172°$
$EG = \sqrt{(0-8)^2 + (\frac{-21}{5} - 2)^2} = \sqrt{\frac{2561}{25}}$
$\Rightarrow \angle GED = \sin^{-1}\frac{r}{EG} = 29.60445°$
$\Rightarrow \angle FED = 2\angle GED = 59.20890°$
$\therefore$ $\angle DFE = 180° - 14.99172° - 59.20890°$
$= 105.6° > 90°$
$\therefore$ YES

16C.56 HKDSE MA 2019 – I – 19

(a) $f(4) = \frac{1}{1+k}\left((4)^2 + (6k-2)(4) + (9k+25)\right)$
$= \frac{1}{1+k}(33+33k) = 33$
Hence, the graph passes through F.

(b) (i) $g(x) = f(-x) + 4$
$= \frac{1}{1+k}\left((-x)^2 + (6k-2)(-x) + (9k+25)\right) + 4$
$= \frac{1}{1+k}\left(x^2 - (6k-2)x + (3k-1)^2 - (3k-1)^2 + (9k+25)\right) + 4$
$= \frac{1}{1+k}\left((x-3k+1)^2 - 9k^2 + 3k + 24\right) + 4$
$= \frac{1}{1+k}\left((x-3k+1)^2 - 3(1+k)(3k-8)\right) + 4$
$= \frac{1}{1+k}(x-3k+1)^2 - 3(3k-8) + 4$
$= \frac{1}{1+k}(x-3k+1)^2 + 28 - 9k$
$\therefore$ $U = (3k-1, 28-9k)$

(ii) As F varies, the circle is the smallest when OU is the diameter.

Method 1
$FO \perp FU \Rightarrow m_{FO}\, m_{FU} = -1$
$\frac{28-9k}{3k-1} \cdot \frac{(28-9k)-33}{(3k-1)-4} = -1$
$(28-9k)^2 - 33(28-9k) = -(3k-1)^2 + 4(3k-1)$
$90k^2 - 225k - 135 = 0$
$k = 3$ or $-\frac{1}{2}$ (rej.)

Method 2
Mid-pt of $OU = \left(2, \frac{33}{2}\right)$
$\sqrt{(3k-1-2)^2 + (28-9k-\frac{33}{2})^2} = \sqrt{2^2 + (\frac{33}{2})^2}$
$(3k-1)^2 - 4(3k-1) + (28-9k)^2 - 33(28-9k) = 0$
$90k^2 - 225k - 135 = 0$
$k = 3$ or $\frac{-1}{2}$ (rej.)

(iii) The fixed point G is the image of F after the above transformations. i.e. $G=(-4,37)$.
Also, $V=(3(3)-1,28-9(3))=(8,1)$

Method 1

$\because\ m_{GF}\cdot m_{GO}=\dfrac{37-33}{-4-4}\cdot\dfrac{37-0}{-4-0}=\dfrac{37}{8}\neq-1$

$\therefore$ G is not on the circle with FO as diameter (which is the circle through F, O and V). $\Rightarrow$ NO

Method 2

The circle through $F(4,33)$, $O(0,0)$ and $V(8,1)$ is

$(x-2)^2+(y-\frac{33}{2})^2=2^2+(\frac{33}{2})^2$

$\Rightarrow x^2+y^2-4x-33y=0$

Put $G(-4,37)$: LHS $=180\neq$ RHS $\Rightarrow$ NO

Method 2'

Let the circle through $F(4,33)$, $O(0,0)$ and $V(8,1)$ be $x^2+y^2+dx+ey+f=0$.

$$\begin{cases}4^2+33^2+4d+33e+f=0\\0^2+0^2+0d+0e+f=0\\8^2+1^2+8d+e+f=0\end{cases}\Rightarrow\begin{cases}d=-4\\e=-33\\f=0\end{cases}$$

Thus, the eqn of circle FOV is $x^2+y^2-4x-33y=0$.

Put $G(-4,37)$: LHS $=180\neq$ RHS $\Rightarrow$ NO

16C.57 HKDSE MA 2020 – I – 14

14a	Let M be the mid-point of AB.
	Then, $GM\perp AB$ (line joining centre to mid-pt. of chord $\perp$ chord).
	Since AB is horizontal, GM is vertical.
	The x-coordinate of $G=\dfrac{-10+30}{2}$ $=10$
	The radius of $C=AG$ $=\sqrt{(-10-10)^2+[0-(-15)]^2}$ $=25$
	Therefore, the equation of C is $(x-10)^2+[y-(-15)]^2=25^2$, i.e. $x^2+y^2-20x+30y-300=0$.
bi	Γ and L are parallel.
ii	Since Γ and L are parallel, we know that the slope of Γ is equal to the slope of L, i.e. $\dfrac{-15-0}{10-30}=\dfrac{3}{4}$.
	Let $P=(x,y)$. $y-0=\dfrac{3}{4}[x-(-10)]$ $3x-4y+30=0$
	Therefore, the equation of Γ is $3x-4y+30=0$.
iii	Let θ be the inclination of AG and ϕ be the inclination of AH.
	Note that $0^\circ\le\theta<180^\circ$ and $0^\circ\le\phi<180^\circ$.
	$\tan\theta=$ The slope of AG; $\tan\theta=\dfrac{-15-0}{10-(-10)}$; $\tan\theta=-\dfrac{3}{4}$; $\theta=180^\circ-36.86989765^\circ$; $\theta=143.1301024^\circ$
	$\tan\phi=$ The slope of AH; $\tan\phi=\dfrac{3}{4}$; $\phi=36.86989765^\circ$

	$\angle BAG+\theta=180^\circ$ (adj. $\angle$s on st. line)
	$\angle BAG+143.1301024^\circ=180^\circ$
	$\angle BAG=36.86989765^\circ$
	$\angle GAH=\angle BAG+\angle BAH$ $=\angle BAG+\phi$ $=36.86989765^\circ+36.86989765^\circ$ $=73.73979529^\circ$ $>70^\circ$
	Therefore, the claim is disagreed with.

16D Loci in the rectangular coordinate plane

16D.1 (HKCEE MA 1981(3) – I – 7)

(a) $P=\left(\dfrac{4(1)+1(16)}{1+4},\dfrac{4(4)+1(-16)}{1+4}\right)=(4,0)$

(b) Put A into the parabola: $(4)^2=4a(1)\Rightarrow a=4$

Hence, the parabola is $y^2=16x$.

Eqn of locus: $(x+a)^2=(x-4)^2+(y-0)^2$

$x^2+8x+16=x^2-8x+16+y^2$

$y^2=16x$

which is the given parabola.

16D.2 HKCEE AM 1987 – II – 10

(a) $(x+1)^2=(x-1)^2+(y-0)^2$

$x^2+2x+1=x^2-2x+1+y^2$

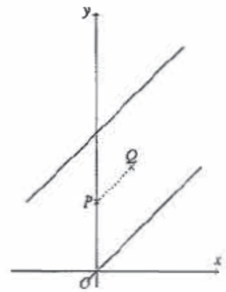
$y^2=4x$

16D.3 (HKCEE AM 1994 – II – 4)

(a) (Note that PR_0 is parallel to the x-axis. Thus:)

Area $=\dfrac{(4-0)(6-4)}{2}=4$

(b) (i) A pair of lines parallel and equidistant to PQ



(ii) $m_{PQ}=\dfrac{6-4}{2-0}=-1$

Since R_0 is a point on the locus (from (a)), the line parallel to PQ and through $R_0(4,4)$ is:

$y-4=1(x-4)\Rightarrow y=x$

Thus, the equations are $y=x$ and $y=x+4+4=x+8$.

16D.4 HKCEE AM 1999 – II – 10

(a) $(x+3)^2+(y-0)^2=3[(x+1)^2+(y-0)^2]$

$x^2+6x+9+y^2=3x^2+6x+3+3y^2$

$2x^2+2y^2=6\Rightarrow x^2+y^2=3$

(b) Slope of segment joining centre and $T=\dfrac{b}{a}$

$\Rightarrow$ Slope of tg $=-\dfrac{a}{b}$

$\therefore$ Eqn of tg: $y-b=-\dfrac{a}{b}(x-a)$

$by-b^2=-ax+a^2$

$ax+by-(a^2+b^2)=0$

$ax+by-3=0$ ($\because (a,b)$ lies on C)

(c) If the tangent in (b) passes through A,

$a(-3)+b(0)-3=0\Rightarrow a=-1$

$\Rightarrow b=\pm\sqrt{3-a^2}=\pm\sqrt{2}$

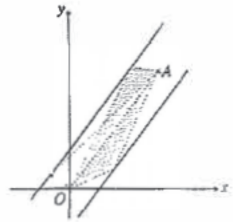
Since S is in Quad II, $S=(a,b)=(-1,\sqrt{2})$

16D.5 (HKCEE AM 2004 – 10)

A pair of straight lines parallel and equidistant to OA

$\because\ OA=\sqrt{3^2+4^2}=5$

$\therefore$ Dist. from the lines to $OA=\dfrac{2\times2}{5}=0.8$



16D.6 (HKCEE AM 2011 – 16)

(a) Centre of $C_1=(0,5)$, Radius of $C_1=\sqrt{5^2-16}=3$

Radius of the unknown circle $=y$

$\because$ It touches C_1 externally

$\therefore\ \sqrt{(x-0)^2+(y-5)^2}=y+3$

$x^2+y^2-10y+25=y^2+6y+9$

$x^2+16=16y\Rightarrow y=\dfrac{1}{16}x^2+1$

(b) (i) Let (h,k) be the centre of C_2.

Then $k=\dfrac{1}{16}h^2+1$.

Radius $=k=\sqrt{(h-20)^2+(k-16)^2}$

$k^2=h^2+k^2-40h-32k+656$

$0=h^2-40h-32\left(\dfrac{1}{16}h^2+1\right)+656$

$0=-h^2-40h+624$

$h=12$ or -52 (rej.)

$\therefore\ (h,k)=\left(12,\dfrac{1}{16}(12)^2+1\right)=(12,10)$

$\Rightarrow$ Eqn of C_2: $(x-12)^2+(y-10)^2=10^2$

$\Rightarrow x^2+y^2-24x-20y+144=0$

(ii) The point of contact is collinear with the 2 centres, which are both points on S. However, for a parabola opening upwards, the line segment joining 2 points on the parabola (we call it a 'secant' line) must lie above the parabola.

$\therefore$ The sentence is not correct.

(c) A circle that satisfies the first two conditions will touch C_1 *externally*. Hence, it cannot satisfy the last condition.

$\therefore$ NO

16D.7 HKDSE MA SP – I – 13

(a) $m_{L_1}=\dfrac{4}{3}\Rightarrow m_{L_2}=-\dfrac{3}{4}$

$\therefore$ Eqn of L_2: $y-9=\dfrac{-3}{4}(x-4)\Rightarrow 3x+4y-48=0$

(b) (i) Γ is the perpendicular bisector of AB.

$\therefore\ \Gamma//L_2$

(ii) *Method 1*

$$\begin{cases}L_1:4x-3y+12=0\\L_2:3x+4y-48=0\end{cases}\Rightarrow A=(3.84,9.12)$$

$B=(0,4)$

$\therefore$ Eqn of Γ is:

$(x-3.84)^2+(y-9.12)^2=(x-0)^2+(y-4)^2$

$-7.68x-18.24y+97.92=-8y+16$

$3x+4y-32=0$

Method 2

y-int of $L_1=4$, y-int of $L_2=12$

$\Rightarrow$ y-intercept of $\Gamma=\dfrac{4+12}{2}=8$

$\therefore$ Eqn of Γ is $y=\dfrac{-3}{4}x+8$

16D.8 HKDSE MA PP – I – 8

(a) $A'(3,4)$, $B'(5,-2)$

(b) Eqn: $(x-3)^2+(y-4)^2=(x-5)^2+(y+2)^2$

$-6x-8y+25=-10x\Rightarrow 4x-8y+25=0$

16D.9 HKDSE MA 2012 – I – 14

(a) (i) $\Gamma//L$

(ii) y-intercept of $\Gamma=\dfrac{(-1)+(-3)}{2}=-2$

$m_L=\dfrac{0+1}{3-0}=\dfrac{1}{3}$

$\therefore$ Eqn of Γ: $y=\dfrac{1}{3}x-2$

(b) (i) Put Q into the eqn of Γ:

RHS $=\dfrac{1}{3}(6)-2=0$ LHS

$\therefore$ Γ passes through Q.

(ii) $QH=QK=$ radius

(In fact, HQK is a diameter of the circle.)

Besides, since A and B lie on L, their perpendicular distances to Γ is the distance between L and Γ.

i.e., The height of $\triangle AQH$ with QH as base and the height of $\triangle BQK$ with QK as base are the same.

$\therefore$ Area of $\triangle AQH$: Area of $\triangle BQK=1:1$

16D.10 HKDSE MA 2013 – I – 14

(a) $R=(6,17)$

(b) (i) *Method 1*

$m_L=\dfrac{-4}{3}$

$\Rightarrow$ Eqn of PR: $y-17=\dfrac{-1}{\frac{-4}{3}}(x-6)\Rightarrow y=\dfrac{3}{4}x+\dfrac{25}{2}$

$$\begin{cases}PR:y=\frac{3}{4}x+\frac{25}{2}\\L:4x+3y+50=0\end{cases}\Rightarrow P=(-14,2)$$

Method 2

Let $P=(a,b)$.

$\because\ PR\perp L$

$\therefore\ m_{PR}=-1\div\dfrac{-4}{3}=\dfrac{3}{4}\Rightarrow\dfrac{b-17}{a-6}=\dfrac{3}{4}$

$$\begin{cases}4a+3b+50=0\\\dfrac{b-17}{a-6}=\dfrac{3}{4}\end{cases}\Rightarrow(a,b)=(-14,2)$$

Hence

$PR=\sqrt{(-14-6)^2+(2-17)^2}=25$

(ii) (1) P, Q and R are collinear.

(2) $QR=$ radius of circle $=\sqrt{6^2+17^2-225}=10$

$\therefore\ \dfrac{\text{Area of }\triangle OPQ}{\text{Area of }\triangle OQR}=\dfrac{PQ}{QR}=\dfrac{25-10}{10}=\dfrac{3}{2}$

16D.11 HKDSE MA 2014 – I – 12

(a) Radius of $C = \sqrt{(6-0)^2+(11-3)^2} = 10$

$\therefore$ Eqn of C: $(x-0)^2+(y-3)^2 = 10^2$

$\Rightarrow x^2+y^2-6y-91=0$

(b) (i) Eqn of Γ:

$(x-6)^2+(y-11)^2 = (x-0)^2+(y-3)^2$

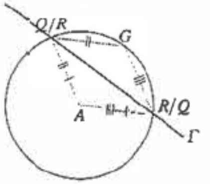
$-12x-22y+157 = -6y+9$

$3x+4y-37=0$

(ii) Γ is the perpendicular bisector of AG.

(iii) The quadrilateral is a rhombus.

$\therefore$ Perimeter $= 4 \times$ Radius $= 40$



16D.12 HKDSE MA 2016 I – 10

(a) Eqn of Γ:

$(x-5)^2+(y-7)^2 = (x-13)^2+(y-1)^2$

$-10x-14y+74 = -26x-2y+170$

$4x-3y-24=0$

(b) $H=(6,0)$, $K=(0,-8)$

Since $\angle HOK = 90°$, HK is a diameter of C.

Diameter $= \sqrt{6^2+8^2} = 10$

$\therefore$ Circumference of $C = 10\pi = 31.4 > 30$

$\therefore$ YES

16D.13 HKDSE MA 2017 – I – 13

(a) Radius $= \sqrt{(-6-2)^2+(5+1)^2} = 10$

$\therefore$ Eqn of C: $(x-2)^2+(y+1)^2 = 10^2$

$\Rightarrow x^2+y^2-4x+2y-95=0$

(b) *Method 1* – From the standard form

$FG = \sqrt{(-3-2)^2+(11+1)^2} = 13 >$ Radius

$\therefore$ Outside

Method 2 – From the general form

Put F: LHS $= (-3)^2+(11)^2-4(-3)+2(11)-95$

$= 69 > 0$

$\therefore$ Outside

(c) (i) F, G and H are collinear.

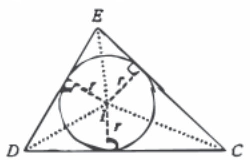
(ii) Req. eqn: $\dfrac{y+1}{x-2} = \dfrac{11+1}{3-2} \Rightarrow 12x+5y-19=0$

16D.14 HKDSE MA 2019 – I – 17

(a) Let I be the in-centre of $\triangle CDE$. Then the perpendiculars from I to CD, DE and EC are all r.

$a = \dfrac{r \cdot CD}{2} + \dfrac{r \cdot DE}{2} + \dfrac{r \cdot EC}{2}$

$= \dfrac{r(CD+DE+EC)}{2} = \dfrac{r(p)}{2} \Rightarrow pr = 2a$



(b) (i) Γ is the angle bisector of $\angle OHK$.

(ii) $OK = 14$

$OH = \sqrt{9^2+12^2} = 15$

$HK = \sqrt{(9-14)^2+(12-0)^2} = 13$

Perimeter of $\triangle OHK = 42$

Area of $\triangle OHK = \dfrac{14 \times 12}{2} = 84$

From (a), radius of inscribed circle $= \dfrac{42 \times 84}{2} = 4$

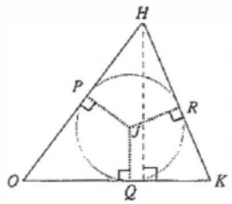
Let the in-centre be $J(h,4)$.

Method 1

By tangent properties,

$OQ = OP = h \Rightarrow \begin{cases} HR = HP = 15-h \\ KR = KQ = 14-h \end{cases}$

$\therefore HK = 13 = (15-h)+(14-h) \Rightarrow h = 8$



Method 2

Let the inscribed circle touch OH at P.

In $\triangle OJP$, $OP^2 = OJ^2 - PJ^2$

$= (\sqrt{h^2+4^2})^2 - 4^2 = h^2$

In $\triangle HJP$, $PH^2 = HJ^2 - PJ^2$

$= (\sqrt{(h-9)^2+(4-12)^2})^2 - 4^2$

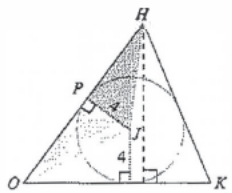
$= h^2 - 18h + 129$

$\therefore OP + PH = OH$

$h + \sqrt{h^2-18h+129} = 15$

$h^2 - 18h + 129 = 225 - 30h + h^2$

$h = 8$



Hence,

$J = (8,4)$

Eqn of HJ (i.e. Γ) is $\dfrac{y-4}{x-8} = \dfrac{12-4}{9-8}$

$\Rightarrow y = 8x - 60$

16E Polar coordinates

16E.1 HKCEE MA 2009 – I – 8

(a) $\angle POQ = 213° - 123° = 90°$

$\therefore \triangle OPQ$ is right-angled.

(b) $k^2+24^2=25^2 \Rightarrow k=7$

$\therefore$ Perimeter $= 7+24+25 = 56$

16E.2 HKDSE MA PP – I – 6

(a) $\angle AOC = 337° - 157° = 180°$

$\therefore A$, O and C are collinear.

(b) $\angle AOB = 247° - 157° = 90°$

$\therefore OB$ is the height of $\triangle ABC$ with AC as base.

$\therefore$ Area $= \dfrac{(13+15) \times 14}{2} = 196$

16E.3 HKDSE MA 2013 – I – 6

(a) L bisects $\angle AOB$.

(b) Suppose L intersects AB at P.

$\angle AOP = \dfrac{130° - 10°}{2} = 60°$, $OP = OA\cos 60° = 13$

$\therefore$ The intersection $= P = (13, 10°+60°) = (13, 70°)$

16E.4 HKDSE MA 2016 – I – 7

(a) $\angle AOB = 135° - 75° = 60°$

(b) $OA = OB = 12$ and $\angle AOB = 60°$

$\Rightarrow \triangle AOB$ is equilateral.

$\therefore$ Perimeter $= 12 \times 3 = 36$

(c) 3

17 Counting Principles and Probability

17A Counting principles

17A.1 HKALE MS 1995 – 3

A teacher wants to divide a class of 18 students into 3 groups, each of 6 students, to do 3 different statistical projects.

(a) In how many ways can the students be grouped?

(b) If there are 3 girls in the class, find the probability that there is one girl in each group.

17A.2 HKALE MS 1999 – 6

At a school sports day, the timekeeping group for running events consists of 1 chief judge, 1 referee and 10 timekeepers. The chief judge and the referee are chosen from 5 teachers while the 10 timekeepers are selected from 16 students.

(a) How many different timekeeping groups can be formed?

(b) If it is possible to have a timekeeping group with all the timekeepers being boys, what are the possible numbers of boys among the 16 students?

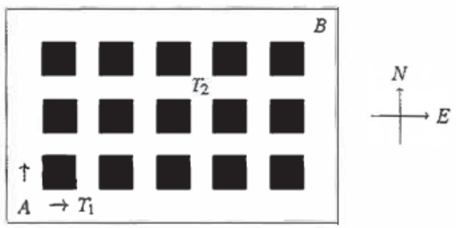
(c) *[Out of syllabus]*

17A.3 HKALE MS 2011 – 5 (Continued from **17B.31**.)

The figure shows a board with routes blocked by shaded squares for an electronic toy car that goes from A to B. At each junction, the toy car will go either East or North as shown by the arrows at A. The toy car will choose randomly a route from A to B. There may be traps being set at some junctions. If the car reaches a trapped junction, it will stop and cannot reach B.

(a) If a trap is set at T_1, how many different routes are there for the toy car to go from A to B?

(b) If a trap is set at T_2, how many different routes are there for the toy car to go from A to B?



17A.4 HKDSE MA 2018 – I – 15

An eight-digit phone number is formed by a permutation of 2, 3, 4, 5, 6, 7, 8 and 9.

(a) How many different eight-digit phone numbers can be formed?

(b) If the first digit and the last digit of an eight-digit phone number are odd numbers, how many different eight digit phone numbers can be formed?

17A.5 HKDSE MA 2019 – I – 15

There are 21 boys and 11 girls in a class. If 5 students are selected from the class to form a committee consisting of at least 1 boy, how many different committees can be formed?

17B Probability (short questions)

17B.1 HKCEE MA 1981(1/3) – I – 3

There are 40 students in a class, including students A and B. If two students are to be chosen at random as class representatives, find the probability that both A and B are chosen.

17B.2 HKCEE MA 1982(1/3) – I – 6

If two dice are thrown once, find the probability that the sum of the numbers on the dice is

(a) equal to 4,

(b) less than 4,

(c) greater than 4.

17B.3 HKCEE MA 1996 – I – 7

The figure shows a circular dartboard. Its surface consists of two concentric circles of radii 12 cm and 2 cm respectively.

(a) Find the area of the shaded region on the dartboard.

(b) Two darts are thrown and hit the dartboard. Find the probability that

(i) both darts hit the shaded region;

(ii) only one dart hits the shaded region.

17B.4 HKCEE MA 1998 – I – 11

There are 8 white socks, 4 yellow socks and 2 red socks in a drawer. A boy randomly takes out 2 socks from the drawer.

(a) Find the probability that the socks taken out are both white.

(b) Find the probability that the socks taken out are of the same colour.

17B.5 HKCEE MA 1999 – I – 12

Mr. Sun is waiting for a bus at a bus stop. It is known that 75% of the buses are air-conditioned, of which 20% have Octopus machines installed. No Octopus machines have been installed on buses without air-conditioning.

(a) Find the probability that the next bus has an Octopus machine installed.

(b) The bus fare is \$3.00. Mr. Sun does not have an Octopus card but has two 1-dollar coins and three 2-dollar coins in his pocket. If he randomly takes out two coins, what is the probability that the total value of these coins is exactly \$3.00?

17B.6 HKCEE MA 2000 – I – 12

A box contains nine hundred cards, each marked with a different 3-digit number from 100 to 999. A card is drawn randomly from the box.

(a) Find the probability that two of the digits of the number drawn are zero.

(b) Find the probability that none of the digits of the number drawn is zero.

(c) Find the probability that exactly one of the digits of the number drawn is zero.

17B.7 HKCEE MA 2004 – I – 8

A box contains nine cards numbered 1, 2, 3, 4, 5, 6, 7, 8 and 9 respectively.

(a) If one card is randomly drawn from the box, find the probability that the number drawn is odd.

(b) If two cards are randomly drawn from the box one by one with replacement, find the probability that the product of the numbers drawn is even.

17B.8 HKCEE MA 2006 – I – 8 (Continued from **18B.11**.)

There are ten cards numbered 2, 3, 5, 8, 11, 11, 12, 15, 19 and k respectively, where k is a positive integer. It is given that the mean of the ten numbers is 11.

(a) Find the value of k.

(b) A card is randomly drawn from the ten cards. Find the probability that the number drawn is a multiple of 3.

17B.9 HKCEE MA 2008 – I – 5

A box contains three cards numbered 2, 3 and 4 respectively while a bag contains two balls numbered 6 and 7 respectively. If one card and one ball are randomly drawn from the box and the bag respectively, find the probability that the sum of the numbers drawn is 10.

17B.10 HKCEE MA 2009 – I – 5

The table below shows the distribution of the ages of all employees in a department of a company.

Age (x) / Employee	$x < 30$	$30 \le x < 40$	$x \ge 40$
Administrative officer	7	21	30
Clerk	53	57	32

If an employee is randomly selected from the department, find the probability that the selected employee is an administrative officer under the age of 40.

17B.11 HKALE MS 1994 – 1

(a) Write down the sample space of the sex patterns of the children of a 2-child family in the order of their ages. (You may use B to denote a boy and G to denote a girl.)

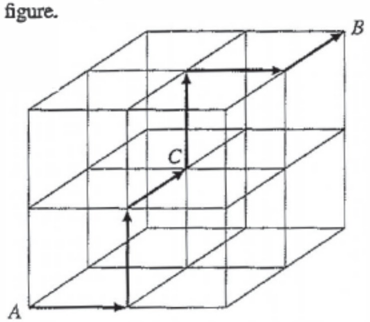
(b) Assume that having a boy or having a girl is equally likely. It is known that a family has two children and they are not both girls.
   (i) Write down the sample space of the sex patterns of the children in the order of their ages.
   (ii) What is the probability that the family has two sons?

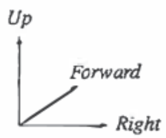
17B.12 HKALE MS 1994 – 3

Jack climbs along a cubical framework from a corner A to meet Jill at the opposite corner B. The framework, shown in the figure, is formed by joining bars of equal length. Jack chooses randomly a path of the shortest length to meet Jill. An example of such a path, which can be denoted by

$$Right - Up - Forward - Up - Right - Forward$$

is also shown in the figure.





(a) Find the number of shortest paths from A to B.

(b) If there is a trap at the centre C of the framework which catches anyone passing through it,
   (i) find the number of shortest paths from A to C,
   (ii) hence find the probability that Jack will be caught by the trap on his way to B.

17B.13 HKALE MS 1994 – 7

In asking some sensitive questions such as "Are you homosexual?", a *randomised response technique* can be applied: The interviewee will be asked to draw a card at random from a box with 1 red card and 2 black cards and then consider the statement 'I am homosexual' if the card is red and the statement 'I am not homosexual' otherwise. He will give the response either 'True' or 'False'. The colour of the card drawn is only known to the interviewee so that nobody knows which statement he has responded to. Suppose in a survey, 790 out of 1200 interviewees give the response 'True'.

(a) Estimate the percentage of persons who are homosexual.

(b) For an interviewee who answered 'True', what is the probability that he is really homosexual?

17B.14 HKALE MS 1995 – 5

An insurance company classifies the aeroplanes it insures into class L (low risk) and class H (high risk), and estimates the corresponding proportions of the aeroplanes as 70% and 30% respectively. The company has also found that 99% of class L and 88% of class H aeroplanes have no accident within a year. If an aeroplane insured by the company has no accident within a year, what is the probability that it belongs to

(a) class H?

(b) class L?

17B.15 HKALE MS 1996 – 6

A company buys equal quantities of fuses, in 100-unit lots, from two suppliers A and B. The company tests two fuses randomly drawn from each lot, and accepts the lot if both fuses are non-defective.
It is known that 4% of the fuses from supplier A and 1% of the fuses from supplier B are defective. Assume that the quality of the fuses are independent of each other.

(a) What is the probability that a lot will be accepted?

(b) What is the probability that an accepted lot came from supplier A?

17B.16 HKALE MS 1997 – 7

A brewery has a backup motor for its bottling machine. The backup motor will be automatically turned on if the original motor breaks down during operating hours. The probability that the original motor breaks down during operating hours is 0.15 and when the backup motor is turn on, it has a probability of 0.24 of breaking down. Only when both the original and backup motors break down is the machine not able to work.

(a) What is the probability that the machine is not working during operating hours?

(b) If the machine is working, what is the probability that it is operated by the original motor?

(c) The machine is working today. Find the probability that the first break down of the machine occurs on the 10th day after today.

17B.17 HKALE MS 1998 – 6

A factory produces 3 kinds of ice cream bars A, B and C in the ratio 1 : 2 : 5. It was reported that some ice cream bars produced on 1 May, 1998 were contaminated. All ice-cream bars produced on that day were withdrawn from sale and a test was carried out. The test results showed that 0.8% of kind A, 0.2% of kind B and 0% of kind C were contaminated.

(a) An ice-cream bar produced on that day is selected randomly. Find the probability that
   (i) the bar is of kind A and is NOT contaminated,
   (ii) the bar is NOT contaminated.

(b) If an ice cream bar produced on that day is contaminated, find the probability that it is of kind A.

17B.18 HKALE MS 1999 5

60% of passengers who travel by train use Octopus. A certain train has 12 compartments and there are 10 passengers in each compartment.

(a) What is the probability that exactly 5 of the passengers in a compartment use Octopus?

(b) What is the expected number of passengers using Octopus in a compartment?

(c) What is the probability that the third compartment is the first one to have exactly 5 passengers using Octopus?

17B.19 HKALE MS 2000 6

Mr. Chan has 6 cups of ice-cream in his refrigerator. There are 5 different flavours as listed: 1 cup of chocolate, 1 cup of mango, 1 cup of peach, 1 cup of strawberry and 2 cups of vanilla. Mr. Chan randomly chooses 3 cups of the ice-cream. Find the probability that

(a) there is no vanilla flavour ice-cream,

(b) there is exactly 1 cup of vanilla flavour ice-cream.

17B.20 HKALE MS 2000 – 8

A department store uses a machine to offer prizes for customers by playing games A or B. The probability of a customer winning a prize in game A is $\frac{5}{9}$ and that in game B is $\frac{5}{6}$. Suppose each time the machine randomly generates either game A or game B with probabilities 0.3 and 0.7 respectively.

(a) Find the probability of a customer winning a prize in 1 trial.

(b) The department store wants to adjust the probabilities of generating game A and game B so that the probability of a customer winning a prize in 1 trial is $\frac{2}{3}$. Find the probabilities of generating game A and game B respectively.

17B.21 HKALE MS 2001 – 6

3 students are randomly selected from 10 students of different weights. Find the probability that

(a) the heaviest student is in the selection,

(b) the heaviest one out of the 3 selected students is the 4th heaviest among the 10 students,

(c) the 2 heaviest students are not both selected.

17B.22 HKALE MS 2001 – 7

In the election of the Legislative Council, 48% of the voters support Party A, 39% Party B and 13% Party C. Suppose on the polling day, 65%, 58% and 50% of the supporting voters of Parties A, B and C respectively cast their votes.

(a) A voter votes on the polling day. Find the probability that the voter supports Party B.

(b) Find the probability that exactly 2 out of 5 voting voters support Party B.

17B.23 HKALE MS 2002 – 5

Twelve boys and ten girls in a class are divided into 3 groups as shown in the table below.

	Group A	Group B	Group C
Number of boys	6	4	2
Number of girls	2	3	5

To choose a student as the class representative, a group is selected at random, then a students is chosen at random from the selected group.

(a) Find the probability that a boy is chosen as the class representative.

(b) Suppose that a boy is chosen as the class representative. Find the probability that the boy is from Group A.

17B.24 HKALE MS 2002 – 8

A flower shop has 13 roses of which 2 are red, 5 are white and 6 are yellow. Mary selects 3 roses randomly and the colours are recorded.

(a) Denote the red rose selected by R, the white rose by W and the yellow rose by Y.
List the sample space (i.e. the set of all possible combinations of the colours of roses selected, for example, $1R2W$ denotes that 1 red rose and 2 white roses are selected).

(b) Find the probability that Mary selects exactly one red rose.

(c) Given that Mary has selected exactly one red rose, find the probability that only one of the other two roses is white.

17B.25 HKALE MS 2003 – 12

A teacher randomly selected 7 students from a class of 13 boys and 17 girls to form a group to take part in a flag selling activity.

(a) Find the probability that the group consists of at least 1 boy and 1 girl.

(b) Given that the group consists of at least 1 boy and 1 girl, find the probability that there are more than 3 girls in the group.

(c) *[Out of syllabus]*

17B.26 HKALE MS 2004 – 6

David has forgotten his uncle's mobile phone number. He can only remember that the phone number is $98677XYZ$, where X, Y and Z are the forgotten digits. Find the probability that

(a) at least 2 of the forgotten digits are different;

(b) the forgotten digits are permutations of 2, 3 and 8;

(c) exactly 2 of the forgotten digits have already appeared among the first five digits of the phone number.

17B.27 HKALE MS 2004 – 10

A certain test gives a positive result in 94% of the people who have disease S. The test gives a positive result in 14% of the people who do not have disease S. In a city, 7.5% of the citizens have disease S.

(a) Find the probability that the test gives a positive result for a randomly selected citizen.

(b) Given that the test gives a positive result for a randomly selected citizen, find the probability that the citizen does not have disease S.

(c) *[Out of syllabus]*

17B.28 HKALE MS 2007 – 6

David has 10 shirts and 3 bags: 1 blue shirt, 4 yellow shirts, 5 white shirts, 1 yellow bag and 2 white bags. He randomly chooses 3 shirts from the 10 shirts and randomly puts the chosen shirts into 3 bags so that each bag contains 1 shirt.

(a) Find the probability that the yellow bag contains the blue shirt and each of the two white bags contains 1 yellow shirt.

(b) Find the probability that each of these three bags contains 1 shirt of a colour different from the bag.

17B.29 HKALE MS 2009 – 5

It is known that 36% of the customers of a certain supermarket will bring their own shopping bags. There are 3 cashiers and each cashier has 5 customers in queue.

(a) Find the probability that among all the customers in queue, at least 4 of them have brought their own shopping bags.

(b) If exactly 4 customers in queue have brought their own shopping bags, what is the probability that each cashier will have at least 1 customer who has brought his/her own shopping bag?

17B.30 HKALE MS 2011 4

Peter and Susan play a shooting game. Each of them will shoot a target twice. Each shot will score 1 point if it hits the target. The one who has a higher score is the winner. It is known that the probabilities of hitting the target in one shot for Peter and Susan are 0.55 and 0.75 respectively.

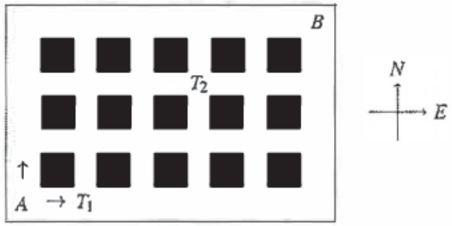
(a) Find the probability that Susan will be the winner.

(b) Given that Peter scores at least 1 point, what is the probability that Susan is the winner?

17B.31 HKALE MS 2011 5

(Continued from **17A.3**.)

The figure shows a board with routes blocked by shaded squares for an electronic toy car that goes from A to B. At each junction, the toy car will go either East or North as shown by the arrows at A. The toy car will choose randomly a route from A to B. There may be traps being set at some junctions. If the car reaches a trapped junction, it will stop and cannot reach B.



(a) If a trap is set at T_1, how many different routes are there for the toy car to go from A to B?

(b) If a trap is set at T_2, how many different routes are there for the toy car to go from A to B?

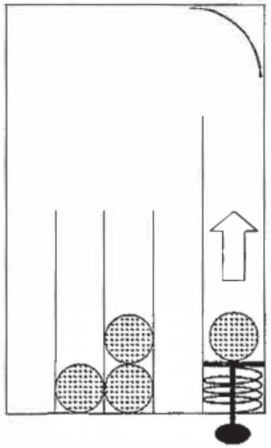
(c) If two traps are set at T_1 and T_2, find the probability that the toy car can reach B from A.

17B.32 HKALE MS 2013 4

In a game, a player will ping 4 balls one by one and each ball will randomly fall into 4 different slots as shown in the figure. A prize will be given if all the 4 balls are aligned in a horizontal or a vertical row.

(a) What is the probability that a player wins the prize?

(b) What is the probability that a player wins the prize given that first two balls are in two different slots?



17B.33 HKDSE MA SP – I – 16

A committee consists of 5 teachers from school A and 4 teachers from school B. Four teachers are randomly selected from the committee.

(a) Find the probability that only 2 of the selected teachers are from school A.

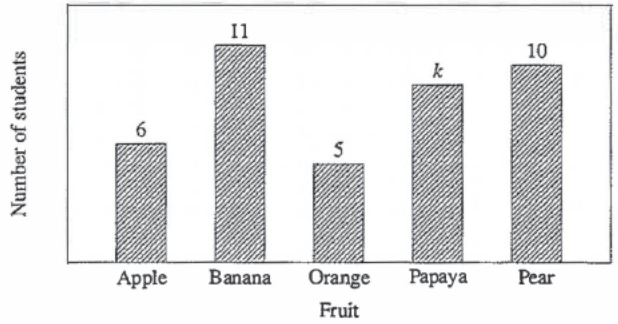
(b) Find the probability that the numbers of selected teachers from school A and school B are different.

17B.34 HKDSE MA PP – I – 13

(To continue as **18A.9**.)

The bar chart below shows the distribution of the most favourite fruits of the students in a group. It is given that each student has only one most favourite fruit.

Distribution of the most favourite fruits of the students in the group



If a student is randomly selected from the group, the probability that the most favourite fruit is apple is $\frac{3}{20}$.

(a) Find k.

(b) Suppose that the above distribution is represented by a pie chart.

17B.35 HKDSE MA PP – I – 16

There are 18 boys and 12 girls in a class. From the class, 4 students are randomly selected to form the class committee.

(a) Find the probability that the class committee consists of boys only.

(b) Find the probability that the class committee consists of at least 1 boy and 1 girl.

17B.36 HKDSE MA 2012 I – 16

There are 8 departments in a company. To form a task group of 16 members, 2 representatives are nominated by each department. From the task group, 4 members are randomly selected.

(a) Find the probability that the 4 selected members are nominated by 4 different departments.

(b) Find the probability that the 4 selected members are nominated by at most 3 different departments.

17B.37 HKDSE MA 2013 – I – 16

A box contains 5 white cups and 11 blue cups. If 6 cups are randomly drawn from the box at the same time,

(a) find the probability that at least 4 white cups are drawn;

(b) find the probability that at least 3 blue cups are drawn.

17B.38 HKDSE MA 2015 – I – 3

Bag A contains four cards numbered 1, 3, 5 and 7 respectively while bag B contains five cards numbered 2, 4, 6, 8 and 10 respectively. If one card is randomly drawn from each bag, find the probability that the sum of the two numbers drawn is less than 9.

17B.39 HKDSE MA 2015 – I – 16

A box contains 5 red bowls, 6 yellow bowls and 3 white bowls. If 4 bowls are randomly drawn from the box at the same time,

(a) find the probability that exactly 2 red bowls are drawn;

(b) find the probability that at least 2 red bowls are drawn.

17B.40 HKDSE MA 2016 – I – 9 (Continued from **18A.10**.)

The frequency distribution table and the cumulative frequency distribution table below show the distribution of the heights of the plants in a garden.

Height (m)	Frequency
0.1 – 0.3	a
0.4 – 0.6	4
0.7 – 0.9	b
1.0 – 1.2	c
1.3 – 1.5	15
1.6 – 1.8	3

Height less than (m)	Cumulative frequency
0.35	2
0.65	x
0.95	13
1.25	y
1.55	37
1.85	z

(a) Find x, y and z.

(b) If a plant is randomly selected from the garden, find the probability that the height of the selected plant is less than 1.25 m but not less than 0.65 m.

17B.41 HKDSE MA 2016 – I – 15

If 4 boys and 5 girls randomly form a queue, find the probability that no boys are next to each other in the queue.

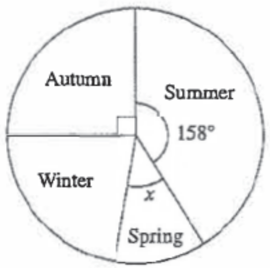
17B.42 HKDSE MA 2017 – I – 7

The pie chart shows the distribution of the seasons of birth of the students in a school.

If a student is randomly selected from the school, then the probability that the selected student was born in spring is $\frac{1}{9}$.

(a) Find x.

(b) In the school, there are 180 students born in winter. Find the number of students in the school.



Distribution of the seasons of birth of the students in a school

17B.43 HKDSE MA 2017 – I – 11 (Continued from **18C.48**.)

The stem-and leaf diagram shows the distribution of the hourly wages (in dollars) of the workers in a group.

It is given that the mean and the range of the distribution are \$70 and \$22 respectively.

Stem (tens)	Leaf (units)
6	1 1 1 3 4 6 8 9 9
7	a 7 7 8
8	1 b

(a) Find the median and the standard deviation of the above distribution.

(b) If a worker is randomly selected from the group, find the probability that the hourly wage of the selected worker exceeds \$70.

17B.44 HKDSE MA 2017 – I – 17

In a bag, there are 4 green pens, 7 blue pens and 8 black pens. If 5 pens are randomly drawn from the bag at the same time,

(a) find the probability that exactly 4 green pens are drawn;

(b) find the probability that exactly 3 green pens are drawn;

(c) find the probability that not more than 2 green pens are drawn.

17B.45 HKDSE MA 2018 – I – 4

A box contains n white balls, 5 black balls and 8 red balls. If a ball is randomly drawn from the box, then the probability of drawing a red ball is $\frac{2}{5}$. Find the value of n.

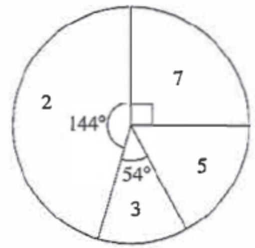
17B.46 HKDSE MA 2019 – I – 8 (Continued from **18B.19**.)

The pie chart below shows the distribution of the numbers of rings owned by the girls in a group.

(a) Write down the mode of the distribution.

(b) Find the mean of the distribution.

(c) If a girl is randomly selected from the group, find the probability that the selected girl owns more than 3 rings.



Distribution of the numbers of rings owned by the girls in the group

17B.47 HKDSE MA 2020 – I – 15

In a box, there are 3 blue plates, 7 green plates and 9 purple plates. If 4 plates are randomly selected from the box at the same time, find

(a) the probability that 4 plates of the same colour are selected; (3 marks)

(b) the probability that at least 2 plates of different colours are selected. (2 marks)

17C Probability (structural questions)

17C.1 HKCEE MA 1980(1/3) – I – 14

The examination for a professional qualification consists of a theory paper and a practical paper. To obtain the qualification, a candidate has to pass both papers. If a candidate fails in either paper, he needs only sit that paper again.

The probabilities of passing the theory paper for two candidates A and B are both $\frac{9}{10}$ and their probabilities of passing the practical paper are both $\frac{2}{3}$. Find the probabilities of the following events:

(a) Candidate A obtaining the qualification by sitting each paper only once.

(b) Candidate A failing in one of the two papers but obtaining the qualification with one re examination.

(c) At least one of the candidates A and B obtaining the qualification without any re examination.

17C.2 HKCEE MA 1983(A/B) I 11

In a short test, there are 3 questions. For each question, 1 mark will be awarded for a correct answer and no marks for a wrong answer. The probability that John correctly answers a question in the test is 0.6. Find the probability that

(a) John gets 3 marks in the test,

(b) John gets no marks in the test,

(c) John gets 1 mark in the test.

(d) John gets 2 marks in the test.

17C.3 HKCEE MA 1984(A/B) I – 11

(a) There are two bags. Each bag contains 1 red, 1 black and 1 white ball. One ball is drawn randomly from each bag. Find the probability that
   (i) the two balls drawn are both red;
   (ii) the two balls drawn are of the same colour;
   (iii) the two balls drawn are of different colours.

(b) A box contains 2 red, 2 black and 3 white balls. One ball is drawn randomly from the box. After putting the ball back into the box, one ball is again drawn randomly. Find the probability that
   (i) the two balls drawn are both red;
   (ii) the two balls drawn are of the same colour;
   (iii) the two balls drawn are of different colours.

17C.4 HKCEE MA 1985(A/B) – I – 10

(a) If two dice are thrown,
   (i) find the probability that the sum of the numbers on the two dice is greater than 9;
   (ii) find the probability that the sum of the numbers on the two dice is greater than 9 or the numbers on the two dice are equal.

(b) In a game, two dice are thrown. In each throw, 2 points are gained if the sum of the numbers on the two dice is greater than 9 or the numbers on the two dice are equal; otherwise 1 point is lost. Using the result in (a)(ii), find the probability of
   (i) losing a total of 2 points in two throws,
   (ii) gaining a total of 1 point in two throws.

17C.5 HKCEE MA 1986(A/B) I 13

A box contains wooden blocks of 5 different shapes A, B, C, D and E. For each shape, there are 5 different colours red, orange, yellow, green and blue. For each colour of each shape, there is one block of each of the sizes L, M and S. (Hint: There are altogether 75 blocks in the box.)

(a) When a block is picked out randomly from the box, what is the probability that it is of
   (i) red colour?
   (ii) blue colour and shape C?
   (iii) size S, shape A or E but not yellow?

(b) Two blocks are drawn at random from the box, one after the other. The first block drawn is put back into the box before the second is drawn. Find the probability that
   (i) the first block drawn is of size L and the second block is of size S,
   (ii) one of the blocks drawn is of size L and the other of size S,
   (iii) the two blocks drawn are of different sizes.

17C.6 HKCEE MA 1987(A/B) – I – 13

P, Q and R are three bags. P contains 1 black ball, 2 green balls and 3 white balls; Q contains 4 green balls; R contains 5 white balls. A ball is drawn at random from P and is put into Q; then a ball is drawn at random from Q and is put into R. Find the probability that

(a) the black ball still remains in P,

(b) the black ball is in Q,

(c) the black ball is in R,

(d) all the balls in R are white.

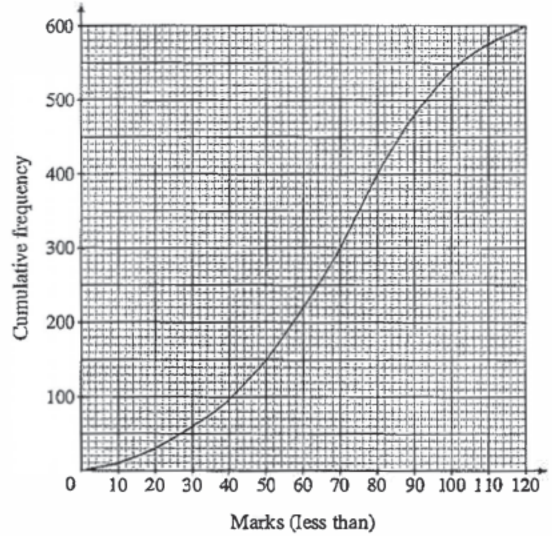
17C.7 HKCEE MA 1988 – I – 11

(Continued from 18C.4.)

The figure shows the cumulative frequency curve of the marks of 600 students in a mathematics contest.

(a) From the curve, find
   (i) the median, and
   (ii) the interquartile range of the distribution of marks.

(b) A student with marks greater than or equal to 100 will be awarded a prize.
   (i) Find the number of students who will be awarded prizes.
   (ii) If one student is chosen at random from the 600 students, find the probability that the student is a prize-winner.
   (iii) If two students are chosen at random, find the probability that
      (1) both of them are prize-winners,
      (2) at least one of them is a prize winner.



17C.8 HKCEE MA 1989 – I – 13

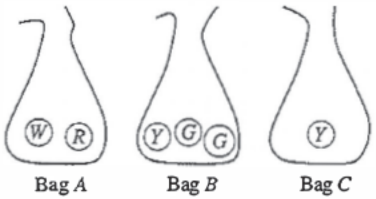
(a) Bag A contains a number of balls. Some are black and the rest are white. A ball is drawn at random from bag A. Let p be the probability that the ball drawn is black and q be the probability that the ball drawn is white. If $p = 3q$, find q.

(b) Bag C contains 10 balls of which n $(2 \le n \le 10)$ balls are black.
   (i) If two balls are drawn at random from bag C, find the probability, in terms of n, that both balls are black.
   (ii) If the probability obtained in (i) is greater that $\frac{1}{3}$, find the possible values of n.

(c) Bag M contains 1 red and 1 green ball. Bag N contains 3 red and 2 green balls. A ball is drawn at random from bag M and put into bag N; then a ball is drawn at random from bag N. Find the probability that the ball drawn from bag N is red.

17C.9 HKCEE MA 1990 – I – 13

The figure shows 3 bags A, B and C.
Bag A contains 1 white ball (W) and 1 red ball (R).
Bag B contains 1 yellow ball (Y) and 2 green balls (G).
Bag C contains only 1 yellow ball (Y).



(a) Peter choose one bag at random and then randomly draws one ball from the bag. Find the probability that
   (i) the ball drawn is green;
   (ii) the ball drawn is yellow.

(b) After Peter has drawn a ball in the way described in (a), he puts it back into the original bag. Next, Alice chooses one bag at random and then randomly draws one ball from the bag. Find the probability that
   (i) the balls drawn by Peter and Alice are both green;
   (ii) the balls drawn by Peter and Alice are both yellow and from the same bag.

17C.10 HKCEE MA 1991 I 10

The practical test for a driving licence consists of two independent parts, A and B. To pass the practical test, a candidate must pass in both parts. If a candidate fails in any one of these parts, the candidate may take that part again. Statistics shows that the passing percentages for Part A and Part B are 70% and 60% respectively.

(a) A candidate takes the practical test. Find the probabilities that the candidate
   (i) fails Part A on the first attempt and passes it on the second attempt,
   (ii) passes Part A in no more than two attempts,
   (iii) passes the practical test in no more than two attempts in each part.

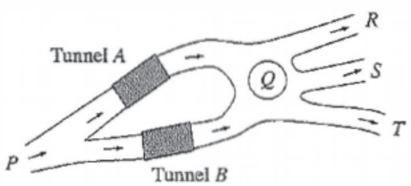
(b) In a sample of 10 000 candidates taking the practical test, how many of them would you expect to pass the practical test in no more than two attempts in each part?

17C.11 HKCEE MA 1992 – I – 10

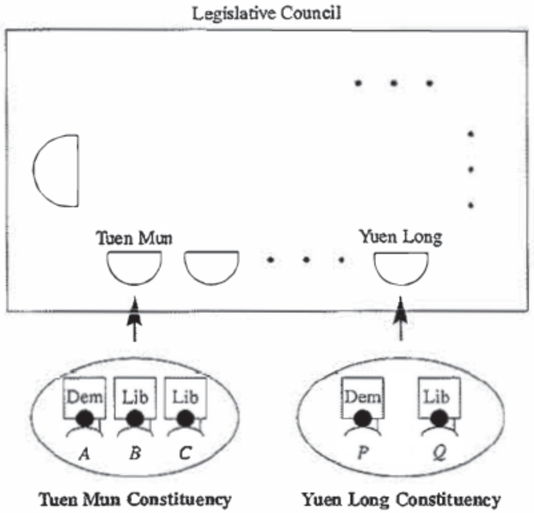
The figure shows a one way road network system from Town P to Towns R, S and T. Any car leaving Town P will pass though either Tunnel A or Tunnel B and arrive at Towns R, S or T via the roundabout Q. A survey shows that $\frac{2}{5}$ of the cars leaving P will pass through Tunnel A. The survey also shows that $\frac{1}{7}$ of all the cars passing through the roundabout Q will arrive at R, $\frac{2}{7}$ at S, and $\frac{4}{7}$ at T.

(a) Find the probabilities that a car leaving P will
   (i) pass through Tunnel B,
   (ii) not arrive at T,
   (iii) arrive at R through Tunnel B,
   (iv) pass through Tunnel A but not arrive at R.

(b) Two cars leave P. Find the probabilities that
   (i) one of them arrives at R and the other one at S,
   (ii) both of them arrive at S, one through Tunnel A and the other one through Tunnel B.



17C.12 HKCEE MA 1993 – I – 13



In a Legislative Council election, each registered voter in a constituency (i.e. district) could select only one candidate in that constituency and cast one vote for that candidate. The candidate who got the greatest number of valid votes won the election in that constituency.

In the Tuen Mun constituency, there were 3 candidates, A, B and C. A belonged to a political party called 'The Democrats'; B and C belonged to a political party called 'The Liberals'.

In the Yuen Long constituency, there were 2 candidates, P and Q. P belonged to 'The Democrats' and Q belonged to 'The Liberals'.

(a) A survey conducted before the election showed that the probabilities of winning the election for A, B and C were respectively 0.65, 0.25 and 0.1 while the probabilities of winning the election for P and Q were respectively 0.45 and 0.55. Calculate from the above data the following probabilities:
   (i) The elections in the Tuen Mun and Yuen Long constituencies would both be won by 'The Democrats'.
   (ii) The elections in the Tuen Mun and Yuen Long constituencies would both be won by the same party.

(b) After the election, it was found that in the Tuen Mun constituency there were 40 000 valid votes of which A got 70%, B got 20% and C got 10%; in the Yuen Long constituency, there were 20 000 valid votes of which P got 40% and Q got 60%. Suppose two votes were chosen at random (one after the other with replacement) from the 60 000 valid votes in the two constituencies. What would be the probability that
   (i) both votes came from the Tuen Mun constituency and were for 'The Democrats',
   (ii) both votes were for 'The Democrats',
   (iii) the votes were for different parties?

17C.13 HKCEE MA 1994 – I – 9

Siu Ming lives in Tuen Mun. He travels to school either by LRT (Light Railway Transit) or on foot. The probability of being late for school is $\frac{1}{7}$ if he travels by LRT and $\frac{1}{10}$ if he travels on foot.

(a) In a certain week, Siu Ming travels to school by LRT on Monday, Tuesday and Wednesday. Find the probability that
  (i) he will be late on all these three days;
  (ii) he will not be late on all these three days.

(b) In the same week, Siu Ming travels to school on foot on Thursday, Friday and Saturday. Find the probability that
  (i) he will be late on Thursday and Friday only in these three days;
  (ii) he will be late on any two of these three days.

(c) On Sunday, Siu Ming goes to school to take part in a basketball match. If he is equally likely to travel by LRT or on foot, find the probability that he will be late on that day.

17C.14 HKCEE MA 1995 – I – 11

If Wai Ming studies in the evening for a test the next day, the probability of him passing the test is $\frac{4}{5}$. If he does not study in the evening for the test, he will certainly fail.

(a) **(You may use Figure (1) to help you answer this part.)**
  (i) If Wai Ming studies in the evening for a test the next day, find the probability p that he will fail the test.
  (ii) If Wai Ming does not study in the evening for a test the next day, find the probability q that he will pass the test and the probability r that he will fail the test.

(b) **(You may use Figure (1) and Figure (2) to help you answer this part.)**
There are four teams competing for the World Women's Volleyball Championship (WWVC) with two games in the semi finals: China against U.S.A. and Japan against Cuba.
The winner of each game will be competing in the final for the Championship. The four teams have an equal chance of beating their opponents.
  (i) Find the probability that China will win the Championship.
  (ii) The final of the WWVC will be shown on television on a Sunday evening and Wai Ming has a test the next day. Wai Ming will definitely watch the TV programme if China gets to the final and the probability of him studying afterwards for the test is $\frac{1}{3}$. If China fails to get to the final, he will not watch that programme at all and will study for the test.
    (1) Find the probability that Wai Ming will study for the test.
    (2) Find the probability that Wai Ming will pass the test.

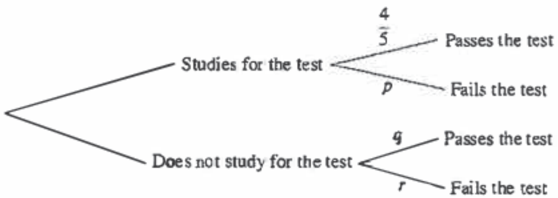


Figure (1)

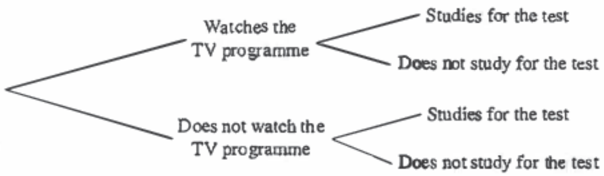


Figure (2)

17C.15 HKCEE MA 1997 – I – 14

In a small pond, there were exactly 40 *small* fish and 10 *large* fish. The ranges of their weights W g are shown in the table.
In the morning on a certain day, a man went fishing in the pond. He caught two fish and their total weight was T g. Suppose each fish was equally likely to be caught.

	Weight (W g)
Small fish	$0 < W \leq 100$
Large fish	$500 \leq W \leq 600$

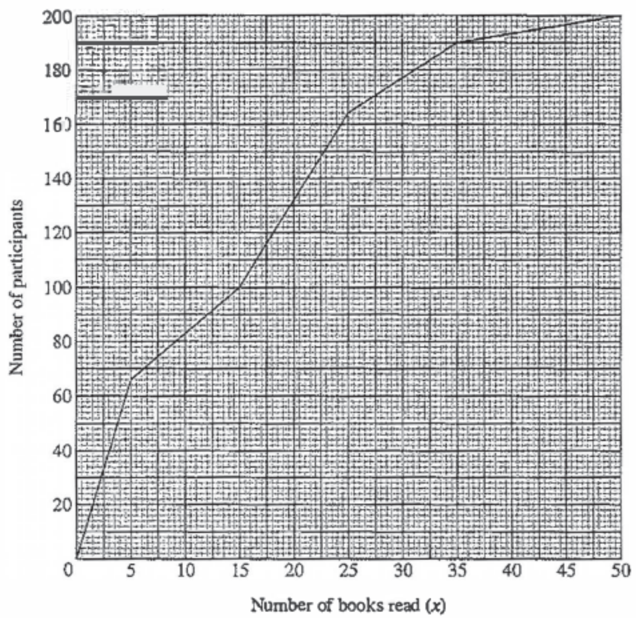
(a) Find the probability that
  (i) $0 < T \leq 200$,
  (ii) $500 \leq T \leq 700$,
  (iii) $1000 \leq T \leq 1200$,
  (iv) $T > 1200$.

(b) Suppose the two fish he caught in the morning were returned alive to the pond. He went fishing again in the pond in the afternoon and also caught two fish.
  (i) If the total weight of the fish caught in the morning was 650 g, find the probability that the difference between the total weights of the fish caught in the morning and in the afternoon is more than 200 g.
  (ii) Find the probability that the difference between the total weights of the fish caught in the morning and in the afternoon is more than 200 g.

17C.16 HKCEE MA 2002 – I – 12

(Continued from **18C.11**.)

The cumulative frequency polygon of the distribution of the numbers of books read by the participants



Two hundred students participated in a summer reading programme. The figure shows the cumulative frequency polygon of the distribution of the numbers of books read by the participants.

(a) The table below shows the frequency distribution of the numbers of books read by the participants. Using the graph in the figure, complete the table.

Number of books read (x)	Number of participants	Award
$0 < x \le 5$	66	Certificate
$5 < x \le 15$		Book coupon
$15 < x \le 25$	64	Bronze medal
$25 < x \le 35$		Silver medal
$35 < x \le 50$	10	Gold medal

(b) Using the graph in the figure, find the inter-quartile range of the distribution.

(c) Two participants were chosen randomly from those awarded with medals. Find the probability that

(i) they both won gold medals;

(ii) they won different medals.

17C.17 HKCEE MA 2003 – I – 16

John will participate in a contest to be held at a university. If John wins the contest, he will go to Canteen X for lunch. Otherwise, he will go to Canteen Y. Table (1) shows the types of set lunches and the prices served in the two canteens. He will choose one type of set lunch randomly.

Canteen	Set lunch	Price ($)
X	A	40
	B	50
Y	C	15
	D	20

Table (1)

Transportation	Cost of a single trip ($)
Bus	4.5
Train	7.5

Table (2)

(a) If the probability of John winning the contest is $\frac{1}{10}$, find the probability that he will spend $15 for lunch.

(b) If John takes a bus leaving at 8:00 a.m. to the university, his probability of winning the contest will be $\frac{1}{10}$. If he misses the bus, he will take a train leaving at 8:20 a.m. Owing to his nervousness, his probability of winning will be reduced to $\frac{2}{25}$.

(i) Suppose John misses the bus, find the probability that he will spend $15 for lunch.

(ii) Table (2) shows the cost of a single trip by bus or train.
It is known that the probability of John taking the bus is twice that of taking the train.

(1) Find the probability that John will spend $15 for lunch after the contest.

(2) If John goes home by train after lunch, find the probability that he will spend more than a total of $30 for the lunch and the transportation of the two trips.

17C.18 HKCEE MA 2005 – I – 11

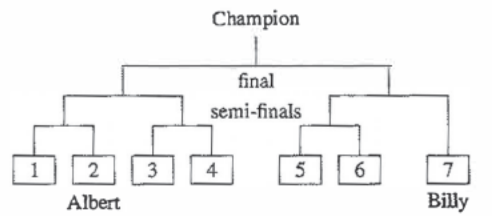
Seven players take part in a men's singles tennis knock out tournament. They are randomly assigned to the positions 1, 2, 3, 4, 5, 6 and 7. It is known that Albert and Billy are in positions 2 and 7 respectively. The winner of each game proceeds to the next round as shown in the figure and the loser is knocked out. Billy goes straight to the semi finals. In each game, each player has an equal chance of beating his opponent.

(a) Write down the probability that Albert will reach the semi-finals.

(b) Find the probability that Albert will be the champion.

(c) Find the probability that Albert will fail to reach the final.

(d) Find the probability that Albert will play against Billy in the final.



17C.19 HKCEE MA 2006 – I – 14 (Continued from 18C.14.)

The stem and leaf diagram below show the distributions of the scores (in marks) of the students of classes A and B in a test, where a, b, c and d are non-negative integers less than 10. It is given that each class consists of 25 students.

Class A

Stem (tens)	Leaf (units)
0	a 9
1	2 5 7 8 8
2	3 3 5 6 7 9
3	2 3 5 6 9 9 9
4	1 2 2 4 b

Class B

Stem (tens)	Leaf (units)
0	c 3 3 4 5
1	1 1 2 2 3 3 5 6 7 8
2	1 1 5 5 5 7 8
3	5 9
4	d

(a) (i) Find the inter-quartile range of the score distribution of the students of class A and the inter quartile range of the score distribution of the students of class B.

(ii) Using the results of (a)(i), state which one of the above score distributions is less dispersed. Explain your answer.

(b) The passing score of the test is 20 marks. From the 50 students, 3 students are randomly selected.

(i) Find the probability that exactly 2 of the selected students pass the test.

(ii) Find the probability that exactly 2 of the selected students pass the test and both of them are in the same class.

(iii) Given that exactly 2 of the selected students pass the test, find the probability that both of them are in the same class.

17C.20 HKCEE MA 2007 – I – 15

The following table shows the results of a survey about the sizes of shirts dressed by 80 students on a certain school day.

Student \ Size	Small	Medium	Large	Total
Boy	8	28	12	48
Girl	20	8	4	32

(a) On that school day, a student is randomly selected from the 80 students.
  (i) Find the probability that the selected student is a boy.
  (ii) Find the probability that the selected student is a boy and he dresses a shirt of large size.
  (iii) Find the probability that the selected student is a boy or the selected student dresses a shirt of large size.
  (iv) Given that the selected student is a boy, find the probability that he dresses a shirt of large size.

(b) On the school day, two students are randomly selected from the 80 students.
  (i) Find the probability that the two selected students both dress shirts of large size.
  (ii) Is the probability of dressing shirts of the same size by the two selected students greater than that of dressing different sizes? Explain your answer.

17C.21 HKCEE MA 2008 I – 1 4 (To continue as **18C.17**.)

The stem-and-leaf diagram below shows the suggested bonuses (in dollars) of the 36 salesgirls of a boutique:

Stem (thousands)	Leaf (hundreds)
2	4 4 7
3	2 5 6 6 8
4	3 3 3 4 4 7 8 8 8
5	0 0 3 4 4 6
6	2 3 3 4 4 9 9
7	0 4 4 8
8	2 3

(a) The suggested bonus of each salesgirl of the boutique is based on her performance. The following table shows the relation between level of performance and suggested bonus:

Level of performance	Suggested bonus ($x)
Excellent	$x > 6500$
Good	$4\ 500 < x \le 6500$
Fair	$x \le 4\ 500$

  (i) From the 36 salesgirl, one of them is randomly selected. Given that the level of performance of the selected salesgirl is good, find the probability that her suggested bonus is less than $5 500.
  (ii) From the 36 salesgirls, two of them are randomly selected.
    (1) Find the probability that the level of performance of one selected salesgirl is excellent and that of the other is good.
    (2) Find the probability that the levels of performance of the two selected salesgirls are different.

17C.22 HKCEE MA 2009 – I – 14

The frequency distribution table shows the lifetime (in hours) of a batch of randomly chosen light bulbs of brand A and a batch of randomly chosen light bulbs of brand B.

Lifetime (x hours)	Frequency	
	Brand A	Brand B
$1000 \le x < 1100$	8	4
$1100 \le x < 1200$	50	12
$1200 \le x < 1300$	42	40
$1\ 300 \le x < 1\ 400$	10	36
$1400 \le x < 1500$	10	28

(a) According to the above frequency distribution, which brand of light bulbs is likely to have a longer lifetime? Explain your answer.

(b) If the lifetime of a light bulb is not less than 1300 hours, then the light bulb is classified as *good*. Otherwise, it is classified as *acceptable*.
  (i) If a light bulb is randomly chosen from the batch of light bulbs of brand A, find the probability that the chosen light bulb is *acceptable*.
  (ii) If two light bulbs are randomly chosen from the batch of light bulbs of brand A, find the probability that at least one of the two chosen light bulbs is *good*.
  (iii) The following 2 methods describe how 2 light bulbs are chosen from the 2 batches of light bulbs.
  Method 1: One batch is randomly selected from the two batches of light bulbs and two light bulbs are then randomly chosen from the selected batch.
  Method 2: One light bulb is randomly chosen from each of the two batches of light bulbs.
  Which one of the above two methods should be adopted in order to have a greater chance of choosing at least one *good* light bulb? Explain your answer.

17C.23 HKCEE MA 2010 – I 1 4

An athlete, Alice, of a school gets the following results (in seconds) in 10 practices of 1500 m race:
279, 280, 264, 267 , 283, 281, 281, 266, 284, 265

(a) Two results are randomly selected from the above results.
  (i) Find the probability that both the best two results are not selected.
  (ii) Find the probability that only one of the best two results is selected.
  (iii) Find the probability that at most one of the best two results is selected.

(b) Another athlete, Betty, of the school gets the following results (in seconds) in 10 practices of 1500 m race: 272, 269, 27 5, 27 4, 27 3, 274, 270, 275, 266, 272
  Alice and Betty will represent the school to participate in the 1500 m race in the inter school athletic meet.
  (i) Which athlete is likely to get a better result? Explain your answer.
  (ii) The best record of the 1500 m race in the past inter school athletic meets is 267 seconds. Which athlete has a greater chance of breaking the record? Explain your answer.

17C.24 HKCEE MA 2011 I 14

In a bank, the queuing times (in minutes) of 12 customers are recorded as follows:

5.1, 5.2, 5.4, 6.1, 6.7, 7.1, 7.4, 7.7, 7.8, 8.4, 9.0, 10.1

It is found that if the queuing time of a customer in the bank is less than 8 minutes, then the probability that the customer makes a complaint is $\frac{1}{6}$. Otherwise, the probability that the customer makes a complaint is $\frac{1}{3}$.

(a) If a customer is randomly selected from the 12 customers, find the probability that the selected customer does not make a complaint.

(b) Two customers are now randomly selected from the 12 customers.

(i) If the queuing time of the selected customer is less than 8 minutes and the queuing time of the other customer is not less than 8 minutes, find the probability that both of them do not make complaints.

(ii) Find the probability that the queuing times of both of the selected customers are not less than 8 minutes and both of them do not make complaints.

(iii) Is the probability of not making complaints by the two selected customers greater than the probability of making complaints by both of them? Explain your answer.

17C.25 HKALE MS 1994 – I 1

A day is regarded as humid if the relative humidity is over 80% and is regarded as dry otherwise. In city K, the probability of having a humid day is 0.7.

(a) Assume that whether a day is dry or humid is independent from day to day.

(i) Find the probability of having exactly 3 dry days in a week.

(ii) *[Out of syllabus]*

(iii) Today is dry. What is the probability of having two or more humid days before the next dry day?

(b) After some research, it is known that the relative humidity in city K depends solely on that of the previous day. Given a dry day, the probability that the following day is dry is 0.9 and given a humid day, the probability that the following day is humid is 0. 8.

(i) If it is dry on March 19, what is the probability that it will be humid on March 20 and dry on March 21?

(ii) If it is dry on March 19, what is the probability that it will be dry on March 21?

(iii) Suppose it is dry on both March 19 and March 21. What is the probability that it is humid on March 20?

17C.26 HKALE MS 1995 – I 1

Madam Wong purchases cartons of oranges from a supplier every day. Her buying policy is to randomly select five oranges from a carton and accept the carton if all five are not rotten. Under usual circumstances, 2% of the oranges are rotten.

(a) Find the probability that a carton of oranges will be rejected by Madam Wong.

(b) *[Out of syllabus]*

(c) Today, Madam Wong has a target of buying 20 acceptable cartons of oranges from the supplier. Instead of applying the stopping rule in (b), she will keep on inspecting the cartons until her target is achieved. Unfortunately, the supplier has a stock of 22 cartons only.

(i) Find the probability that she can achieve her target.

(ii) Assuming she can achieve her target, find the probability that she needs to inspect 20 cartons only.

(d) The supplier would like to import oranges of better quality so that each carton will have at least a 95% probability of being accepted by Madam Wong. If r% of these oranges are rotten, find the greatest acceptable value of r.

17C.27 HKALE MS 1998 3 (Continued from 18B.12.)

40 students participate in a 5-day summer camp. The stem-and-leaf diagram below shows the distribution of heights in cm of these students.

(a) Find the median of the distribution of heights.

(b) A student is to be selected randomly to hoist the school flag every day during the camp. Find the probability that

(i) the fourth day will be the first time that a student taller than 170 cm will be selected,

(ii) out of the 5 selected students, exactly 3 are taller than 170 cm.

Stem (tens)	Leaf (units)
13	8
14	1 5 6 9
15	0 1 3 4 4 4 5 5 6 7 8 8 9
16	1 1 2 3 3 4 5 6 7 7 8 8
17	0 2 2 3 4 5 6 7
18	1 4

17C.28 HKALE MS 1998 – 5

John and Mary invite 8 friends to their Christmas party.

(a) When playing a game, all of the 10 participants are arranged in a row. Find the number of arrangements that can be made if

(i) there is no restriction,

(ii) John and Mary are next to each other.

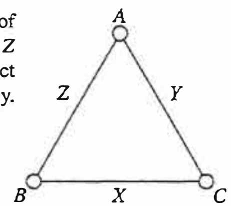
(b) By the end of the party, the participants are arranged in 2 rows of 5 in order to take a photograph. Find the number of arrangements that can be made if

(i) there is no restriction,

(ii) John and Mary are next to each other.

17C.29 HKALE MS 1999 7

Three control towers A, B and C are in telecommunication contact by means of three cables X, Y and Z as shown in the figure. A and B remain in contact only if Z is operative or if both cables X and Y are operative. Cables X, Y and Z are subject to failure in any one day with probabilities 0.015, 0.025 and 0.030 respectively. Such failures occurs independently.



(a) Find, to 4 significant figures, the probability that, on a particular day,

(i) both cables X and Z fail to operate,

(ii) all cables X, Y and Z fail to operate,

(iii) A and B will not be able to make contact.

(b) Given that cable X fails to operate on a particular day, what is the probability that A and B are not able to make contact?

(c) Given that A and B are not able to make contact on a particular day, what is the probability that cable X has failed?

17C.30 HKALE MS 2002 – 7

(Continued from **18B.13.**)

Twenty two students in a class attended an examination. The stem and leaf diagram below shows the distri bution of the examination marks of these students.

Stem (tens)	Leaf (units)
3	5 7
4	2 4 6
5	0 3 4 4 4 5
6	1 2 5 5 8
7	3 8 9
8	4 8
9	5

(a) Find the mean of the examination marks.

(b) Two students left the class after the examination and their marks are deleted from the stem and leaf diagram. The mean of the remaining marks is then increased by 1.2 and there are two modes. Find the two deleted marks.

(c) Two students are randomly selected from the remaining 20 students. Find the probability that their marks are both higher than 75.

17C.31 HKALE MS 2003 11

In a game, two boxes A and B each contains n balls which are numbered $1, 2, \ldots, n$. A player is asked to draw a ball randomly from each box. If the number drawn from box A is greater than that from box B, the player wins a prize.

(a) Find the probability that the two numbers drawn are the same.

(b) Let p be the probability that a player wins the prize.

(i) Find, in terms of p only, the probability that the number drawn from box B is greater than that from box A.

(ii) Using the result of (i), express p in terms of n.

(iii) If the above game is designed so that at least 46% of the players win the prize, find the least value of n.

(c) Two winners, John and Mary, are selected to play another game. They take turns to throw a fair six sided die. The first player who gets a number '6' wins the game. John will throw the die first.

(i) Find the probability that John will win the game on his third throw.

(ii) Find the probability that John will win the game.

(iii) Given that Mary has won the game, find the probability that Mary did not win the game before her third throw.

17C.32 HKALE MS 2004 11

A manufacturer of brand C potato chips runs a promotion plan. Each packet of brand C potato chips contains either a red coupon or a blue coupon. Four red coupons can be exchanged for a toy. Five blue coupons can be exchanged for a lottery ticket. It is known that 30% of the packets contain red coupons and the rest contain blue coupons.

(a) Find the probability that a lottery ticket can be exchanged only when the 6th packet of brand C potato chips has been opened.

(b) A person buys 10 packets of brand C potato chips.

(i) Find the probability that at least 1 toy can be exchanged.

(ii) Find the probability that exactly 1 toy and exactly 1 lottery ticket can be exchanged.

(iii) Given that at least 1 toy can be exchanged, find the probability that exactly 1 lottery ticket can also be exchanged.

(c) Two persons buy 10 packets of brand C potato chips each. Assume that they do not share coupons or exchange coupons with each other.

(i) Find the probability that they can each get at least 1 toy.

(ii) Find the probability that one of them can get at least 1 toy and the other can get 2 lottery tickets.

17C.33 HKALE MS 2005 – 6

Mrs. Wong has 12 bottles of fruit juice in her kitchen: 1 bottle of grape juice, 6 bottles of apple juice and 5 bottles of orange juice. She randomly chooses 4 bottles to serve her friends, Ann, Billy, Christine and Donald.

(a) Find the probability that exactly 2 bottles of orange juice are chosen by Mrs. Wong.

(b) Suppose that each of the four friends randomly selects a bottle of fruit juice from the 4 bottles offered by Mrs. Wong.

(i) If only 2 of the bottles of fruit juice offered by Mrs. Wong are orange juice, find the probability that both Ann and Billy select orange juice.

(ii) Find the probability that fewer than 4 of the bottles of fruit juice offered by Mrs. Wong are orange juice and both Ann and Billy select orange juice.

17C.34 HKALE MS 2010 5

(Continued from **18B.14.**)

The following stem-and-leaf diagram shows the distribution of the test scores of 21 students taking a statistics course. Let $\bar{x}$ be the mean of these 21 scores.
It is known that if the smallest value of these 21 scores is removed, the range is decreased by 27 and the mean is increased by 2.

Stem (Tens)	Leaf (Units)
2	a
3	
4	9
5	0 0 1 3 7 7
6	0 2 3 5 5 5 9
7	0 3 4 9
8	2 b

(a) Find the values of a, b and $\bar{x}$.

(b) The teacher wants to select 6 students to participate in a competition by first excluding the student with the lowest score. If the students are randomly selected, find the probability that there will be

(i) no students with score higher than 70 begin selected;

(ii) at least 2 students with scores higher than 70 being selected.

17C.35 HKALE MS 2012 6

(Continued from **18C.35.**)

An educational psychologist adopts the Internet Addiction Test to measure the students' level of Internet ad diction. The scores of a random sample of 30 students are presented in the following stem and leaf diagram. Let σ be the standard deviation of the scores. It is known that the mean of the scores is 71 and the range of the scores is 56.

Stem (tens)	Leaf (units)
3	a
4	
5	2 4 6 8
6	0 1 3 5 6 7 8 8 9
7	1 2 2 4 5 5 6 8
8	0 2 3 5 8
9	0 2 b

(a) Find the values of a, b and σ.

(b) The psychologist classifies those scoring between 73 and 100 as excessive Internet users. If 4 students are selected randomly from the excessive Internet users among the students, find the probability that 3 of them will have scores higher than 80.

(c) *[Out of syllabus]*

17C.36 HKALE MS 2013 – 11

According to the school regulation, air conditioners can only be switched on if the temperature at 8 am exceeds 26°C. From past experience, the probability that the temperature at 8 am does NOT exceed 26°C is q ($q > 0$). Assume that there are five school days in a week. For two consecutive school days, the probability that the air conditioners are switched on for not more than one day is $\frac{7}{16}$.

(a) (i) Show that the probability that the air-conditioners are switched on for not more than one day on two consecutive school days is $2q - q^2$.

(ii) Find the value of q.

(b) The air conditioners are said to be *fully engaged* in a week if the air conditioners are switched on for all five school days in a week.

(i) Find the probability that the fifth week is the second week that the air conditioners are *fully engaged.*

(ii) ***[Out of syllabus]***

(c) On a certain day, the temperature at 8 am exceeds 26°C and all the 5 classrooms on the first floor are reserved for class activities after school. There are 2 air-conditioners in each classroom. The number of air conditioners being switched off in the classroom after school depends on the number of students staying in the classroom. Assume that the number of students in each classroom is independent.

Case	I	II	III
Number of air conditioners being switched off	2	1	0
Probability	0.25	0.3	0.45

(i) What is the probability that all air-conditioners are switched off on the first floor after school?

(ii) Find the probability that there are exactly 2 classrooms with no air-conditioners being switched off and at most 1 classroom with exactly 1 air conditioner being switched off on the first floor after school.

(iii) Given that there are 6 air-conditioners being switched off on the first floor after school, find the probability that at least 1 classroom has no air conditioners being switched off.

17C.37 HKDSE MA 2013 – I – 10

(Continued from 18C.41.)

The ages of the members of Committee A are shown as follows:

17	18	21	21	22	22	23	23	23	31
31	34	35	36	47	47	58	68	69	69

(a) Write down the median and the mode of the ages of the members of Committee A.

(b) The stem-and leaf diagram shows the distribution of the ages of the members of Committee B. It is given that the range of this distribution is 47.

(i) Find a and b.

(ii) From each committee, a member is randomly selected as the representative of that committee. The two representatives can join a competition when the difference of their ages exceeds 40. Find the probability that these two representatives can join the competition.

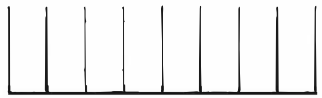
Stem (tens)	Leaf (units)
2	a 5 6 7
3	3 3 8
4	3
5	1 2 9
6	7 b

17C.38 HKDSE MA 2014 – I – 19

Ada and Billy play a game consisting of two rounds. In the first round, Ada and Billy take turns to throw a fair die. The player who first gets a number '3' wins the first round. Ada and Billy play the first round until one of them wins. Ada throws the die first.

(a) Find the probability that Ada wins the first round of the game.

(b) In the second round of the game, balls are dropped one by one into a device containing eight tubes arranged side by side (see the figure). When a ball is dropped into the device, it falls randomly into one of the tubes. Each tube can hold at most three balls.



The player of this round adopts one of the following two options.

Option 1: Two balls are dropped one by one into the device. If the two balls fall into the same tube, then the player gets 10 tokens. If the two balls fall into two adjacent tubes, then the player gets 5 tokens. Otherwise, the player gets no tokens.

Option 2: Three balls are dropped one by one into the device. If the three balls fall into the same tube, then the player gets 50 tokens. If the three balls fall into three adjacent tubes, then the player gets 10 tokens. If the three balls fall into two adjacent tubes, then the player gets 5 tokens. Otherwise, the player gets no tokens.

(i) If the player of the second round adopts Option 1, find the expected number of tokens got.

(ii) Which option should the player of the second round adopt in order to maximise the expected number of tokens got? Explain your answer.

(iii) Only the winner of the first round plays the second round. It is given that the player of the second round adopts the option which can maximise the expected number of tokens got. Billy claims that the probability of Ada getting no tokens in the game exceeds 0.9. Is the claim correct? Explain your answer.

17 Probability

17A Counting principles

17A.1 HKALE MS 1995 – 3

(a) No of ways $= C_6^{18}C_6^{12}C_6^{6} = 17153136$

(b) Required p $= \dfrac{(C_5^{15}C_1^{3})(C_5^{10}C_1^{2})(C_5^{5}C_1^{1})}{17153136} = \dfrac{4040536}{17153136} = \dfrac{9}{34}$

17A.2 HKALE MS 1999 – 6

(a) No of ways $= P_2^5 C_{10}^{16} = 160160$

(b) 10, 11, 12, 13, 14, 15, 16

17A.3 HKALE MS 2011 – 5

(Each route is a 8-step route consisting of 3 N's and 5 E's, such as NNNEEEEE or NNENEEEE.)

(a) *Method 1*
The routes are all possible routes subtracted by the routes going through T_1.
$\therefore$ No of routes $= C_3^8 - C_3^7 = 21$
Method 2
The routes are all the routes that start from the junction 1N from A.
$\therefore$ No of routes $= C_2^7 = 21$

(b) No of ways $= C_3^8 - C_2^5 \times C_1^3 = 26$

17A.4 HKDSE MA 2018 – I – 15

(a) Required no $= 8! = 40320$

(b) Required no $= P_2^4 \times 6! = 8640$

17A.5 HKDSE MA 2019 – I – 15

Required no $= C_5^{21+11} - C_5^{11} = 200914$

17B Probability (short questions)

17B.1 HKCEE MA 1981(1/3) – I – 3

Method 1 Required p $= \dfrac{1}{C_2^{40}} = \dfrac{1}{780}$

Method 2 Required p $= \dfrac{2}{40} \times \dfrac{1}{39} = \dfrac{1}{780}$

17B.2 HKCEE MA 1982(1/3) – I – 6

(a) Required p $= \dfrac{3}{36} = \dfrac{1}{12}$

(b) Required p $= \dfrac{3}{36} = \dfrac{1}{12}$

(c) Required p $= 1 - \dfrac{1}{12} - \dfrac{1}{12} = \dfrac{5}{6}$

17B.3 HKCEE MA 1996 – I – 7

(a) Area $= \pi(12)^2 - \pi(2)^2 = 140\pi$ (cm^2)

(b) (i) Required p $= \dfrac{140\pi}{144\pi} \times \dfrac{140\pi}{144\pi} = \dfrac{1225}{1296}$

(ii) Required p $= \dfrac{140\pi}{144\pi} \times \dfrac{4\pi}{144\pi} \times 2 = \dfrac{35}{648}$

17B.4 HKCEE MA 1998 – I – 11

(a) Required p $= \dfrac{8}{14} \times \dfrac{7}{13} = \dfrac{4}{13}$

(b) Required p $= \dfrac{4}{13} + \dfrac{4}{14} \times \dfrac{3}{13} + \dfrac{2}{14} \times \dfrac{1}{13} = \dfrac{5}{13}$

17B.5 HKCEE MA 1999 – I – 12

(a) Required p $= 75\% \times 20\% = 0.15$

(b) Required p $= \dfrac{2}{5} \times \dfrac{3}{4} + \dfrac{3}{5} \times \dfrac{2}{4} = \dfrac{3}{5}$

17B.6 HKCEE MA 2000 – I – 12

(a) Required p $= \dfrac{9}{900} = \dfrac{1}{100}$

(b) Required p $= \dfrac{9 \times 9 \times 9}{900} = \dfrac{81}{100}$

(c) Required p $= 1 - \dfrac{1}{100} - \dfrac{81}{100} = \dfrac{9}{50}$

17B.7 HKCEE MA 2004 – I – 8

(a) Required p $= \dfrac{5}{9}$

(b) Required p $= 1 - P(\text{both odd}) = 1 - \left(\dfrac{5}{9}\right)^2 = \dfrac{22}{27}$

17B.8 HKCEE MA 2006 – I – 8

(a) Sum $= 11 \times 10 \Rightarrow k = 24$

(b) Required p $= \dfrac{4}{10} = \dfrac{2}{5}$

17B.9 HKCEE MA 2008 – I – 5

Favourable outcomes: 4&6, 3&7

$\therefore$ Required p $= \dfrac{2}{3 \times 2} = \dfrac{1}{3}$

17B.10 HKCEE MA 2009 – I – 5

Required p $= \dfrac{7+21}{7+21+30+53+57+32} = \dfrac{7}{50}$

17B.11 HKALE MS 1994 – 1

(a) BB, BG, GB, GG

(b) (i) BB, BG, GB

(ii) Required p $= \dfrac{1}{3}$

17B.12 HKALE MS 1994 – 3

The shortest paths must consist of 6 steps, among which 2 are 'up', 2 are 'forward' and 2 are 'right'.

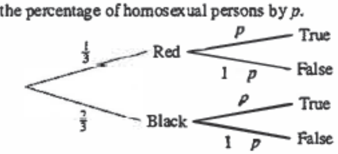
(a) No of ways $= C_2^6 C_2^4 C_2^2 = 90$

(b) (i) No of ways $= C_1^3 C_1^2 C_1^1 = 6$

(ii) Required p $= \dfrac{6}{90} = \dfrac{1}{15}$

17B.13 HKALE MS 1994 – 7

(a) Let the percentage of homosexual persons by p.

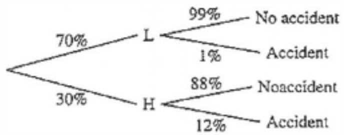


$\dfrac{790}{1200} = \dfrac{1}{3}(p) + \dfrac{2}{3}(1-p) \Rightarrow p = 0.025$

$\therefore$ 2.5% are homosexual.

(b) $P(\text{Homo}|\text{True}) = \dfrac{\frac{1}{3}(p)}{\frac{790}{1200}} = 1.27\%$ (3 s.f.)

17B.14 HKALE MS 1995 – 5



$P(\text{No accident}) = 70\% \times 99\% + 30\% \times 88\% = 0.957$

(a) $P(\text{H}|\text{No accident}) = \dfrac{30\% \times 88\%}{0.957} = 0.276$ (3 s.f.)

(b) *Method 1*
$P(\text{L}|\text{No accident}) = \dfrac{70\% \times 99\%}{0.957} = 0.724$ (3 s.f.)
Method 2
$P(\text{L}|\text{No accident}) = 1 - 0.276 = 0.724$ (3 s.f.)

17B.15 HKALE MS 1996 – 6

(a) $P(\text{Accepted}) = P(A)P(\text{Accepted}|A) + P(B)P(\text{Accepted}|B)$
$= \dfrac{1}{2} \times (1-4\%)^2 + \dfrac{1}{2} \times (1-1\%)^2 = 0.95085$

(b) $P(A|\text{Accepted}) = \dfrac{\frac{1}{2} \times (1-4\%)^2}{0.95085} = 0.485$ (3 s.f.)

17B.16 HKALE MS 1997 – 7

(a) $P(\text{Not working}) = 0.15 \times 0.24 = 0.036$

(b) $P(\text{Working}) = 1 - 0.036 = 0.964$
$\Rightarrow$ Required p $= \dfrac{0.85}{0.964} = 0.882$ (3 s.f.)

(c) Required p $= (0.964)^9(0.036) = 0.0259$ (3 s.f.)

17B.17 HKALE MS 1998 – 6

(a) (i) Required p $= \dfrac{1}{8} \times (1 - 0.8\%) = 0.124$

(ii) Required p $= \dfrac{1}{8}(1 - 0.8\%) + \dfrac{2}{8}(1 - 0.2\%) + \dfrac{5}{8}$
$= 0.9985$

(b) Required p $= \dfrac{\frac{1}{8}(0.8\%)}{1 - 0.9985} = \dfrac{2}{3}$

17B.18 HKALE MS 1999 – 5

(a) Required p $= C_5^{10}(60\%)^5(1-60\%)^5 = 0.200658$
$= 0.201$ (3 s.f.)

(b) Expected no $= 10 \times 60\% = 6$

(c) Required p $= (1 - 0.200658)^2(0.200658) = 0.128$ (3 s.f.)

17B.19 HKALE MS 2000 – 6

(a) Required p $= \dfrac{C_3^4}{C_3^6} = \dfrac{1}{5}$

(b) Required p $= \dfrac{C_2^4 C_1^2}{C_3^6} = \dfrac{3}{5}$

17B.20 HKALE MS 2000 – 8

(a) Required p $= 0.3 \times \dfrac{5}{9} + 0.7 \times \dfrac{5}{6} = 0.75$

(b) Let a be the probability of generating game A.
$a \times \dfrac{5}{9} + (1 - a) \times \dfrac{5}{6} = \dfrac{2}{3} \Rightarrow a = 0.6$
$\therefore P(\text{Game } A) = 0.6,\ P(\text{Game } B) = 0.4$

17B.21 HKALE MS 2001 – 6

(a) Required p $= \dfrac{C_2^9}{C_3^{10}} = \dfrac{3}{10}$

(b) Required p $= \dfrac{C_2^6}{C_3^{10}} = \dfrac{1}{8}$

(c) *Method 1*
~~Required~~ p $= \dfrac{C_3^8 + C_1^2 C_2^8}{C_3^{10}} = \dfrac{14}{15}$
Method 2
Required p $= 1 - P(\text{2 heaviest selected}) = 1 - \dfrac{C_1^8}{C_3^{10}} = \dfrac{14}{15}$

17B.22 HKALE MS 2001 – 7

(a) Required p $= \dfrac{58\% \times 39\%}{65\% \times 48\% + 58\% \times 39\% + 50\% \times 13\%}$
$= 0.375$

(b) Required p $= (0.375)^2(1 - 0.375)^2 = 0.0343$ (3 s.f.)

17B.23 HKALE MS 2002 – 5

(a) Required p $= \dfrac{1}{3} \times \dfrac{6}{8} + \dfrac{1}{3} \times \dfrac{4}{7} + \dfrac{1}{3} \times \dfrac{2}{7} = \dfrac{15}{28}$

(b) Required p $= \dfrac{\frac{1}{3} \times \frac{6}{8}}{\frac{15}{28}} = \dfrac{7}{15}$

17B.24 HKA LBMS 2002 – 8

(a) 3W, 3Y, 2R1W, 2R1Y, 1R2W, 1R2Y, 2W1Y, 1W2Y, 1R1W1Y

(b) Required p $= \dfrac{C_1^2 C_2^{11}}{C_3^{13}} = \dfrac{5}{13}$

(c) *Method 1*

Required p $= \dfrac{P(\text{1R1W1Y})}{P(\text{Exactly 1R})} = \dfrac{C_1^2 C_1^5 C_1^6}{C_1^2 C_2^{11}} = \dfrac{6}{11}$

Method 2

Required p $= P(\text{Exactly 1W1Y after 1R is selected})$

$= \dfrac{C_1^5 C_1^6}{C_2^{11}} = \dfrac{6}{11}$

17B.25 HKA MS 2003 – 12

(a) Required p $= 1 - P(\text{No boy}) - P(\text{No girl})$

$= 1 - \dfrac{C_7^{17}}{C_7^{30}} - \dfrac{C_7^{13}}{C_7^{30}} = \dfrac{38743}{39150} = 0.990$

(b) Required p $= \dfrac{P(\text{4 or 5 or 6 girls})}{\text{Prob in (a)}}$

$= \dfrac{\dfrac{C_4^{17}C_3^{13} + C_5^{17}C_2^{13} + C_6^{17}C_1^{13}}{C_7^{30}}}{\dfrac{38743}{39150}} = 0.657$

17B.26 HKA LBMS 2004 – 6

(a) Required p $= 1 - P(\text{all 3 same}) = 1 - \left(\dfrac{1}{10}\right)^2 = \dfrac{99}{100}$

(b) Required p $= \dfrac{3!}{10^3} = \dfrac{3}{500}$

(c) Required p $= C_2^3 \left(\dfrac{4}{10}\right)^2 \left(\dfrac{6}{10}\right) = \dfrac{36}{125}$

17B.27 HKA LBMS 2004 10

(a) Required p $= 7.5\% \times 94\% + (1 - 7.5\%) \times 14\% = 0.2$

(b) Required p $= \dfrac{(1 - 7.5\%) \times 14\%}{0.2} = 0.6475$

17B.28 HKA LBMS 2007 – 6

(a) *Method 1* Required p $= \dfrac{C_1^1 C_2^4}{C_3^{10}} \times \dfrac{1}{3} = \dfrac{1}{60}$

Method 2 Required p $= \dfrac{1}{10} \times \dfrac{4}{9} \times \dfrac{3}{8} = \dfrac{1}{60}$

(b) *Method 1* Required p $= \dfrac{1}{60} + \dfrac{C_1^5 C_2^5}{C_1^{10} C_2^9} = \dfrac{7}{45}$

Method 2 Required p $= \dfrac{1}{60} + \dfrac{5}{10} \times \dfrac{5}{9} \times \dfrac{4}{8} = \dfrac{7}{45}$

17B.29 HKA LBMS 2009 – 5

(a) Required p $= 1 - C_1^{15}(36\%)(64\%)^{14} - C_2^{15}(36\%)^2(64\%)^{13}$
$C_3^{15}(36\%)^3(64\%)^{12} = 0.847$

(b) Required p

$= \dfrac{C_1^5(36\%)(64\%)^4 \times C_1^5(36\%)(64\%)^4 \times C_2^5(36\%)^2(64\%)^3 \times 3}{C_4^{15}(36\%)^4(64\%)^{11}}$

$= \dfrac{50}{91} = 0.549$

17B.30 HKA LBMS 2011 – 4

(a) Required p

$= 0.75^2(1 - 0.55^2) + 0.75(1 - 0.75)(1 - 0.55)^2$

$= 0.468$

(b) Required p $= \dfrac{0.75^2 \times 0.55(1 \quad 0.55) \cdot 2}{1 - (1 - 0.55)^2} = 0.349$

17B.31 HKA LE MS 2011 – 5

(Each route is a 8-step route consisting of 3 N's and 5 E's, such as NNNEEEEE or NNENEEEE.)

(a) *Method 1*

The routes are all possible routes subtracted by the routes going through T_1.

$\therefore$ No of routes $= C_3^8 - C_3^7 = 21$

Method 2

The routes are all the routes that start from the junction 1N from A.

$\therefore$ No of routes $= C_2^7 = 21$

(b) No of ways $= C_3^8 - C_2^5 \times C_1^3 = 26$

(c) Required p $= \dfrac{C_2^7 - C_1^4 \times C_1^3}{C_3^8} = \dfrac{9}{56}$

17B.32 HKA LE MS 2013 – 4

(a) Required p $= \left(\dfrac{1}{4}\right)^4 \times 4 + \dfrac{4!}{4^4} = \dfrac{7}{64}$

(b) Required p $= \dfrac{\frac{4!}{4^4}}{\frac{4}{4} \times \frac{3}{4}} = \dfrac{1}{8}$

17B.33 HKDSE MA SP – I – 16

(a) Required p $= \dfrac{C_2^5 \times C_2^4}{C_4^9} = \dfrac{10}{21}$

(b) Required p $= 1 \quad \dfrac{10}{21} = \dfrac{11}{21}$

17B.34 HKDSE MA PP – I – 13

(a) Number of students $= 6 \div \dfrac{3}{20} = 40$

$\Rightarrow k = 40 - 6 - 11 \quad 5 \quad 10 = 8$

17B.35 HKDSE MA PP – I 16

(a) Required p $= \dfrac{C_4^{18}}{C_4^{30}} = \dfrac{68}{609}$

(b) Required p $= 1 - \dfrac{68}{609} - \dfrac{C_4^{12}}{C_4^{30}} = \dfrac{530}{609}$

17B.36 HKDSE MA 20 1 2 – I – 16

(a) Required p $= \dfrac{C_4^8 \times (C_1^2)^4}{C_4^{16}} = \dfrac{8}{13}$

(b) Required p $= 1 - \dfrac{8}{13} = \dfrac{5}{13}$

17B.37 HKDSE MA 2013 – I – 16

(a) Required p $= \dfrac{C_4^5 C_2^{11} + C_5^5 C_1^{11}}{C_6^{16}} = \dfrac{1}{28}$

(b) Required p $= 1 - \dfrac{1}{28} = \dfrac{27}{28}$

17B.38 HKDSE MA 2015 – I – 3

Required p $= \dfrac{1+2+3}{4 \times 5} = \dfrac{3}{10}$

17B.39 HKDSE MA 2015 – I – 16

(a) Required p $= \dfrac{C_2^5 C_2^9}{C_4^{14}} = \dfrac{360}{1001}$

(b) Required p $= 1 - \dfrac{C_4^9}{C_4^{14}} - \dfrac{C_1^5 C_3^9}{C_4^{14}} \quad \dfrac{5}{11}$

17B.40 HKDSE MA 2016 – I – 9

(a) $x = 2 + 4 = 6$

$y = 37 - 15 = 22$

$z = 37 + 3 = 40$

(b) Required p $= \dfrac{22 \quad 6}{40} = \dfrac{2}{5}$

17B.41 HKDSE MA 2016 – I – 15

Required p $= \dfrac{P_4^6 P_5^5}{(4+5)!} = \dfrac{5}{42}$

17B.42 HKDSE MA 2017 – I – 7

(a) $x = 360° \times \dfrac{1}{9} = 40°$

(b) No of students $= 180 \div \dfrac{360° - 90° - 158° \quad 40°}{360°} = 900$

17B.43 HKDSE MA 2017 – I 11

(a) $(80 + b) - 61 = 22 \Rightarrow b = 3$

$\dfrac{61 + \cdots + (70 + a) + \cdots + 83}{15} = 70 \Rightarrow a = 2$

$\therefore$ Median = \$69, SD = \$7.33

(b) Required p $= \dfrac{6}{15} = \dfrac{2}{5}$

17B.44 HKDSE MA 2017 – I – 17

(a) Required p $= \dfrac{C_1^{7+8}}{C_5^{4+7+8}} = \dfrac{5}{3876}$

(b) Required p $= \dfrac{C_3^4 C_2^{15}}{C_5^{19}} = \dfrac{35}{969}$

(c) Required p $= 1 - \dfrac{5}{3876} - \dfrac{35}{969} = \dfrac{3731}{3876}$

17B.45 HKDSE MA 2018 – I – 4

$\dfrac{8}{n+5+8} = \dfrac{2}{5} \Rightarrow n = 7$

17B.46 HKDSE MA 2019 – I – 8

(a) 2

(b) Mean $= 2 \times \dfrac{144°}{360°} + 3 \times \dfrac{54°}{360°} + 5 \times \dfrac{72°}{360°} + 7 \times \dfrac{90°}{360°} = 4$

(c) Required p $= \dfrac{72 + 90}{360} = \dfrac{9}{20}$

17B.47 HKDSE MA 2020 – I – 15

15a	The required probability $= \dfrac{C_4^7 + C_4^9}{C_4^{4+7+9}}$ $= \dfrac{161}{3876}$
b	The required probability $= 1 - \dfrac{161}{3876}$ $= \dfrac{3715}{3876}$

17C Probability (structural questions)

17C.1 HKCEE MA 1980(1/3) I – 14

(a) Required p $= \dfrac{9}{10} \times \dfrac{2}{3} = \dfrac{3}{5}$

(b) Required p $= \dfrac{1}{10} \times \dfrac{9}{10} \times \dfrac{2}{3} + \dfrac{9}{10} \times \dfrac{1}{3} \times \dfrac{2}{3} = \dfrac{13}{50}$

(c) Required p $= 1 - \left(1 \quad \dfrac{3}{5}\right)^2 = \dfrac{21}{25}$

17C.2 HKCEE MA 1983(A /B) – I – 11

(a) Required p $= 0.6^3 = 0.216$

(b) Required p $= (1 - 0.6)^3 = 0.064$

(c) Required p $= C_1^3(0.6)(0.4)^2 = 0.288$

(d) *Method 1*

Required p $= 1 \quad 0.216 - 0.064 \quad 0.288 = 0.432$

Method 2 Required p $= C_2^3(0.6)^2(0.4) = 0.432$

17C.3 HKCEE MA 1984(A/B) – I – 11

(a) (i) Required p $= \left(\dfrac{1}{3}\right)^2 = \dfrac{1}{9}$

(ii) Required p $= \dfrac{1}{9} \times 3 = \dfrac{1}{3}$

(iii) Required p $= 1 - \dfrac{1}{3} = \dfrac{2}{3}$

(b) (i) Required p $= \left(\dfrac{2}{7}\right)^2 = \dfrac{4}{49}$

(ii) Required p $= \dfrac{4}{49} + \left(\dfrac{2}{7}\right)^2 + \left(\dfrac{3}{7}\right)^2 = \dfrac{17}{49}$

(iii) Required p $= 1 - \dfrac{17}{49} = \dfrac{32}{49}$

17C.4 HKCEE MA 19 85(A /B) I 10

(a) (i) Required p $= \dfrac{3+2+1}{36} = \dfrac{6}{36} = \dfrac{1}{6}$

(ii) Required p $= \dfrac{6+4}{36} = \dfrac{5}{18}$

(b) (i) Required p $= \left(1 - \dfrac{5}{18}\right)^2 = \dfrac{169}{324}$

(ii) Required p $= 2 \times \dfrac{5}{18} \times \dfrac{13}{18} = \dfrac{65}{162}$

17C.5 HKCEE MA 19 86(A/B) – I – 13

(a) (i) Required p $= \dfrac{1}{5}$

(ii) Required p $= \dfrac{3}{75} = \dfrac{1}{25}$

(iii) Required p $= \dfrac{2 \times 4}{75} = \dfrac{8}{75}$

(b) (i) Required p $= \dfrac{1}{3} \times \dfrac{1}{3} = \dfrac{1}{9}$

(ii) Required p $= \dfrac{1}{9} \times 2! = \dfrac{2}{9}$

(iii) Required p $= 1 - \dfrac{1}{9} \times 3 = \dfrac{2}{3}$

17C.6 HKCEE MA 1 987(A /B) – I – 13

(a) Required p $= \dfrac{5}{6}$

(b) Required p $= \dfrac{1}{6} \times \dfrac{4}{5} = \dfrac{2}{15}$

(c) Required p $= \dfrac{1}{6} \times \dfrac{1}{5} = \dfrac{1}{30}$

(d) Required p $= \dfrac{3}{6} \times \dfrac{1}{5} = \dfrac{1}{10}$

17C.7 HKCEE MA 1988 – I – 11

(a) (i) Median = 70 marks
(ii) IQR = 86 − 50 = 36 (marks)

(b) (i) Number of students = 600 − 540 = 60
(ii) Required p $= \frac{60}{600} = \frac{1}{10}$
(iii) (1) Required p $= \frac{C_2^{60}}{C_2^{600}} = \frac{59}{5990}$
(2) Required p $= 1 - \frac{C_2^{540}}{C_2^{600}} = \frac{1139}{5990}$

17C.8 HKCEE MA 1989 – I – 13

(a) $\begin{cases} p = 3q \\ p + q = 1 \end{cases} \Rightarrow q = 0.25$

(b) (i) Required p $= \frac{n}{10} \times \frac{n-1}{9} = \frac{n(n-1)}{90}$
(ii) $\frac{n(n-1)}{90} > \frac{1}{3} \Rightarrow n^2 - n - 30 > 0$
$\Rightarrow n < -5$ or $n > 6$
$\therefore$ Possible n's = 7, 8, 9, 10

(c) Required p $= \frac{1}{2} \times \frac{4}{6} + \frac{1}{2} \times \frac{3}{6} = \frac{7}{12}$

17C.9 HKCEE MA 1990 – I – 13

(a) (i) Required p $= \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$
(ii) Required p $= \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} = \frac{4}{9}$

(b) (i) Required p $= \frac{2}{9} \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{81}$
(ii) Required p $= \left(\frac{1}{3} \times \frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{10}{81}$

17C.10 HKCEE MA 1991 – I – 10

(a) (i) Required p = (1 − 70%)(70%) = 0.21
(ii) Required p = 70% + 0.21 = 0.91
(iii) Required p = 0.91 × [60% + (1 − 60%)(60%)]
= 0.7644

(b) Expected number = 10000 × 0.7644 = 7644

17C.11 HKCEE MA 1992 – I – 10

(a) (i) Required p $= 1 - \frac{2}{5} = \frac{3}{5}$
(ii) Required p $= 1 - \frac{4}{7} = \frac{3}{7}$
(iii) Required p $= \frac{3}{5} \times \frac{1}{7} = \frac{3}{35}$
(iv) Required p $= \frac{2}{5}\left(1 - \frac{1}{7}\right) = \frac{12}{35}$

(b) (i) Required p $= \frac{1}{7} \times \frac{2}{7} \times 2 = \frac{4}{49}$
(ii) Required p $= \left(\frac{2}{5} \times \frac{2}{7}\right) \times \left(\frac{3}{5} \times \frac{2}{7}\right) \times 2 = \frac{48}{1225}$

17C.12 HKCEE MA 1993 – I – 13

(a) (i) Required p = 0.65 × 0.45 = 0.2925
(ii) Required p = 0.2925 + (0.25 + 0.1) × 0.55 = 0.485

(b) (i) Required p $= \left(\frac{40000 \times 70\%}{60000}\right)^2 = \frac{49}{225}$
(ii) Req. p $= \left(\frac{40000 \times 70\% + 20000 \times 40\%}{60000}\right)^2 - \frac{9}{25}$
(iii) Required p $= 1 - \frac{9}{25} - \left(\frac{60000 - 36000}{60000}\right)^2 = \frac{12}{25}$

17C.13 HKCEE MA 1994 – I – 9

(a) (i) Required p $= \left(\frac{1}{7}\right)^3 = \frac{1}{343}$
(ii) Required p $= \left(\frac{6}{7}\right)^3 = \frac{216}{343}$

(b) (i) Required p $= \left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right) = \frac{9}{1000}$
(ii) Required p $= \frac{9}{1000} \times 3 = \frac{27}{1000}$

(c) Required p $= \frac{1}{2} \times \frac{1}{7} + \frac{1}{2} \times \frac{1}{10} = \frac{17}{140}$

17C.14 HKCEE MA 1995 – I – 11

(a) (i) $p = 1 - \frac{4}{5} = \frac{1}{5}$
(ii) $q = 0,\ r = 1$

(b) (i) Required p $= \frac{1}{2} \times \frac{1}{2}$
(ii) (1) Required p $= \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} = \frac{2}{3}$
(2) Required p $= \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$

17C.15 HKCEE MA 1997 – I – 14

(a) (i) Required p $= \frac{C_2^{40}}{C_2^{50}} = \frac{156}{245}$
(ii) Required p $= \frac{C_1^{10}C_1^{40}}{C_2^{50}} = \frac{16}{49}$
(iii) Required p $= \frac{C_2^{10}}{C_2^{50}} = \frac{9}{245}$
(iv) Required p = 0

(b) (i) Required p $= \frac{156}{245} + \frac{9}{245} = \frac{33}{49}$
(ii) Required p $= 1 - \left(\frac{156}{245}\right)^2 - \left(\frac{16}{49}\right)^2 - \left(\frac{9}{245}\right)^2$
= 0.487

17C.16 HKCEE MA 2002 – I – 12

(a)

$0 < x \le 5$	66	Certificate
$5 < x \le 15$	34	Book coupon
$15 < x \le 25$	64	Bronze medal
$25 < x \le 35$	26	Silver medal
$35 < x \le 50$	10	Gold medal

(b) IQR = 23 − 4 = 19

(c) Number of medallists = 200 − 100 = 100
(i) Required p $= \frac{C_2^{10}}{C_2^{100}} = \frac{1}{110}$
(ii) Required p $= 1 - \frac{1}{110} - \frac{C_2^{36}}{C_2^{100}} - \frac{C_2^{64}}{C_2^{100}} = \frac{1282}{2475}$

17C.17 HKCEE MA 2003 – I – 16

(a) Required p $= \frac{9}{10} \times \frac{1}{2} = \frac{9}{20}$

(b) (i) Required p $= \frac{23}{25} \times \frac{1}{2} = \frac{23}{50}$
(ii) (1) Required p $= \frac{2}{3} \times \frac{9}{20} + \frac{1}{3} \times \frac{23}{50} = \frac{34}{75}$
(2) Required p $= 1 - \frac{34}{75} = \frac{41}{75}$

17C.18 HKCEE MA 2005 – I – 11

(a) Required p $= \frac{1}{2}$

(b) Required p $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

(c) Required p $= 1 - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$

(d) Required p $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

17C.19 HKCEE MA 2006 – I – 14

(a) (i) Class A: IQR = 39 − 18 = 21 (marks)
Class B: IQR = 25 − 11 = 14 (marks)
(ii) $\because$ IQR of B < IQR of A
$\therefore$ Class B is less dispersed.

(b) (i) Required p $= \frac{C_2^{18+10}C_1^{22}}{C_3^{50}} = \frac{297}{700}$
(ii) Required p $= \frac{(C_2^{18} + C_2^{10})C_1^{22}}{C_3^{50}} = \frac{1089}{4900}$
(iii) Required p $= \frac{\frac{1089}{4900}}{\frac{297}{4900}} = \frac{11}{21}$

17C.20 HKCEE MA 2007 – I – 15

(a) (i) Required p $= \frac{48}{80} = \frac{3}{5}$
(ii) Required p $= \frac{12}{80} = \frac{3}{20}$
(iii) Required p $= \frac{48+4}{80} = \frac{13}{20}$
(iv) Required p $= \frac{12}{48} = \frac{1}{4}$

(b) (i) Required p $= \frac{C_2^{16}}{C_2^{80}} = \frac{3}{79}$
(ii) $\because$ P(same size) $= \frac{C_2^{28}}{C_2^{80}} + \frac{C_2^{36}}{C_2^{80}} + \frac{3}{79} = \frac{141}{395} < \frac{1}{2}$
$\therefore$ NO

17C.21 HKCEE MA 2008 – I – 14

(a) (i) Required p $= \frac{9}{15} = \frac{3}{5}$
(ii) (1) Required p $= \frac{8 \times 15}{C_2^{36}} = \frac{4}{21}$
(2) Required p $= 1 - \frac{C_2^{8}}{C_2^{36}} - \frac{C_2^{15}}{C_2^{36}} - \frac{C_2^{13}}{C_2^{36}} = \frac{419}{630}$

17C.22 HKCEE MA 2009 – I – 14

(a) For Brand A,
mean $= \frac{1050 \cdot 8 + 1150 \cdot 50 + 1250 \cdot 42 + 1350 \cdot 10 + 1450 \cdot 10}{120}$
= 1220 (h)
For Brand B,
mean $= \frac{1050 \cdot 4 + 1150 \cdot 12 + 1250 \cdot 40 + 1350 \cdot 36 + 1450 \cdot 28}{120}$
= 1310 (h) > 1220 (h)
$\therefore$ Brand B

(b) (i) Required p $= \frac{8+50+42}{120} = \frac{5}{6}$
(ii) Required p $= 1 - \frac{5}{6} \times \frac{99}{119} = \frac{73}{238}$
(iii) Method 1: required p $= \frac{1}{2} \cdot \frac{73}{238} + \frac{1}{2}\left(1 - \frac{C_2^{56}}{C_2^{120}}\right)$
$= \frac{779}{1428}$
Method 2: required p $= 1 - \frac{5}{6} \times \frac{56}{120} = \frac{11}{18} > \frac{779}{1428}$
$\therefore$ Method 2

17C.23 HKCEE MA 2010 – I – 14

(a) (i) Required p $= \frac{C_2^8}{C_2^{10}} = \frac{28}{45}$
(ii) Required p $= \frac{C_1^2 C_1^8}{C_2^{10}} = \frac{16}{45}$
(iii) *Method 1* Required p $= \frac{28}{45} + \frac{16}{45} = \frac{44}{45}$
Method 2 Required p $= 1 - \frac{1}{C_2^{10}} = \frac{44}{45}$

(b) (i) Alice's mean = 275 s, Betty's mean = 272 s < 275 s
$\therefore$ Betty
(ii) Alice got 3 results < 267 s but Betty only got 1.
$\therefore$ Alice

17C.24 HKCEE MA 2011 – I – 14

(a) Required p $= \frac{9}{12}\left(1 - \frac{1}{6}\right) + \frac{3}{12}\left(1 - \frac{1}{3}\right) = \frac{19}{24}$

(b) (i) Required p $= \frac{5}{6} \times \frac{2}{3} = \frac{5}{9}$
(ii) Required p $= \left(\frac{3}{12} \cdot \frac{2}{3}\right) \times \left(\frac{2}{11} \cdot \frac{2}{3}\right) = \frac{2}{99}$
(iii) P(both not making complaints)
$= \left(\frac{9}{12} \cdot \frac{5}{6}\right) \cdot \left(\frac{8}{11} \cdot \frac{5}{6}\right) + 2\left(\frac{3}{12} \cdot \frac{9}{11} \cdot \frac{5}{9}\right) + \frac{2}{99}$
$= \frac{62}{99} > \frac{1}{2} \Rightarrow$ YES

17C.25 HKALE MS 1994 – 11

(a) (i) Required p $= C_3^7(30\%)^3(70\%)^4 = 0.227$
(iii) Required p
= 1 − P(next day dry) − P(next day humid, then dry)
= 1 − 30% − (70%)(30%) = 0.49

(b) (i) Required p = (1 − 0.9)(1 − 0.8) = 0.02
(ii) Required p = P(20 dry, 21 dry) + P(20 hmd. 21 dry)
= 0.02 + (0.9)(0.9) = 0.83
(iii) Required p $= \frac{0.02}{0.83} = 0.0241$

17C.26 HKALE MS 1995 – 11

(a) Required p = 1 − (1 − 2%)^5 = 0.096079 = 0.0961 (3 s.f.)

(c) (i) *Method 1*
Required p
= P(22 good) + P(21 good 1 bad) + P(20 good 2 bad)
$= (0.903921)^{22} + C_1^{22}(0.903921)^{21}(0.096079)$
$+ C_2^{22}(0.903921)^{20}(0.096079)^2$
= 0.64455 = 0.645 (3 s.f.)
Method 2
Required p = P(1st 20 accepted)
+ P(1 rejected in 1st 20, 21st accepted)
+ P(2 rejected in 1st 21, 22nd accepted)
$= (0.903921)^{20}$
$+ C_1^{20}(0.096079)(0.903921)^{20}$
$+ C_2^{21}(0.096079)^2(0.903921)^{20}$
= 0.64455 = 0.645 (3 s.f.)
(ii) Required p $= \frac{(0.903921)^{20}}{0.64455} = 0.206$

(d) $(1 - r\%)^5 \ge 0.95 \Rightarrow 1 - r\% \ge \sqrt[5]{0.95} \Rightarrow r \le 1.02$
Hence, the greatest acceptable value of r is 1.02.

17C.27 HKALE MS 1998 – 3

(a) Median $= (161+162) \div 2 = 161.5$ (cm)

(b) (i) Required p $= \left(\frac{31}{40}\right)^3\left(\frac{9}{40}\right) = 0.105$

(ii) Required p $= C_3^5\left(\frac{31}{40}\right)^2\left(\frac{9}{40}\right)^3 = 0.0684$

17C.28 HKALE MS 1998 – 5

(a) (i) No of arrangements $= 10! = 3628800$

(ii) No of arrangements $= 9! \times 2! = 725760$

(b) (i) No of arrangements $= 10! = 3628800$

(ii) *Method 1*

No of arrangements $= (9! - 8!) \times 2! = 645120$

Method 2

No of arrangements $= C_3^8 \times 4! \times 2! \times 5! \times 2! = 645120$

Method 3

No of arrangements $= 8! \times 2 \times 8 = 645120$

17C.29 HKALE MS 1999 – 7

(a) (i) Required p $= 0.015 \times 0.030 = 0.00045$

(ii) Required p $= 0.015 \times 0.025 \times 0.030 = 0.00001125$

(iii) Required p $= 0.00045 + 0.025 \times 0.030 - 0.00001125$
$= 0.00118875 = 0.001189$ (4 s.f.)

(b) Required p $= 0.030$

(c) Requi redp $= \frac{0.015 \times 0.030}{0.00118875} = 0.379$

17C.30 HKALE MS 2002 – 7

(a) Mean $= 61$

(b) Si ncethere are two modes, one deleted mark is 54.
The other mark $= 61 \times 22 - (61+1.2) \times 20 \quad 54 = 44$

(c) Required p $= \frac{C_2^5}{C_2^{20}} = \frac{1}{19}$

17C.31 HKALE MS 2003 – 11

(a) Required p $= \frac{1}{n}$

(b) (i) Required p $= p$

(ii) $p + p + \frac{1}{n} = 1 \Rightarrow p = \left(1 - \frac{1}{n}\right) \div 2 = \frac{1}{2} - \frac{1}{2n}$

(iii) $\frac{1}{2} - \frac{1}{2n} \geq 0.46 \Rightarrow n \geq 12.5 \Rightarrow$ Least $n = 13$

(c) (i) Required p $= \left(\frac{5}{6}\right)^4 \frac{1}{6} = \frac{625}{7776}$

(ii) Required p $= \frac{1}{6} + \left(\frac{5}{6}\right)^2\frac{1}{6} + \left(\frac{5}{6}\right)^4\frac{1}{6} + \left(\frac{5}{6}\right)^6\frac{1}{6} + \ldots$
$\frac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^2} = \frac{6}{11}$

(iii) Required p $= \frac{\left(1-\frac{6}{11}\right) - \left(\frac{5}{6}\right)\frac{1}{6} \quad \left(\frac{5}{6}\right)^3\frac{1}{6}}{1-\frac{6}{11}} = \frac{625}{1296}$

17C.32 HKALE MS 2004 – 11

(a) Required p $= C_4^5(70\%)^4(30\%) \times 0.7 = 0.252105$

(b) (i) Required p $= 1 - (0.7)^{10} - C_1^{10}(0.7)^9(0.3)$
$- C_2^{10}(0.7)^8(0.3)^2 - C_3^{10}(0.7)^7(0.3)^3$
$= 0.350389 = 0.350$ (3 s.f.)

(i i) Required p $= C_4^{10}(0.7)^6(0.3)^4 + C_5^{10}(0.7)^5(0.3)^5$
$= 0.303040 = 0.303$ (3 s.f.)

(iii) Required p $= \frac{0.303040}{0.350389} = 0.865$

(c) (i) Required p $= (0.350389)^2 = 0.123$

(ii) Required p $= (0.350389)(0.7)^{10} \times 2 = 0.0198$

17C.33 HKALE MS 2005 6

(a) Required p $= \frac{C_2^5 C_2^7}{C_4^{12}} = \frac{14}{33}$

(b) (i) *Method 1* Required p $= \frac{P_2^2 \times P_2^2}{P_4^4} = \frac{1}{6}$

Method 2 Required p $= \frac{C_2^2}{C_2^4} = \frac{1}{6}$

(ii) *Method 1*

Required p $= \frac{14}{33} \times \frac{1}{6} + \frac{C_3^5 C_1^7}{C_4^{12}} \times \frac{P_2^3 \times P_2^2}{P_4^4} = \frac{14}{99}$

Method 2 Required p $= \frac{14}{33} \times \frac{1}{6} + \frac{C_3^5 C_1^7}{C_4^{12}} \times \frac{C_2^3}{C_2^4} = \frac{14}{99}$

17C.34 HKALE MS 2010 – 5

(a) $49 - (20+a) = 27 \Rightarrow a = 2$

$\frac{49 + \quad + (80+b)}{20} \quad \frac{22+49+\cdots+(80+b)}{21} = 2$

$\frac{1274+b}{20} - \frac{1296+b}{21} = 2$

$b = 6$

$\bar{x} = (1296+6) \div 21 = 62$

(b) (i) Required p $= \frac{C_6^{15}}{C_6^{20}} \quad \frac{1001}{7752}$

(ii) Required p $= 1 - \frac{1001}{7752} - \frac{C_1^5 C_5^{15}}{C_6^{20}} = \frac{937}{1938}$

17C.35 HKALE MS 2012 – 6

(a) $\frac{(30+a)+52+\cdots+92+(90+b)}{30} = 71$

$2120 + a + b = 2130$

$a + b = 10$

$(90+b) - (30+a) = 56 \Rightarrow a - b = 4$

Solving, $a = 7$, $b = 3$

$\Rightarrow \sigma = 12.7$

(b) Required p $= \frac{C_3^7 C_1^6}{C_4^{13}} = \frac{42}{143}$

17C.36 HKALE MS 2013 – 11

(a) (i) Required p $= 1 - (1-q)^2 = 2q - q^2$

(ii) $2q - q^2 = \frac{7}{16} \Rightarrow q = 0.25$ or 1.75 (rejected)

(b) (i) P(a week is *fully engaged*) $= (1-q)^5 = 0.75^5$

$\therefore$ Required p $= C_1^4(0.75^5)(1-0.75^5)^3 \times 0.75^5$
$= 0.0999$

(c) (i) Required p $= 0.25^5 = \frac{1}{1024}$

(ii) Required p $= C_2^5(0.45)^2\left(0.25^3 + C_1^3(0.25)^2(0.3)\right)$
$= 0.1455$

(iii) P(6 a/c switched off) $= C_1^5(0.25)(0.3)^4$
$+ C_2^5(0.25)^2 C_2^3(0.3)^2(0.45) + C_3^5(0.25)^3(0.45)^2$
$= 0.117703125$

$\therefore$ Required p $= 1 - \frac{C_1^5(0.25)(0.3)^4}{0.117703125} = 0.914$

17C.37 HKDSE MA 2013 – I – 10

(a) Median $= 31$
Mode $= 23$

(b) (i) $(60+b) - (20+a) = 47 \Rightarrow b - a = 7$

$\because 0 \leq a \leq 5$ and $7 \leq b \leq 9$

$\therefore (a, b) = (0, 7), (1, 8)$ or $(2, 9)$

(ii) Required p $= \frac{3+3+3+3+2+9+9}{20 \times 13} = \frac{8}{65}$

17C.38 HKDSE MA 2014 – I – 19

(a) Required p $= \frac{1}{6} + \left(\frac{5}{6}\right)^2\frac{1}{6} + \left(\frac{5}{6}\right)^4\frac{1}{6} + \left(\frac{5}{6}\right)^6\frac{1}{6} + \ldots$

$= \frac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^2} = \frac{6}{11}$

(b) (i) Expected no $= 10 \times \frac{1}{8} + 5 \times \frac{7 \cdot 2!}{8^2} = \frac{75}{32}$

(ii) Expected no of tokens with Option 2

$= 50 \times \frac{1}{8^2} + 10 \times \frac{6 \cdot 3!}{8^3} + 5 \times \frac{7 \times 2 \times C_2^3}{8^3} = \frac{485}{256}$

$< \frac{75}{32}$

Option 1

(iii) P(Ada getting no tokens) $= 1 - \frac{1}{6} \times \left(\frac{1}{8} + \frac{7 \cdot 2!}{8^2}\right)$

$= \frac{13}{16} < 0.9$

$\therefore$ NO

18 Statistics

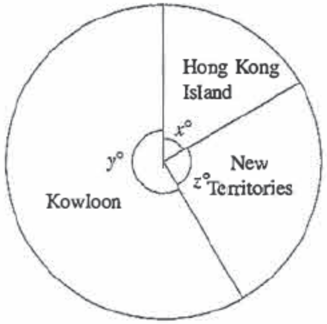
18A Presentation of data

18A.1 HKCEE MA 1982(1) – I – 7

In a certain school, the numbers of students living on Hong Kong Island, in Kowloon and the New Territories are in the ratios 2 : 7 : 3. The pie-chart in the figure shows the distribution.

(a) Find x, y and z.

(b) If the number of students living on Hong Kong Island is 240, find the total number of students in the school.



18A.2 HKCEE MA 1982(3) – I – 12

Figure (1)

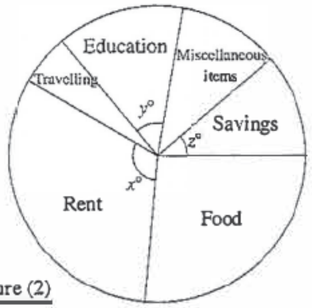


Figure (2)

(a) The pie chart in Figure (1) shows how Mr Wong's income was distributed between his expenses and savings for March. If his rent is \$2000, find Mr Wong's income for that month.

(b) The table below shows the percentage changes when each item of Mr Wong's expenses in April is compared with that in March.

Item	Food	Rent	Travelling	Education	Miscellaneous items	Savings
Percentage Change	Increased by 10%	Increased by 30%	Increased by 30%	No change	No change	?

The pie chart in Figure (2) shows how Mr Wong's income was distributed between his expenses and savings for April.

(i) Suppose that Mr Wong's income in March and April were the same.

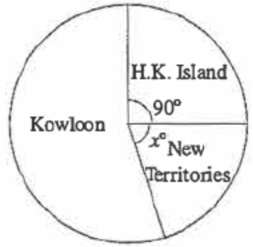
(1) Find x, y and z in Figure (2).

(2) Calculate the percentage change in Mr Wong's savings for April when compared with those for March.

(ii) If Mr Wong's income in April actually increased by 37.5%, what percentage of his income in April was spent on food?

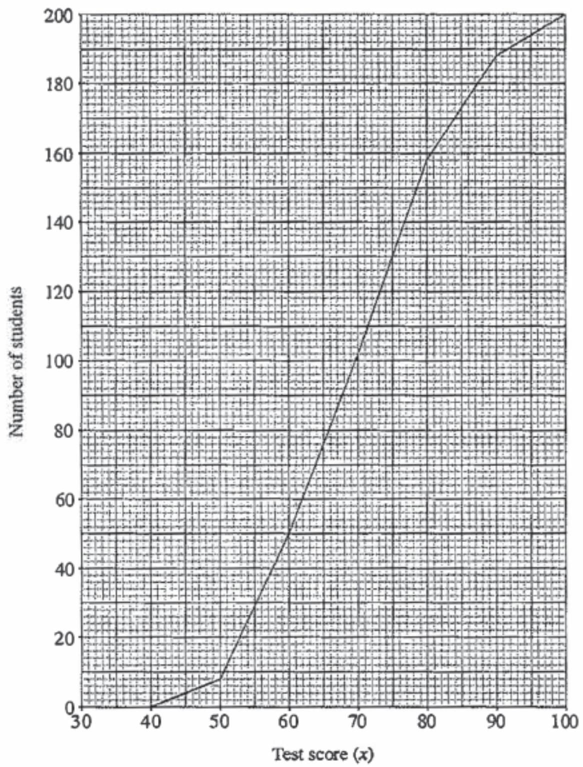
18A.3 HKCEE MA 1985(A/B) – I – 7

The pie-chart in the figure shows the distribution of traffic accidents in Hong Kong in 1983. There were 4200 traffic accidents on H.K. Island, 9240 accidents in Kowloon and n accidents in the New Territories. Find n and x.



18A.4 HKCEE MA 1998 – I – 10

The cumulative frequency polygon of the distribution of test scores of 200 students



Two hundred students took a test in Mathematics. The figure shows the cumulative frequency polygon of the distribution of the test scores.

(a) Complete the tables below.

Test score (x)	Cumulative frequency
$x \le 50$	8
$x \le 60$	50
$x \le 70$	
$x \le 80$	
$x \le 90$	188
$x \le 100$	200

Test score (x)	Frequency
$40 < x \le 50$	8
$50 < x \le 60$	42
$60 < x \le 70$	
$70 < x \le 80$	
$80 < x \le 90$	30
$90 < x \le 100$	12

(b) If the passing score is 55, estimate the passing percentage of the students in the test.

18A.5 HKCEE MA 1999 – I – 11

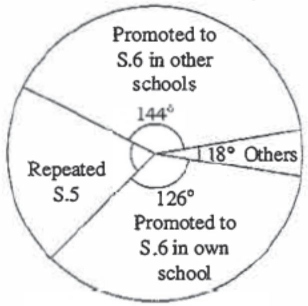
A school conducted a survey on the placement of her S.5 graduates last year. There were 200 graduates, of which 120 were boys and 80 were girls. The placement of the boys was shown in the figure.

(a) Find the number of boys who repeated S.5.

(b) Among all the boys promoted to S.6, what percentage of them was promoted in their own school?

(c) The result of the survey also showed that 22.5% of the girls were promoted to S.6 in their own school. Find the percentage of graduates promoted to S.6 in their own school.

Placement of the 120 boys



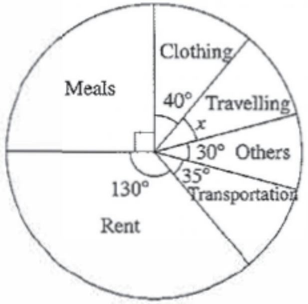
18A.6 HKCEE MA 2006 – I – 9

In the figure, the pie chart shows the expenditure of Ada in February 2006. It is given that she spent \$1750 on transportation in that month. Find

(a) x,

(b) her total expenditure in that month,

(c) her expenditure on travelling in that month.

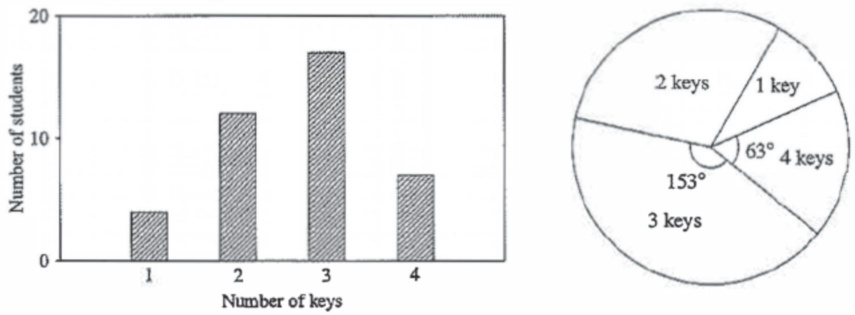


The expenditure of Ada in February 2006

18A.7 HKCEE MA 2007 – I – 12

The bar chart and pie chart in the figure show the distribution of the numbers of keys owned by the students in class A. The numbers of students having 2 keys, 3 keys and 4 keys are 12, 17 and k respectively.

Distribution of the numbers of keys owned by the students in class A



(a) Find the value of k.

(b) Find the number of students in class A.

(c) Find the probability that a randomly selected student in class A has only 1 key.

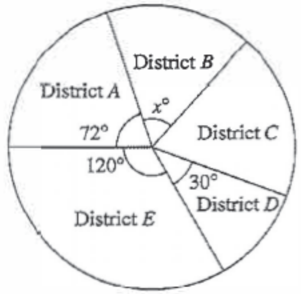
(d) It is given that the numbers of students in class A and class B are the same. The distributions of the numbers of keys owned by the students in class A and class B are also the same. The two classes are now combined to form a group. On each of the bar chart and the pie chart in the figure, is there a modification needed in order that the statistical chart can show the distribution of the numbers of keys owned by the students in this group? If your answer is 'yes', write down the modification needed.

18A.8 HKDSE MA SP – I – 9

In the figure, the pie chart shows the distribution of the numbers of traffic accidents occurred in a city in a year. In that year, the number of traffic accidents occurred in District A is 20% greater than that in District B.

(a) Find x.

(b) Is the number of traffic accidents occurred in District A greater than that in District C? Explain your answer.

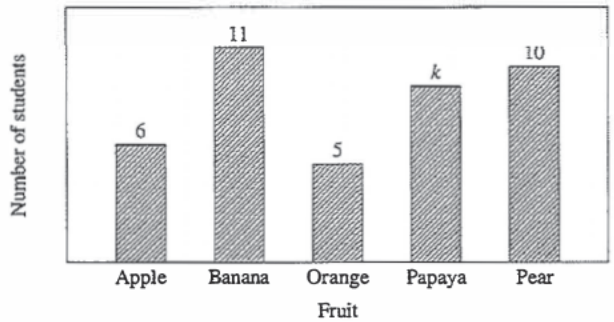


The distribution of the numbers of traffic accidents occurred in the city

18A.9 HKDSE MA PP – I – 13 (Continued from **17B.34**.)

The bar chart below shows the distribution of the most favourite fruits of the students in a group. It is given that each student has only one most favourite fruit.

Distribution of the most favourite fruits of the students in the group



If a student is randomly selected from the group, the probability that the most favourite fruit is apple is $\frac{3}{20}$.

(a) Find k.

(b) Suppose that the above distribution is represented by a pie chart.

(i) Find the angle of the sector representing that the most favourite fruit is orange.

(ii) Some new students now join the group and the most favourite fruit of each of these students is orange. Will the angle of the sector representing that the most favourite fruit is orange be doubled? Explain your answer.

18A.10 HKDSE MA 2016 – I – 9 (To continue as **17B.40**.)

The frequency distribution table and the cumulative frequency distribution table below show the distribution of the heights of the plants in a garden.

Height (m)	Frequency
0.1 – 0.3	a
0.4 – 0.6	4
0.7 – 0.9	b
1.0 – 1.2	c
1.3 – 1.5	15
1.6 – 1.8	3

Height less than (m)	Cumulative frequency
0.35	2
0.65	x
0.95	13
1.25	y
1.55	37
1.85	z

(a) Find x, y and z.

18B Measures of central tendency

18B.1 (HKCEE MA 1983(B) I 3)

The table shows the distribution of the marks of 1000 students in a mathematics test:

Class of Marks	Number of Students
40 – 49	100
50 – 59	300
60 – 69	400
70 – 79	200

(a) Find the class mark of the class 50 – 59.

(b) Estimate the mean of the distribution of marks.

18B.2 HKCEE MA 1984(A/B) – I 2

The table shows the distribution of the marks of a group of students in a short test:

If the mean of the distribution is 3, find the value of x.

Marks	1	2	3	4	5
Number of Students	10	10	5	20	x

18B.3 HKCEE MA 1986(A/B) I – 3

The table shows the number of students in three classes of a school and their average marks in a test.

If the overall average mark of the three classes is 60, find x.

Class	No. of Students	Average Mark
F.5A	40	61
F.5B	x	70
F.5C	35	50

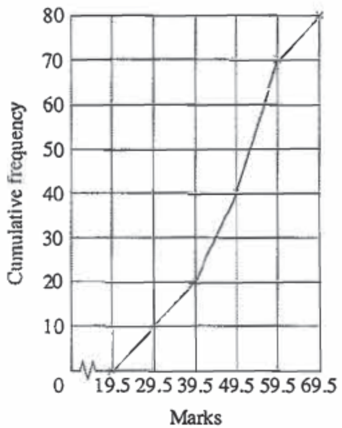
18B.4 HKCEE MA 1991 – I – 1

In the figure, the cumulative frequency polygon shows the distribution of the marks of 80 students in a Mathematics test.

(a) From the figure, write down the median of the distribution.

(b) Complete the table below.
Hence find the mean mark of the students in the test.

Marks	No. of students
20 – 29	
30 – 39	
40 – 49	
50 – 59	
60 – 69	



18B.5 HKCEE MA 1992 I 8

In a sports competition, the mean score of a team of m men and n women is 70.

(a) Find the total score of the team in terms of m and n.

(b) If the mean score of the men is 75 and the mean score of the women is 62, find the ratio $m : n$.

(c) If there are altogether 39 persons in the team, find the number of men.

18B.6 HKCEE MA 1994 I 1(d)

The marks scored by eleven students in a mathematics quiz are as follows:

10 20 30 45 50 60 65 65 65 70 70.

Find (i) the mean, (ii) the mode and (iii) the median of the above marks.

18B.7 HKCEE MA 1996 I 14

A youth centre has done a survey on the amount of money $\$x$ teenagers spent on buying clothes for Christmas. The results of the survey are shown in Tables (1) and (2).

Table (1) **The amount of money spent by *boys* on buying clothes for Christmas**

x	Frequency	Percentage (%)
0	70	20.0
$0 < x \le 200$	17	4.9
$200 < x \le 400$	48	13.7
$400 < x \le 600$	83	[illegible]
$600 < x \le 800$	92	26.3
$800 < x \le 1000$	36	10.3
$x > 1000$	4	1.1
Total frequency =	350	

Table (2) **The amount of money spent by *girls* on buying clothes for Christmas**

x	Frequency	Percentage (%)
0	81	15.0
$0 < x \le 200$	51	9.4
$200 < x \le 400$	135	25.0
$400 < x \le 600$	87	16.1
$600 < x \le 800$	74	13.7
$800 < x \le 1000$	56	10.4
$x > 1000$	57	10.5
Total frequency =	541	

(a) A number in Table (1) was accidentally covered in ink. What should this number be?

(b) Explain why the sum of the percentages in Table (2) is 100.1 instead of 100.

(c) The cumulative frequency polygon of the distribution of x ($x \le 1000$) for girls is drawn in Figure (3).

  (i) Construct the cumulative frequency table of the distribution of x ($x \le 1000$) for boys.

  (ii) On the same graph (Figure (3)), draw the cumulative frequency polygon of the distribution in (i).

  (iii) Find the medians of x for boys and girls respectively in this survey.

  (iv) Estimate the total number of teenagers in this survey spending not more than \$700 on buying clothes for Christmas.

(d) By considering the percentages in Tables (1) and (2), find evidence to support the statement:
"In this survey, more boys did not spend any money on buying clothes for Christmas."
Explain briefly why we have to consider the percentages instead of the frequencies.

The cumulative frequency polygon of the distribution of x ($x \le 100$) for girls

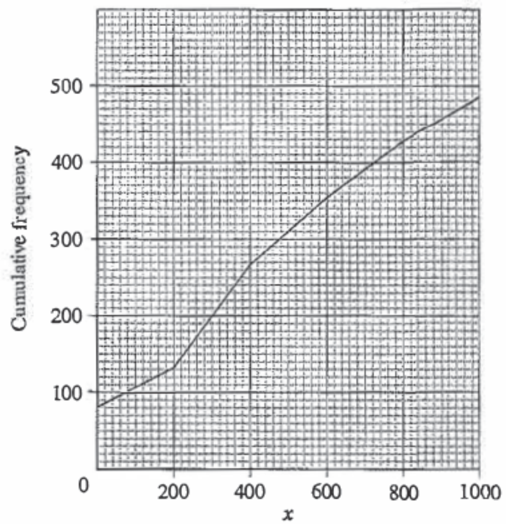


Figure (3)

18B.8 HKCEE MA 1999 – I – 8

The heights of 6 students are x cm, 161 cm, 168 cm, 159 cm, 161 cm and 152 cm. The mean height of these students is 158 cm.

(a) Find x.

(b) Find the median of the heights of these students.

18B.9 HKCEE MA 2000 – I – 11

The figure shows the cumulative frequency polygon of the distribution of the lengths of 75 songs.

(a) Complete the tables below.

Length (t seconds)	Cumulative frequency
$t \le 220$	3
$t \le 240$	16
$t \le 260$	46
$t \le 280$	
$t \le 300$	75

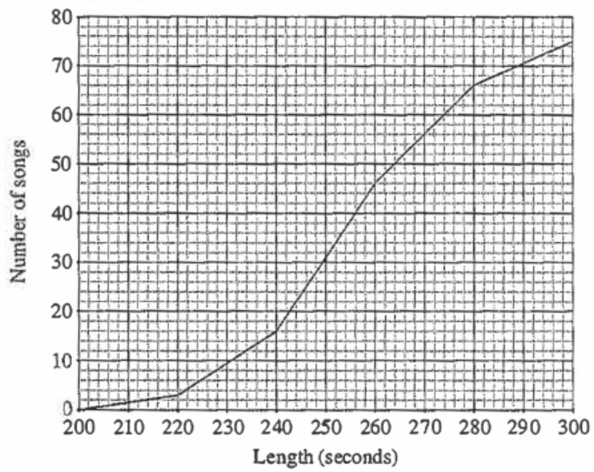
Length (t seconds)	Frequency
$200 < t \le 220$	3
$220 < t \le 240$	13
$240 < t \le 260$	30
$260 < t \le 280$	
$280 < t \le 300$	9

(b) Find an estimate of the mean of the distribution.

(c) Estimate from the cumulative frequency polygon the median of the distribution.

(d) What percentage of these songs have lengths greater than 220 seconds but not greater than 260 seconds?

The cumulative frequency polygon of the distribution of the lengths of 75 songs



18B.10 HKCEE MA 2003 – I – 11

(a) For the set of data 10, 10, 11, 12, 13, 16, find

(i) the mode,

(ii) the median,

(iii) the mean,

(iv) the range.

(b) Four unknown data are combined with the six data in (a) to form a set of ten data.

(i) Find the least and the greatest possible values of the median of the combined set of ten data.

(ii) If the mean of the four unknown data is 11, find the mean of the combined set of ten data.

18B.11 HKCEE MA 2006 – I – 8 (To continue as **17B.8**.)

There are ten cards numbered 2, 3, 5, 8, 11, 11, 12, 15, 19 and k respectively, where k is a positive integer. It is given that the mean of the ten numbers is 11.

(a) Find the value of k.

18B.12 HKALE MS 1998 – 3 (To continue as **17C.27**.)

40 students participate in a 5-day summer camp. The stem and leaf diagram below shows the distribution of heights in cm of these students.

(a) Find the median of the distribution of heights.

Stem (tens)	Leaf (units)
13	8
14	1 5 6 9
15	0 1 3 4 4 4 5 5 6 7 8 8 9
16	1 1 2 3 3 4 5 6 7 7 8 8
17	0 2 2 3 4 5 6 7
18	1 4

18B.13 HKALE MS 2002 – 7 (To continue as **17C.30**.)

Twenty two students in a class attended an examination. The stem-and-leaf diagram below shows the distribution of the examination marks of these students.

(a) Find the mean of the examination marks.

(b) Two students left the class after the examination and their marks are deleted from the stem-and-leaf diagram. The mean of the remaining marks is then increased by 1.2 and there are two modes. Find the two deleted marks.

Stem (tens)	Leaf (units)
3	5 7
4	2 4 6
5	0 3 4 4 4 5
6	1 2 5 5 8
7	3 8 9
8	4 8
9	5

18B.14 HKALE MS 2010 – 5 (To continue as **17C.34**.)

The following stem-and-leaf diagram shows the distribution of the test scores of 21 students taking a statistics course. Let $\bar{x}$ be the mean of these 21 scores.

It is known that if the smallest value of these 21 scores is removed, the range is decreased by 27 and the mean is increased by 2.

(a) Find the values of a, b and $\bar{x}$.

Stem (Tens)	Leaf (Units)
2	a
3	
4	9
5	0 0 1 3 7 7
6	0 2 3 5 5 5 9
7	0 3 4 9
8	2 b

18B.15 HKDSE MA SP – I – 14

The data below show the percentages of customers who bought newspaper A from a magazine stall in city H for five days randomly selected in a certain week:

62% 63% 55% 62% 58%

(a) Find the median and the mean of the above data.

(b) Let a% and b% be the percentages of customers who bought newspaper A from the stall for the other two days in that week. The two percentages are combined with the above data to form a set of seven data.

(i) Write down the least possible value of the median of the combined set of seven data.

(ii) It is known that the median and the mean of the combined set of seven data are the same as that found in (a). Write down one pair of possible values of a and b.

(c) The stall-keeper claims that since the median and the mean found in (a) exceed 50%, newspaper A has the largest market share among the newspapers in city H. Do you agree? Explain your answer.

18B.16 HKDSE MA 2012 – I – 10

Tom conducts a survey on the numbers of hours spent on doing homework in a week by secondary students. Questionnaires are sent out and twenty of them are returned. The stem-and-leaf diagram below shows the numbers of hours recorded in the twenty questionnaires:

Stem (tens)	Leaf (units)
1	0 0 1 1 2 3 4 5 5 6 6 7 7
2	0 0 0 5 8
3	4 6

(a) Find the mean and the median of the numbers of hours recorded in the twenty questionnaires.

(b) Tom receives four more questionnaires. He finds that the mean of the numbers of hours recorded in these four questionnaires is 18. It is found that the numbers of hours recorded in two of these four questionnaires are 19 and 20.

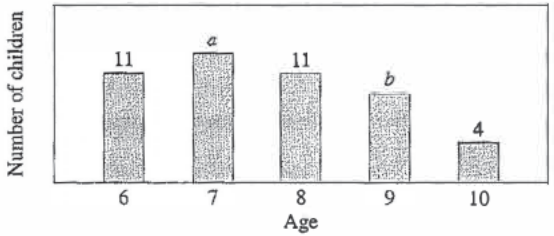
  (i) Write down the mean of the numbers of hours recorded in the twenty four questionnaires.

  (ii) Is it possible that the median of the numbers of hours recorded in the twenty-four questionnaires is the same as the median found in (a)? Explain your answer.

18B.17 HKDSE MA 2016 – I – 12

The bar chart below shows the distribution of the ages of the children in a group, where $a > 11$ and $4 < b < 10$. The median of the ages of the children in the group is 7.5.

Distribution of the ages of the children in the group



(a) Find a and b.

(b) Four more children now join the group. It is found that the ages of these four children are all different and the range of the ages of the children in the group remains unchanged. Find

  (i) the greatest possible median of the ages of the children in the group,

  (ii) the least possible mean of the ages of the children in the group.

18B.18 HKDSE MA 2018 – I – 11

The following table shows the distribution of the numbers of children of some families:

Number of children	0	1	2	3	4
Number of families	k	2	9	6	7

It is given that k is a positive integer.

(a) If the mode of the distribution is 2, write down

  (i) the least possible value of k;

  (ii) the greatest possible value of k.

(b) If the median of the distribution is 2, write down

  (i) the least possible value of k;

  (ii) the greatest possible value of k.

(c) If the mean of the distribution is 2, find the value of k.

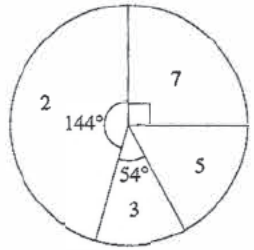
18B.19 HKDSE MA 2019 – I 8

(To continue as **17B.46**.)

The pie chart below shows the distribution of the numbers of rings owned by the girls in a group.

(a) Write down the mode of the distribution.

(b) Find the mean of the distribution.



Distribution of the numbers of rings owned by the girls in the group

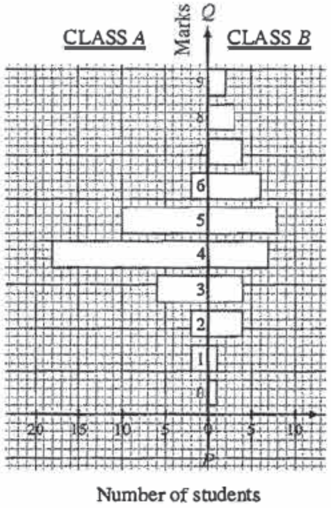
18C Measures of dispersion

18C.1 HKCEE MA 1980(3) I – 8

Two classes, A and B, each of 40 students, took a test. In the test, students may score 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 marks. In the figure, the distribution of marks of class A is shown in the bar chart on the left of PQ and that of class B is shown on the right.

(a) Find, by inspection, which class has a greater standard deviation of marks.

(b) If 70 students from the two classes pass the test, what is the minimum mark that a student should get in order to obtain a pass?



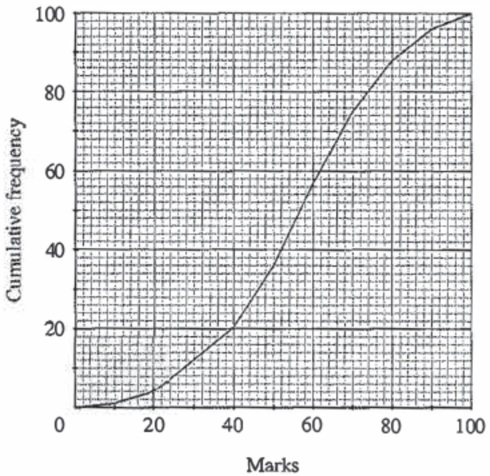
18C.2 HKCEE MA 1981(1) – I 6

The figure shows the cumulative frequency polygon of the marks obtained by 100 students taking a mathematics test.

(a) If 75% of the students pass the test, what is the pass mark, correct to the nearest integer?

(b) If the pass mark were 40, how many students would pass the test?

(c) Find the inter quartile range.



18C.3 HKCEE MA 1983(A) I – 3

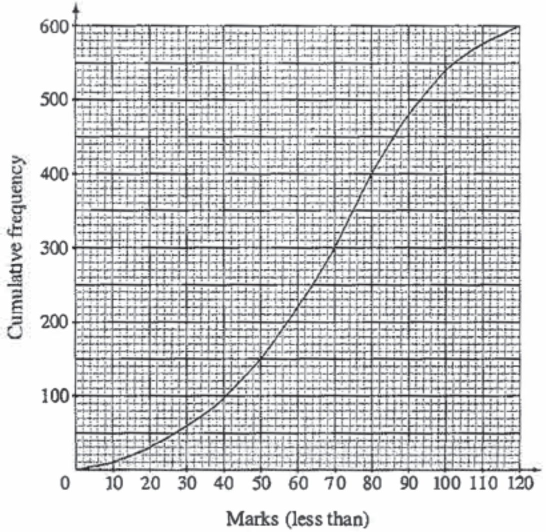
Given five real numbers $a-6$, a, $a+2$, $a+3$, $a+6$, find

(a) the mean,

(b) the standard deviation.

18C.4 HKCEE MA 1988 – I 11 (To continue as 17C.7.)

The figure below shows the cumulative frequency curve of the marks of 600 students in a mathematics contest.



(a) From the curve, find
   (i) the median, and
   (ii) the interquartile range of the distribution of marks.

18C.5 HKCEE MA 1990 I 12

(a) The distribution of the monthly salaries of 100 employees in a firm is shown in the histogram in the figure.
   (i) Find the modal class, median, mean, interquartile range and ***mean deviation (out of syllabus)*** of the monthly salaries of the 100 employees.
   (ii) Now the firm employs 10 more employees whose monthly salaries are all \$6500. Will the standard deviation of the monthly salaries of all the employees in the firm become greater, smaller or remain unchanged? Explain briefly.

(b) The mean of 7 numbers $x_1, x_2, \ldots, x_7$ is $\bar{x}$ and the *squares* of the deviations from $\bar{x}$ are 9, 4, 1, 0, 1, 4, 9 respectively. Find the standard deviation of the 7 numbers. ***[not mandatory]***

Distribution of monthly salaries of 100 employees

18C.6 HKCEE MA 1993 – I – 7

The following frequency table shows the distribution of the scores of 200 students in a Mathematics examination.

Frequency Table

Score	Frequency
0 – 9	20
10 – 19	40
20 – 29	60
30 39	50
40 49	20
50 59	10

Cumulative Frequency Table

Score (less than)	Cumulative Frequency
9.5	
19.5	
29.5	
39.5	
49.5	
59.5	

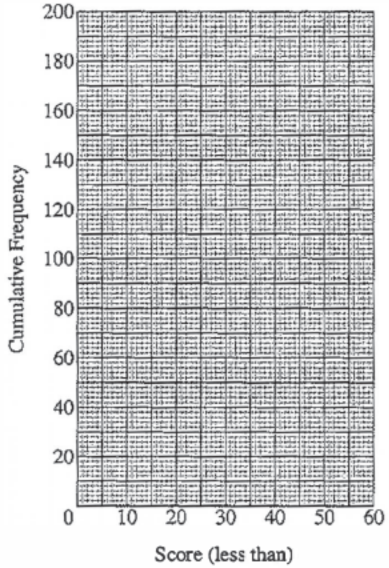
(a) Fill in the cumulative frequency table.

(b) (i) Draw the cumulative frequency polygon on the graph paper and determine the interquartile range.

(ii) If the pass percentage is set at 60%, determine the pass score from the cumulative frequency polygon.

(c) Find the mean and standard deviation of the distribution of scores. (Working steps need not be shown.)

(d) The teacher found that the scores were too low. He added 20 to each score. Write down the mean and the standard deviation of the new set of scores.



18C.7 HKCEE MA 1995 I – 9

The cumulative frequency polygon in the figure shows the distribution of the yearly average scores of all the Secondary 2 students in School A.

(a) Find

(i) the total number of Secondary 2 students in School A;

(ii) the median of the yearly average scores, correct to the nearest integer.

(b) The students will be allocated to 3 different groups in Secondary 3 according to their yearly average scores. The top 25% will be in Group I and the bottom 25% will be in Group III. The rest will be in Group II. Find, correct to the nearest integer,

(i) the minimum yearly average score for students to be allocated to Group I;

(ii) the minimum yearly average score for students to be allocated to Group II.

(c) Fill in the class marks and frequencies in the table.

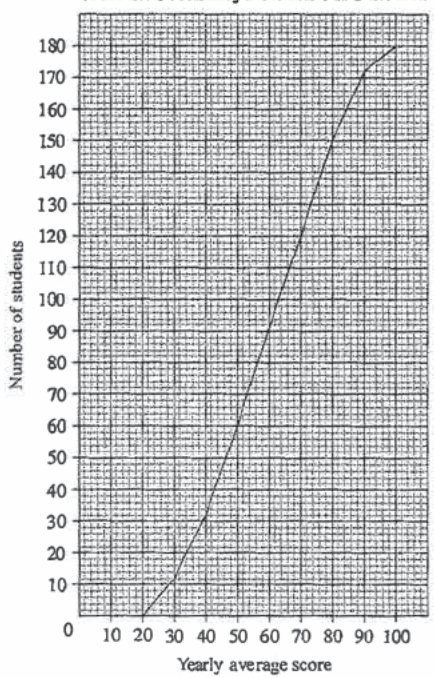
(d) From the table, find the mean and standard deviation of the yearly average scores. (Working need not be shown.)

(e) Find the percentage of students whose yearly average scores are within one standard deviation from the mean. (The distribution of the yearly average scores is not necessarily a normal distribution.)

The frequency distribution table of the yearly average scores of all the Secondary 2 students in School A

Yearly average score (x)	Class mark	Frequency
$20 < x \le 30$	25	
$30 < x \le 40$		20
$40 < x \le 50$		
$50 < x \le 60$		32
$60 < x \le 70$		
$70 < x \le 80$		30
$80 < x \le 90$		22
$90 < x \le 100$	95	

The cumulative frequency polygon of the yearly average scores of all the Secondary 2 students in School A



18C.8 HKCEE MA 1997 – I – 11

The following are the marks scored by a class of 35 students in a Mathematics test:

0	0	5	8	11	12	41	42	45	48
50	62	70	73	73	73	77	78	80	80
82	82	82	83	83	85	85	87	90	90
95	95	95	95	98					

(a) Find the mean, mode, median and standard deviation of the above marks. (Working need not be shown.)

(b) Explain briefly why the mean may not be a suitable measure of central tendency of the distribution of marks in the Mathematics test.

(c) The mean and standard deviation of the marks scored by the same class of students in an English test are 63 and 15 respectively.

(i) The standard score of a student in the English test was 0.4. Find the mark the student scored in this test.

(ii) Assume that the marks in the English test are normally distributed and the marks scored by Lai Wah in both the Mathematics and English tests are 78.

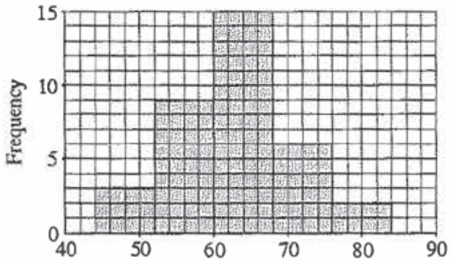
(1) What percentage of her classmates scored fewer marks than Lai Wah in the Mathematics test?

(2) Relative to her classmates, did Lai Wah perform better in the English test than in the Mathematics test?

(iii) The English teacher later found that a student was given 10 marks fewer in the English test. Find the mean of the marks in the English test after the wrong mark has been corrected.

18C.9 HKCEE MA 2001 – I – 10

Distribution of scores of 40 students



The histogram in the figure shows the distribution of scores of a class of 40 students in a test.

(a) Complete the table.

Score (x)	Class mark	Frequency
$44 \le x < 52$		3
$52 \le x < 60$		
	64	15
$68 \le x < 76$		11
	80	

(b) Estimate the mean and standard deviation of the distribution.

(c) Susan scores 76 in this test. Find her standard score.

(d) Another test is given to the same class of students. It is found that the mean and standard deviation of the scores in this second test are 58 and 10 respectively. Relative to her classmates, if Susan performs equally well in these two tests, estimate her score in the second test.

18C.10 HKCEE MA 2002 – I – 5

For the set of data 4, 4, 5, 6, 8, 12, 13, 13, 13, 18, find

(a) the mean,

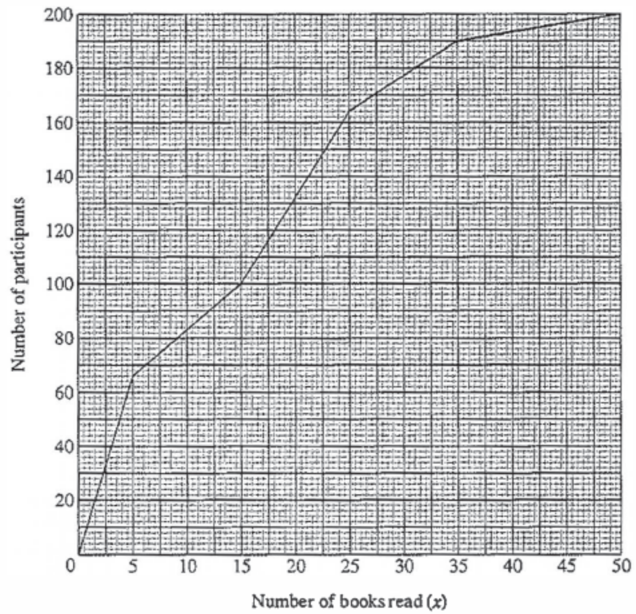
(b) the mode,

(c) the median,

(d) the standard deviation.

18C.11 HKCEE MA 2002 I – 12 (To continue as **17C.16**.)

The cumulative frequency polygon of the distribution of the numbers of books read by the participants



Two hundred students participated in a summer reading programme. The figure shows the cumulative frequency polygon of the distribution of the numbers of books read by the participants.

(a) The table below shows the frequency distribution of the numbers of books read by the participants. Using the graph in the figure, complete the table.

Number of books read (x)	Number of participants	Award
$0 < x \le 5$	66	Certificate
$5 < x \le 15$		Book coupon
$15 < x \le 25$	64	Bronze medal
$25 < x \le 35$		Silver medal
$35 < x \le 50$	10	Gold medal

(b) Using the graph in the figure, find the inter quartile range of the distribution.

18C.12 HKCEE MA 2004 – I – 11

A large group of students sat in a Mathematics test consisting of two papers, Paper I and Paper II. The table below shows the mean, median, standard deviation and range of the test marks of these students in each paper:

Test paper	Mean	Median	Standard deviation	Range
Paper I	46.1 marks	46 marks	15.2 marks	91 marks
Paper II	60.3 marks	60 marks	11.6 marks	70 marks

A student, John, scored 54 marks in Paper I and 66 marks in Paper II.

(a) Assume that the marks in each paper of the Mathematics test are normally distributed. Relative to other students, did John perform better in Paper II than in Paper I? Explain your answer.

(b) In a mark adjustment, the Mathematics teacher added 4 marks to the test mark of Paper I for each of these students. Write down the mean, the median and the range of the test marks of Paper I after the mark adjustment.

18C.13 HKCEE MA 2005 – I – 15

The scores (in marks) obtained by a class of 20 students in a music test are shown below:

84	86	90	93	100
103	120	120	120	121
122	134	134	136	137
144	146	146	146	158

(a) Find the mean, the ***mean deviation (out of syllabus)*** and the standard deviation of the above scores.

(b) Mary is one of the students in the class and her standard score in the music test is 1. Is Mary one of the top 20% students of the class in the music test? Explain your answer.

(c) (i) If one student in the class withdraws, find the probability that the mean of the scores obtained by the remaining 19 students in the music test is 122 marks.

(ii) If two students in the class withdraw, find the probability that the mean of the scores obtained by the remaining 18 students in the music test is 122 marks.

18C.14 HKCEE MA 2006 – I – 14 (To continue as 17C.19.)

The stem and leaf diagrams below show the distributions of the scores (in marks) of the students of classes A and B in a test, where a, b, c and d are non negative integers less than 10. It is given that each class consists of 25 students.

Class A

Stem (tens)	Leaf (units)
0	a 9
1	2 5 7 8 8
2	3 3 5 6 7 9
3	2 3 5 6 9 9 9
4	1 2 2 4 b

Class B

Stem (tens)	Leaf (units)
0	c 3 3 4 5
1	1 1 2 2 3 3 5 6 7 8
2	1 1 5 5 5 7 8
3	5 9
4	d

(a) (i) Find the inter quartile range of the score distribution of the students of class A and the inter quartile range of the score distribution of the students of class B.

(ii) Using the results of (a)(i), state which one of the above score distributions is less dispersed. Explain your answer.

18C.15 HKCEE MA 2007 – I – 4

The stem and leaf diagram below shows the distribution of weights (in kg) of 15 teachers in a school.

Stem (tens)	Leaf (units)
5	0 5 5 5 8
6	2 3 7 8 8 9
7	1 3 3 5

Find the median, the range and the standard deviation of the distribution.

18C.16 HKCEE MA 2008 – I – 10

The frequency distribution table and the cumulative frequency distribution table below show the distribution of the weights of the 50 babies born in a hospital during the last week, where a, b, c, k, l and m are integers.

Weight (kg)	Frequency
2.6 – 2.8	a
2.9 – 3.1	12
3.2 – 3.4	b
3.5 – 3.7	10
3.8 – 4.0	c

Weight less than (kg)	Cumulative Frequency
2.85	4
3.15	k
3.45	37
3.75	l
4.05	m

(a) Find a, b and c.

(b) Find estimates of the mean and the standard deviation of the weights of the 50 babies born in the hospital during the last week.

18C.17 HKCEE MA 2008 – I – 14 (Continued from 17C.21.)

The stem-and-leaf diagram below shows the suggested bonuses (in dollars) of the 36 salesgirls of a boutique:

Stem (thousands)	Leaf (hundreds)
2	4 4 7
3	2 5 6 6 8
4	3 3 3 4 4 7 8 8 8
5	0 0 3 4 4 6
6	2 3 3 4 4 9 9
7	0 4 4 8
8	2 3

(a) The suggested bonus of each salesgirl of the boutique is based on her performance. The following table shows the relation between level of performance and suggested bonus:

Level of performance	Suggested bonus ($x)
Excellent	$x > 6500$
Good	$4500 < x \le 6500$
Fair	$x \le 4500$

(i) From the 36 salesgirl, one of them is randomly selected. Given that the level of performance of the selected salesgirl is good, find the probability that her suggested bonus is less than $5500.

(ii) From the 36 salesgirls, two of them are randomly selected.

(1) Find the probability that the level of performance of one selected salesgirl is excellent and that of the other is good.

(2) Find the probability that the levels of performance of the two selected salesgirls are different.

(b) (i) Find the median and the inter quartile range of the suggested bonuses of the 36 salesgirls.

(ii) The boutique has made a considerable profit and so the manager wants to raise the suggested bonus of each of the 36 salesgirls such that the median of the suggested bonuses will be increased by 20% and the inter-quartile range will remain unchanged. Describe how the manager should raise the suggested bonus of each of the 36 salesgirls.

18C.18 HKCEE MA 2009 – I – 10

The stem-and leaf diagram below shows the distribution of the typing speed (in words per minute) of 20 students in a school before training.

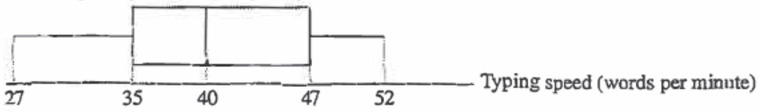
Stem (tens)	Leaf (units)
1	2 2 6 7
2	1 1 3 3 3 5 7 8
3	2 4 5 5 8 9 9 9

(a) Find the median, the range and the inter-quartile range of the above distribution.

(b) The box-and-whisker diagram below shows the distribution of the typing speed (in words per minute) of the 20 students after the training.

(i) Is the distribution of the typing speed after the training more dispersed than that before the training? Explain your answer.

(ii) The trainer claims that not less than half of these students show improvement in their typing speed after the training. Do you agree? Explain your answer.



18C.19 HKCEE MA 2010 – I – 11

The stem-and leaf diagram shows that ages of the players of a football team:

Stem (tens)	Leaf (units)
1	8 9 9
2	0 1 1 1 3 3 5 6 6 7 7 8 8 8 8
3	0 0 1 1

(a) Find the mean, the median and the range of the ages of the players of the football team.

(b) As the two oldest players leave the team, three new players join the football team. After the three players join the football team, the manager of the team finds that the mean age of the players of the football team is the same as the mean found in (a).

(i) Find the mean age of the three new players.

(ii) Furthermore, the manager finds that the median and the range of the ages of the players of the football team are the same as the median and the range found in (a) respectively. Write down two sets of possible ages of the three new players.

18C.20 HKCEE MA 2011 I – 10

The student union of a school conducts two surveys to measure the extent of the students' satisfaction on the services provided by the school library. A score from 0 to 100 is used to measure the extent of satisfaction on the services, with 0 indicating absolute dissatisfaction and 100 indicating absolute satisfaction. The stem-and-leaf diagram below shows the distribution of scores rated by 32 students in the first survey.

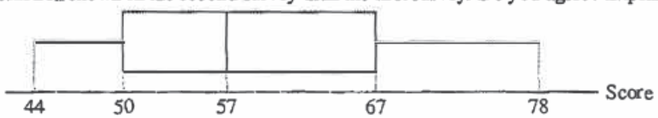
Stem (tens)	Leaf (units)
2	3 3
3	2 4 6 6
4	2 3 3 5 5 7
5	1 6 6 7 7 8 8 8 8
6	3 3 5 5 6 6 7 7 9 9
7	5

(a) Find the median, the range and the inter-quartile range of the above distribution.

(b) After six months, the student union conducts the second survey to these 32 students. The box-and-whisker diagram below shows the distribution of scores rated by these students in the second survey.

(i) Is the distribution of scores in the second survey less dispersed than the first survey? Explain your answer.

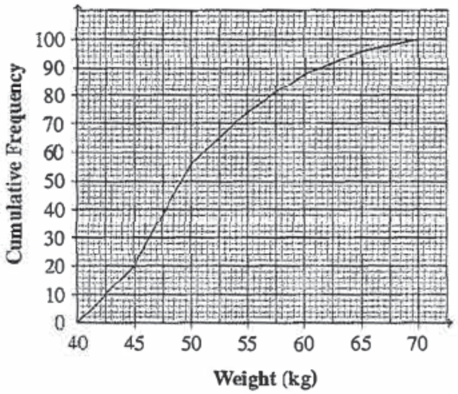
(ii) The chairman of the student union claims that at least 25% of these students have a greater extent of satisfaction shown in the second survey than the first survey. Do you agree? Explain your answer.



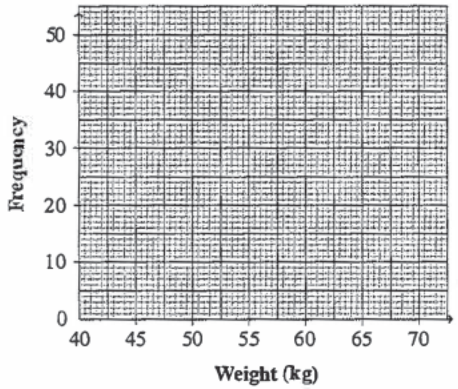
18C.21 HKALE MS 1994 – 4

The figure shows the cumulative frequency polygon of weights (in kg) for a group of 100 students.

Cumulative frequency polygon of weights for a group of 100 students



Weights of a group of 100 students



(a) Use the graph paper provided to draw a histogram of the weights.

(b) Determine the inter-quartile range of the weights from the cumulative frequency polygon.

(c) Determine the mean weight from the histogram.

18C.22 HKALE MS 1995 – 1

The numbers of hours spent by 25 students in studying for an examination are as follows:

11	8	25	21	18	25	7	32	29	18
18	18	22	12	5	30	19	15	20	50
25	10	26	23	12					

Stem (in 10)	Leaf (in 1)
0	5 7 8
1	
2	
3	
4	
5	

(a) Complete the stem and leaf diagram for the above data.

(b) Find the mode, the median and the interquartile range of the numbers of hours spent by the 25 students.

18C.23 HKALE MS 1996 1

A stem-and-leaf diagram for the test scores of 30 students is shown.

Stem (tens)	Leaf (units)
1	0
2	
3	0 2
4	4 5 8 9
5	0 1 2 6 8 8 9 9
6	2 3 3 5 5 8
7	1 2 2 4 4 4
8	2 5
9	1

(a) Find the mean, mode and interquartile range of these scores.

(b) If the score 71 is an incorrect record and the correct score is 11, which of the statistics in (a) will have different values? Find the correct values of these statistics.

18C.24 HKALE MS 1997 – 2

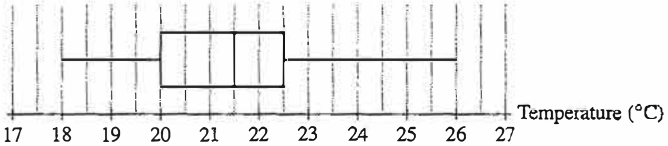
In an experiment, temperatures of a certain liquid under various experimental settings are measured. The box and whisker diagram for these temperatures (in °C) is constructed below.

(a) Find the range (in °C) of the temperatures.

(b) The temperature C (in °C) can be converted to the temperature F (in °F) according to the formula $F=\frac{9}{5}C+32$.

(i) Find the median and interquartile range of the temperatures in °F.

(ii) If the mean and standard deviation of the temperatures are 22°C and 2°C respectively, find their values in °F.



18C.25 HKALE MS 1999 – 3

A test was carried out to see how quickly a class of students reacted to a visual instruction to press a particular key when they played a computer game. Their reaction times, measured in tenths of a second, are recorded and the statistics for the whole class are summarised below.

	Lower quartile	Upper quartile	Median	Minimum	Maximum
Boys	8	14	11	5	17
Girls	9	16	11	7	21

(a) Draw two box-and whisker diagrams comparing the reaction times of boys and girls.

(b) Suppose a boy and a girl are randomly selected from the class. Which one will have a bigger chance of having a reaction time shorter than 1.1 seconds? Explain.

18C.26 HKALE MS 2000 – 5

A fitness centre advertised a programme specifically designed for women weighing 70 kg or more, and claimed that their individual weights could be reduced by at least 20 kg on completion of the programme. 21 women joined the programme and their weights in kg when they started are shown.

Stem (tens)	Leaf (units)
7	0 0 2 3 5 5 7
8	1 1 4 5 6 6 7 8
9	0 2 5 8 9 9

(a) Find the median and the interquartile range of these weights.

(b) On completion of the programme, the median, lower quartile and upper quartile of the weights of these women are 73 kg, 68 kg and 77 kg respectively. The lightest and heaviest women weigh 60 kg and 82 kg respectively. Draw two box and-whisker diagrams comparing the weights of these women before and after the programme.

(c) Referring to the box and-whisker diagram in (b), someone claimed that none of these women had reduced their individual weights by 20 kg or more on completion of the programme. Determine whether this claim is correct or not. Explain your answer briefly.

18C.27 HKALE MS 2001 – 3

The ages of 35 members of a golf club are shown below. It is known that the median and the range of the ages are 36 and 48 respectively, and the ages of the two eldest members differ by 1.

(a) Find the unknown digits a, b and c.

(b) The three members whose ages correspond to the three unknown digits a, b and c are replaced with three new members with ages 12, 38 and 68 respectively. Draw two box and whisker diagrams comparing the age distributions of the members before and after replacement.

Stem (tens)	Leaf (units)
1	a 8 8 9 9
2	0 1 2 3 3 4 7 8
3	1 2 2 5 b 9 9
4	0 2 5 5 6
5	2 2 5 5 8 8
6	0 1 c 6

18C.28 HKALE MS 2003 5

A researcher conducted a study on the time (in minutes) spent on using the Internet by university students. Thirty questionnaires were sent out and only 19 were returned. The results are as follows:

12	13	14	15	15	21	25	29
36	37	38	41	47	49	49	49
52	54	57					

(a) Construct a stem and leaf diagram for these data.

(b) Suppose that the research has received eight more questionnaires. Three of them show that the time spent on using the Internet is one hour. The other show that the time spent is more than one hour.

(i) Find the revised median and the revised interquartile range of the time spent.

(ii) Describe briefly the change in the mean and the change in the range of the time spent.

18C.29 HKALE MS 2004 – 5

Some statistics from a survey on the monthly incomes (in thousands of dollars) of a group of university graduates are summarised in the table.

(a) Using the above information, construct a box and whisker diagram to describe the distribution of the monthly incomes.

(b) A student proposes to model the distribution of the monthly incomes of the group of university graduates by a normal distribution with mean and standard deviation given in the table.

(i) *[Out of syllabus]*

(ii) Is the model proposed by the student appropriate? Explain your answer.

Minimum	8
Maximum	52
Lower quartile	10
Median	17
Upper quartile	20
Mean	17.94
Standard deviation	4.7

18C.30 HKALE MS 2005 – 4

The stem and leaf diagram below shows the distribution of heights in cm of 32 students.
It is found that three records less than 150 cm are incorrect. Each of them should be 10 cm greater than the original record. Find the change in each of the following statistics after correcting the three records:

(a) the mean,

(b) the median,

(c) the mode,

(d) the range,

(e) the interquartile range.

Stem (tens)	Leaf (units)
14	5 5 6 6
15	1 2 2 4 4 5 5 7 7 7 7 7 9
16	0 2 2 5 6 7 8 8 9
17	0 1 2 3 4 4

18C.31 HKALE MS 2006 – 4

The stem-and-leaf diagram shows the distribution of the numbers of books read by 24 students of a school in the first term.

Stem (tens)	Leaf (units)
0	3 4 6 7
1	1 2 2 3 5 6 7 8 8 9
2	1 3 4 5 5 7 8 9
3	0 0

(a) Find the median and the interquartile range of the numbers of books read.

(b) The librarian of the school ran a reading award scheme in the second term. The following table shows some statistics of the distribution of the numbers of books read by these 24 students in the second term:

Minimum	Lower quartile	Median	Upper quartile	Maximum
8	26	35	41	46

(i) Draw two box-and-whisker diagrams of the same scale to compare the numbers of books read by these students in the first term and in the second term.

(ii) The librarian claims that not less than 50% of these students read at least 5 more books in the second term than that in the first term. Do you agree? Explain your answer.

18C.32 HKALE MS 2007 – 4

Albert conducted a survey on the time spent (in hours) on watching television by 16 students. The data recorded are 3.7, 1.2, 2.1, 5.1, 2.1, 4.7, 1.9, 2.4, 2.4, 2.9, 3.6, 2.3, 3.9, 2.2, 1.8 and k, where k is the missing datum.

(a) Albert assumes that the range of these data is 5.3 hours.

(i) Find the value of k.

(ii) Construct a stem-and leaf diagram for these data.

(iii) Find the mean and the median of these data.

(b) Albert finds that the assumption in (a) is incorrect and he can only assume that the range of these data is greater than 5.3 hours. Describe the change in the mean and the change in the median of these data due to the revision of Albert's assumption.

18C.33 HKALE MS 2008 6

A test is taken by a class of 18 students. The marks are as follows:

55	82	74	70	91	75	79	89	68
79	59	72	79	73	60	71	82	k

where k is Jane's mark.
It is known that the mean mark of the class is the same irrespective of including or excluding Jane's.

(a) Find the value of k.

(b) If 3 student marks are selected randomly from the set of the 18 student marks, find the probability that exactly 1 of them is the mode of the set of the 18 student marks.

(c) A student mark is classified as an *outlier* if it lies outside the interval $(\mu-2\sigma, \mu+2\sigma)$, where μ is the mean and σ is the standard deviation of the set of marks.

(i) Find all the *outlier(s)* of the set of the 18 student marks.

(ii) In order to assess the students' performance in the test, all *outliers* are removed from the set. Describe the change in the median and the standard deviation of the student marks due to such removal.

18C.34 HKALE MS 2011 6

The revision times (in minutes) of 19 students are represented by the stem and leaf diagram in the figure. It is known that the mean revision time is $(40+b)$ minutes.

(a) Find a and b.

(b) Find the standard deviation of the revision times for the students.

(c) The revision times of 2 more students are added. If both the range and the mean do not change after the inclusion of the 2 data, find the range of possible values of the standard deviation of the revision times for the 21 students.

Tens	Units
2	6 7
3	0 0 a 2 9 9
4	b 3 3 3 6 8 8
5	6 9
6	5 9

18C.35 HKALE MS 2012 – 6 (To continue as **17C.35**.)

An educational psychologist adopts the Internet Addiction Test to measure the students' level of Internet ad diction. The scores of a random sample of 30 students are presented in the following stem and-leaf diagram. Let σ be the standard deviation of the scores. It is known that the mean of the scores is 71 and the range of the scores is 56.

(a) Find the values of a, b and σ.

Stem (tens)	Leaf (units)
3	a
4	
5	2 4 6 8
6	0 1 3 5 6 7 8 8 9
7	1 2 2 4 5 5 6 8
8	0 2 3 5 8
9	0 2 b

18C.36 HKDSE MA PP – I – 9

The following table shows the distribution of the numbers of online hours spent by a group of children on a certain day.

Number of online hours	2	3	4	5
Number of children	r	8	12	s

It is given that r and s are positive numbers.

(a) Find the least possible value and the greatest possible value of the inter quartile range of the distribution.

(b) If $r=9$ and the median of the distribution is 3, how many possible values of s are there? Explain your answer.

18C.37 HKDSE MA PP I – 15

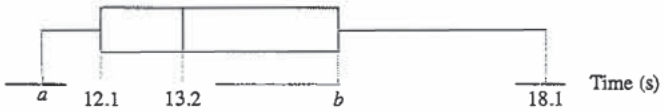
The mean score of a class of students in a test is 48 marks. The scores of Mary and John in the test are 36 marks and 66 marks respectively. The standard score of Mary in the test is -2.

(a) Find the standard score of John in the test.

(b) A student, David, withdraws from the class and his test score is then deleted. It is given that his test score is 48 marks. Will there be any change in the standard score of John due to the deletion of the test score of David? Explain your answer.

18C.38 HKDSE MA 2012 – I – 7

The box and whisker diagram below shows the distribution of the times taken by a large group of students of an athletic club to finish a 100 m race:



The inter quartile range and the range of the distribution are 3.2 s and 6.8 s respectively.

(a) Find a and b.

(b) The students join a training program. It is found that the longest time taken by the students to finish a 100 m race after the training is 2.9 s less than that before the training. The trainer claims that at least 25% of the students show improvement in the time taken to finish a 100 m race after the training. Do you agree? Explain your answer.

18C.39 HKDSE MA 2012 – I – 15

The standard deviation of the test scores obtained by a class of students in a Mathematics test is 10 marks. All the students fail in the test, so the test score of each student is adjusted such that each score is increased by 20% and then extra 5 marks are added.

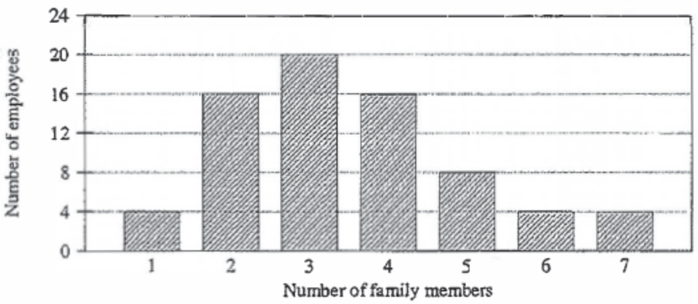
(a) Find the standard deviation of the test scores after the score adjustment.

(b) Is there any change in the standard score of each student due to the score adjustment? Explain your answer.

18C.40 HKDSE MA 2013 – I – 9

The bar chart shows the distribution of the numbers of family members of the employees of company D.

Distribution of the numbers of family members of the employees of company D



(a) Find the mean, the inter quartile range and the standard deviation of the above distribution.

(b) An employee leaves company D. The number of family members of this employee is 7. Find the change in the standard deviation of the numbers of family members of the employees of company D due to the leaving of this employee.

18C.41 HKDSE MA 2013 – I – 10

(To continue as 17C.37.)

The ages of the members of Committee A are shown as follows:

17	18	21	21	22	22	23	23	23	31
31	34	35	36	47	47	58	68	69	69

(a) Write down the median and the mode of the ages of the members of Committee A.

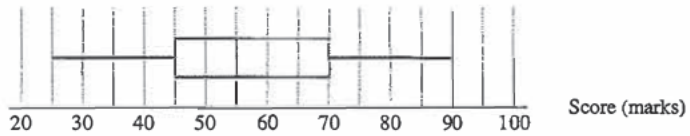
(b) The stem and leaf diagram shows the distribution of the ages of the members of Committee B. It is given that the range of this distribution is 47.

(i) Find a and b.

Stem (tens)	Leaf (units)
2	a 5 6 7
3	3 3 8
4	3
5	1 2 9
6	7 b

18C.42 HKDSE MA 2013 – I – 15

The box and whisker diagram below shows the distribution of the scores (in marks) of the students of a class in a test. Susan gets the highest score while Tom gets 65 marks in the test. The standard scores of Susan and Tom in the test are 3 and 0.5 respectively.



(a) Find the mean of the distribution.

(b) Susan claims that the standard scores of at least half of the students in the test are negative. Do you agree? Explain your answer.

18C.43 HKDSE MA 2014 – I – 4

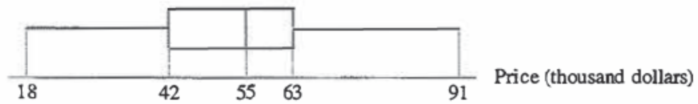
The table below shows the distribution of the numbers of calculators owned by some students.

Number of calculators	0	1	2	3
Number of students	7	14	15	4

Find the median, the mode and the standard deviation of the above distribution.

18C.44 HKDSE MA 2014 – I – 11

There are 33 paintings in an art gallery. The box-and whisker diagram below shows the distribution of the prices (in thousand dollars) of the paintings in the art gallery. It is given that the mean of this distribution is 53 thousand dollars.



(a) Find the range and the inter quartile range of the above distribution.

(b) Four paintings of respective prices (in thousand dollars) 32, 34, 58 and 59 are now donated to a museum. Find the mean and the median of the prices of the remaining paintings in the art gallery.

18C.45 HKDSE MA 2015 I 12

The stem-and-leaf diagram shows the distribution of the weights (in kg) of the students in a football club.

Stem (tens)	Leaf (units)
4	0 2 3 3 3 3 9
5	1 1 2 2 3 7 9
6	3 5 8 9
7	8 9

(a) Find the mean, the median and the range of the above distribution.

(b) Two more students now join the club. It is found that both the mean and the range of the distribution of the weights are increased by 1 kg. Find the weight of each of these students.

18C.46 HKDSE MA 2015 I – 15

The table below shows the means and the standard deviations of the scores of a large group of students in a Mathematics examinations and a Science examination:

Examination	Mean	Standard deviation
Mathematics	66 marks	12 marks
Science	52 marks	10 marks

The standard score of David in the Mathematics examination is -0.5.

(a) Find the score of David in the Mathematics examination.

(b) Assume that the scores in each of the above examinations are normally distributed. David gets 49 marks in the Science examination. He claims that relative to other students, he performs better in the Science examination than in the Mathematics examination. Is the claim correct? Explain your answer.

18C.47 HKDSE MA 2016 – I – 16

In a test, the mean of the distribution of the scores of a class of students is 61 marks. The standard scores of Albert and Mary are -2.6 and 1.4 respectively. Albert gets 22 marks. A student claims that the range of the distribution is at most 59 marks. Is the claim correct? Explain your answer.

18C.48 HKDSE MA 2017 I 11 (To continue as **17B.43**.)

The stem-and-leaf diagram shows the distribution of the hourly wages (in dollars) of the workers in a group.

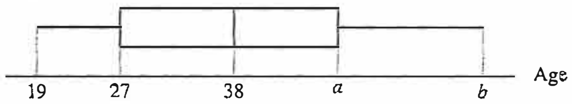
It is given that the mean and the range of the distribution are \$70 and \$22 respectively.

Stem (tens)	Leaf (units)
6	1 1 1 3 4 6 8 9 9
7	a 7 7 8
8	1 b

(a) Find the median and the standard deviation of the above distribution.

18C.49 HKDSE MA 2018 – I – 10

The box-and whisker diagram below shows the distribution of the ages of the clerks in team X of a company. It is given that the range and the inter-quartile range of this distribution are 43 and 21 respectively.



(a) Find a and b.

(b) There are five clerks in team Y of the company and three of them are of age 38. It is given that the range of the ages of the clerks in team Y is 20. Team X and team Y are now combined to form a section. The manager of the company claims that the range of the ages of the clerks in the section and the range of the ages of the clerks in team X must be the same. Do you agree? Explain your answer.

18C.50 HKDSE MA 2019 – I 12

The stem-and leaf diagram shows the distribution of the results (in seconds) of some boys in a 400 m race.

It is given that the inter-quartile range of the distribution is 8 seconds.

Stem (tens)	Leaf (units)
5	a
6	0 0 3 c c 8 9 9 9
7	0 1 1 1 2 2 5 6 9
8	b

(a) Find c.

(b) It is given that the range of the distribution exceeds 34 seconds and the mean of the distribution is 69 seconds. Find

(i) a and b,

(ii) the least possible standard deviation of the distribution.

18C.51 HKDSE MA 2020 – I – 9

The table below shows the distribution of the numbers of subjects taken by a class of students.

Number of subjects taken	4	5	6	7
Number of students	8	12	16	4

(a) Write down the mean, the median and the standard deviation of the above distribution.

(b) A new student now joins the class. The number of subjects taken by the new student is 5 . Find the change in the median of the distribution due to the joining of this student.

(5 marks)

18C.52 HKDSE MA 2020 – I –11

The stem-and-leaf diagram below shows the distribution of the weights (in grams) of the letters in a bag.

Stem (tens)	Leaf (units)
1	1 2 3 3
2	3 3 4 5 6 9 9
3	1 6 7 8 8 8
4	2
5	0 w

It is given that the range of the above distribution is the triple of its inter-quartile range.

(a) Find w. (4 marks)

(b) If a letter is randomly chosen from the bag, find the probability that the weight of the chosen letter is not less than the mode of the distribution. (2 marks)

18 Statistics

18A Presentation of data

18A.1 HKCEE MA 1982(1) – I – 7

(a) $x = 360 \times \frac{2}{2+7+3} = 60$
Similarly, $y = 210$, $z = 90$

(b) Total no. of students $= 240 \times \frac{2+7+3}{2} = 1440$

18A.2 HKCEE MA 1982(3) – I – 12

(a) Income $= \$2000 \times \frac{360°}{\underline{100°}} = \$7200$

(b) (i) (1) $x = 100(1+30\%) = 130$
$y = 50$
$z = 360 - 90(1+10\%) - 130$
$20(1+30\%) - 50 - 40 = 15$

(2) % change $= \frac{15-60}{60} \times 100\% = -75\%$

(ii) Income in April $= \$7200 \times (1+37.5\%) = \$9900$
Expense on food $= \$7200 \times \frac{90°}{\cancel{360°}} \times (1+10\%)$
$= \$1980$
$\therefore$ Required % $= \frac{1980}{\cancel{9900}} \times 100\% = 20\%$

18A.3 HKCEE MA 1985(A/B) – I – 7

$\angle$ of sector representing Kowloon $= 90° \times \frac{9240}{4200} = 198°$
$\therefore x° = 360° - 90° - 198° \Rightarrow x = 72$
$n = 4200 \times \frac{72°}{90°} = 3360$

18A.4 HKCEE MA 1998 – I – 10

(a)

$x \le 70$	102
$x \le 80$	158

$60 < x \le 70$	52
$70 < x \le 80$	56

(b) The line $x = 55$ meets the c.f. polygon at around $(55, 29)$.
$\therefore$ Passing % $= \frac{200-29}{200} \times 100\% = 85.5\%$

18A.5 HKCEE MA 1999 – I – 11

(a) $\angle$ of sector representing 'Repeated S.5' $= 72°$
$\therefore$ No of boys who repeated S.5 $= 120 \times \frac{72°}{\cancel{360°}} = 24$

(b) Required % $= \frac{126°}{126° + 144°} \times 100\% = 46\frac{2}{3}\%$

(c) No of boys promoted to S.6 in own school $= 120 \times \frac{126°}{\cancel{360°}}$
$= 42$
No of girls promoted to S.6 in own school $= 80 \times 22.5\%$
$= 18$
$\therefore$ Required % $= \frac{42+18}{200} \times 100\% = 30\%$

18A.6 HKCEE MA 2006 – I – 9

(a) $x = 360° - 40° - 90° - 130° - 35° - 30° = 35°$

(b) Total expenditure $= \$1750 \times \frac{360°}{35°} = \$18000$

(c) Expenditure on travelling $=$ Expenditure on transportation
$= \$1750$

18A.7 HKCEE MA 2007 – I – 12

(a) $k = 17 \times \frac{63°}{\underline{153°}} = 7$

(b) No of students $= 17 \times \frac{360°}{153°} = 40$

(c) No of students with 1 key $= 40 - 12 - 17 - 7 = 4$
$\therefore$ Required p $= \frac{4}{40} = \frac{1}{10}$

(d) Bar: Yes. Scales on the vertical axis should be doubled.
Pie: No

18A.8 HKDSE MA SP – I – 9

(a) $72 = (1+20\%)x \Rightarrow x = 60$

(b) $\angle$ of sector representing District C $= 78° > 72°$
$\therefore$ NO.

18A.9 HKDSE MA PP – I – 13

(a) Number of students $= 6 \div \frac{3}{20} = 40$
$\Rightarrow k = 40 - 6 - 11 \quad 5 \quad 10 = 8$

(b) (i) Required $\angle = 360° \times \frac{5}{40} = 45°$

(ii) Suppose n new students will double the $\angle$ for orange.
$\frac{5+n}{40+n} = \frac{45° \times 2}{\underline{360°}} \Rightarrow n = \frac{20}{3}$
But since n must be an integer, there is no n satisfying the condition. NO.

18A.10 HKDSE MA 2016 – I – 9

(a) $x = 2 + 4 = 6$
$y = 37 - 15 = 22$
$z = 37 + 3 = 40$

18B Measures of central tendency

18B.1 HKCEE MA 1983(B) – I – 3

(a) Class mark $= 54.5$

(b) Mean $= (44.5 \times 100 + 54.5 \times 300 + 64.5 \times 400$
$+ 74.5 \times 200) \div 1000 = 61.5$

18B.2 HKCEE MA 1984(A/B) – I – 2

$1 \times 10 + 2 \times 10 + 3 \times 5 + 4 \times 20 + 5x = 3(10 + 10 + 5 + 20 + x)$
$125 + 5x = 135 + 3x \Rightarrow x = 5$

18B.3 HKCEE MA 1986(A/B) – I – 3

$61 \times 40 + 70x + 50 \times 35 = 60(40 + x + 35)$
$4190 + 70x = 4500 + 60x \Rightarrow x = 31$

18B.4 HKCEE MA 1991 – I – 1

(a) 49.5

(b)

20 – 29	10
30 – 39	10
40 – 49	20
50 – 59	30
60 – 69	10

$\therefore$ Mean mark $= (24.5 \times 10 + 34.5 \times 10 + 44.4 \times 20$
$+ 54.5 \times 30 + 64.5 \times 10) \div 80 = 47$

18B.5 HKCEE MA 1992 – I – 8

(a) $70(m+n)$

(b) $70(m+n) = 75m + 62n \Rightarrow 5m = 8n \Rightarrow m : n = 8 : 5$

(c) No of men $= 39 \times \frac{8}{8+5} = 24$

18B.6 HKCEE MA 1994 – I – 1(d)

Mean $= 50$ Mode $= 65$ Median $= 60$

18B.7 HKCEE MA 1996 – I – 14

(a) 100 $\quad 20.0 - 4.9 - 13.7 - 26.3 - 10.3 - 1.1 = 23.7$

(b) Some round-off errors have accumulated.

(c) (i)

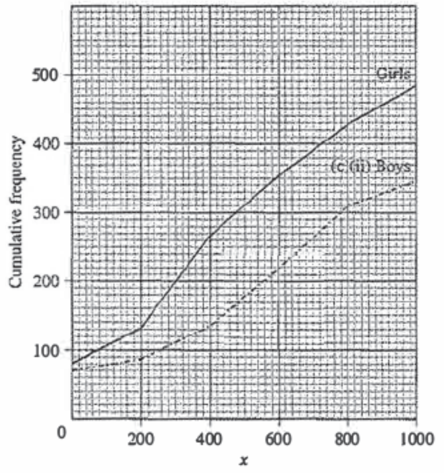
x	c.f.
$x \le 0$	70
$x \le 200$	87
$x \le 400$	135
$x \le 600$	218
$x \le 800$	310
$x \le 1000$	346

(ii) (See below)

(iii) For boys, median $= 490$; for girls, median $= 410$

(iv) Draw the vertical line $x = 700$. It meets the polygons at around $(700, 390)$ (girls) and $(700, 265)$ (boys).
$\therefore$ Required no $= 390 + 265 = 655$

(d) Referring to the first row of each table, the percentage of boys spending \$0 (20.0%) is indeed higher than the percentage of girls spending \$0 (15.0%).
However, the percentages have to be considered instead of the frequencies because the *total frequencies* of boys and of girls are *different*.



18B.8 HKCEE MA 1999 – I – 8

(a) $x + 161 + 168 + 159 + 161 + 152 = 158 \times 6 \Rightarrow x = 147$

(b) Median $= (159 + 161) \div 2 = 160$ (cm)

18B.9 HKCEE MA 2000 – I – 11

(a)

$t \le 280$	66

$260 < t \le 280$	20

(b) Mean $= (210 \times 3 + 230 \times 13 + 250 \times 30$
$+ 270 \times 20 + 290 \times 9) \div 75 = 255$ (s, 3 s.f.)

(c) Median $= 254$ seconds

(d) Required % $= \frac{13+30}{75} \times 100\% = 57.3\%$ (3 s.f.)

18B.10 HKCEE MA 2003 – I – 11

(a) (i) 10
(ii) 11.5
(iii) 12
(iv) $16 - 10 = 6$

(b) (i) (When all 4 new data are large,)
Least possible median $= (13 + 16) \div 2 = 14.5$
(When all 4 new data are small,)
Greatest possible median $= 10$
(ii) New mean $= (12 \times 6 + 11 \times 4) \div 10 = 11.6$

18B.11 HKCEE MA 2006 – I – 8

(a) $11 \times 10 = 86 + k \Rightarrow k = 24$

18B.12 HKALE MS 1998 – 3

(a) Median $= (161 + 162) \div 2 = 161.5$ (cm)

18B.13 HKALE MS 2002 – 7

(a) Mean $= 61$

(b) Since there are two modes, one deleted mark is 54.
The other mark $= 61 \times 22 - (61 + 1.2) \times 20 - 54 = 44$

18B.14 HKALE MS 2010 – 5

(a) $49-(20+a)=27 \Rightarrow a=2$

$\dfrac{49+\cdots+(80+b)}{20}-\dfrac{22+49+\cdots+(80+b)}{21}=2$

$\dfrac{1274+b}{20}-\dfrac{1296+b}{21}\quad 2$

$b=6$

$\bar{x}=(1296+6)\div 21=62$

18B.15 HKDSE MA SP – I – 14

(a) Median $=62\%$
Mean $=(55+58+62+62+63)\div 5=60(\%)$

(b) (i) 58% (when the new data are small)
(ii) (Mean unchanged $\Rightarrow$ Mean of a and $b=60$)
(Median unchanged $\Rightarrow$ $a\le 62$ and $b\ge 62$)
Possible pairs: $(57,63)$, $(56,64)$, $(55,65)$, etc.

(c) Possible reasons for NO:
- The week may not be randomly chosen.
- Only one stall is considered.

Possible reasons for YES:
- The week may be randomly chosen.
- There may only be very few stalls in H.

18B.16 HKDSE MA 2012 – I – 10

(a) Mean $=10+10+\cdots+36)\div 20=18$
Median $=16$

(b) (i) New mean $=$ Original mean $=18$
(ii) Let the new data be 19, 20, a and b.
Mean $=18 \Rightarrow a+b=18\times 4 \quad 19 \quad 20=33$
Since 19 and 20 exceed the original median, a and b must not exceed the original median if the median is unchanged. $\Rightarrow a+b\le 16+16=32$
Hence it is not possible.

18B.17 HKDSE MA 2016 – I – 12

(a) Median $=7.5 \Rightarrow$ No of 6 and 7 = No of 8, 9 and 10
$11+a=11+b+4$
$a=b+4$
$\because a>11$ and $4<b<10$
$\therefore (a,b) \quad (12,8)$ or $(13,9)$

(b) (i) Greatest possible median $=8$
(when the 4 new ages are 7, 8, 9 and 10)
(ii) Mean is least when the 4 new ages are 6, 7, 8 and 9.
If $(a,b)=(12,8)$, mean $=(6\times 12+7\times 13+8\times 12+9\times 9+10\times 4)\div(12+13+12+9+4)=7.6$
If $(a,b)=(13,9)$, mean $=(6\times 12+7\times 14+8\times 12+9\times 10+10\times 4)\div(12+14+12+10+4)=7.62$
$\therefore$ Least possible mean $=7.6$

18B.18 HKDSE MA 2018 – I – 11

(a) (i) 1
(ii) 8

(b) (i) 3 (when the '9th 2' is the median)
(ii) 19 (when the '1st 2' is the median)

(c) $(0\times k+1\times 2+2\times 9+3\times 6+4\times 7)\div(k+2+9+6+7)=2 \Rightarrow k=9$

18B.19 HKDSE MA 2019 – I – 8

(a) 2

(b) Mean $=2\times\dfrac{144^\circ}{360^\circ}+3\times\dfrac{54^\circ}{360^\circ}+5\times\dfrac{72^\circ}{360^\circ}+7\times\dfrac{90^\circ}{360^\circ}=4$

18C Measures of Dispersion

18C.1 HKCEE MA 1980(3) – I – 8

(a) Class B (since its dispersion is greater)

(b) 10 students fail the test.
$\Rightarrow$ Students getting 0, 1 and 2 marks fail the test.
$\Rightarrow$ Min mark to pass test $=3$

18C.2 HKCEE MA 1981(1) – I – 6

(a) The line $y=25$ meets the polygon at around $(43,25)$.
$\therefore$ Pass mark $=43$

(b) The line $x=40$ meets the polygon at around $(40,20)$.
$\therefore$ $100-20=80$ students would pass.

(c) IQR $=70-43=27$

18C.3 HKCEE MA 1983(A) – I – 3

(a) Mean $=[(a\quad 6)+a+(a+2)+(a+3)+(a+6)]\div 5$
$=a+1$

(b) SD $=$ SD of $\{\quad 6,0,2,3,6\}=4$

18C.4 HKCEE MA 1988 – I – 11

(a) (i) Median $=70$ marks
(ii) IQR $=86 \quad 50=36$ (marks)

(b) (i) Number of students $=600-540=60$
(ii) Required p $=\dfrac{60}{600}=\dfrac{1}{10}$
(iii) (1) Required p $=\dfrac{C_2^{60}}{C_2^{600}}=\dfrac{59}{5990}$
(2) Required p $=1-\dfrac{C_2^{540}}{C_2^{600}}=\dfrac{1139}{5990}$

18C.5 HKCEE MA 1990 – I – 12

(a) (i) Modal class $=\$6000 \quad \$7000$
Median $=\$6500$
Mean $=\$6500$ (since the distribution is symmetric)
IQR $=Q_3-Q_1=\dfrac{7500+8500}{2}\quad\dfrac{4500+5500}{2}$
$=(\$)3000$
(ii) $\because$ More data are close to the mean
$\therefore$ SD becomes smaller.

(b) SD $=\sqrt{\dfrac{9+4+1+0+1+4+9}{7}}=2$

18C.6 HKCEE MA 1993 – I – 7

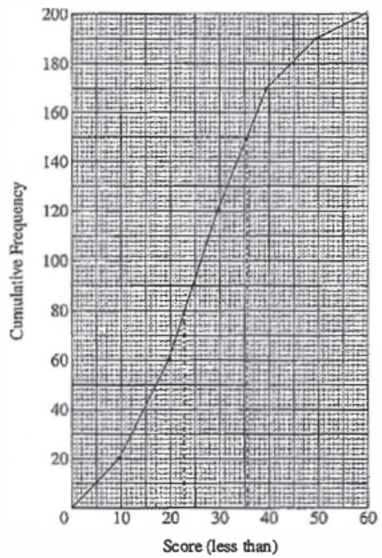
(a)

9.5	**20**
19.5	**60**
29.5	**120**
39.5	**170**
49.5	**190**
59.5	**200**

(b) (i) (See below)
Hence, $IQR=36-17=19$ (or $35 \quad 17=18$)
(ii) $\because$ $200\times 60\%=80$ students pass the test.
$\therefore$ The passing score should be 23.

(c) SD $=12.9$

(d) SD $=12.9$ (i.e. unchanged)



18C.7 HKCEE MA 1995 – I – 9

(a) (i) 180
(ii) 60

(b) (25% of students $=45$)
(i) The horizontal line at 135 meets the graph at around $(75,135)$.
$\therefore$ 75
(ii) The horizontal line at 45 meets the graph at around $(44,45)$.
$\therefore$ 44

(c)

$20<x\le 30$	25	**12**
$30<x\le 40$	35	20
$40<x\le 50$	**45**	**28**
$50<x\le 60$	55	32
$60<x\le 70$	**65**	28
$70<x\le 80$	**75**	30
$80<x\le 90$	**85**	22
$90<x\le 100$	95	**8**

(d) Mean $=59.6$, SD $=19.0$ (3 s.f.)

(e) $\bar{x}\quad\sigma=40.6$, $\bar{x}+\sigma=78.6$
No. of students within this range $=146 \quad 34=112$
$\therefore$ Required % $=\dfrac{112}{180}\times 100\%=62.2\%$

18C.8 HKCEE MA 1997 – I – 11

(a) Mean $=64.4$, Mode $=95$, Median $=78$, SD $=30.6$

(b) There are several extremely small data.

(c) (i) Required mark $=63+0.4\times 15=69$
(ii) (1) Required % $=\dfrac{17}{35}\times 100\%=48.6\%$ (3 s.f.)
(2) *Method 1 – Standard score*
S.S. in Maths $=\dfrac{78-64.4}{30.6}=0.44$
S.S. in Eng $=\dfrac{78-63}{15}=1>0.44$
$\therefore$ Performance in Eng was better.
Method 2 – Use distribution
In Maths, her score was the median. Thus, not more than half of the classmates perform worse than her.
In Eng, her score was above the mean. Thus, more than half of the classmates perform worse than her.
$\therefore$ Performance in Eng was better.
(iii) New mean $=(63\times 35+10)\div 35=63.3$

18C.9 HKCEE MA 2001 – I – 10

(a)

Score (x)	Class mid-value (Class mark)	Frequency
$44\le x<52$	**48**	3
$52\le x<60$	**56**	**9**
$60\le x<68$	64	15
$68\le x<76$	**72**	11
$76\le x<84$	80	**2**

(b) Mean $=64$
SD $=8$

(c) S.S. $=(76\quad 64)\div 8=1.5$

(d) Let x be her score in the second test.
$1.5=\dfrac{x-58}{10} \Rightarrow x=73$
The required score is 73.

18C.10 HKCEE MA 2002 – I – 5

(a) 9.6
(b) 13
(c) 10
(d) 4.59

18C.11 HKCEE MA 2002 – I – 12

(a)

$0<x\le 5$	66	Certificate
$5<x\le 15$	**34**	Book coupon
$15<x\le 25$	64	Bronze medal
$25<x\le 35$	**26**	Silver medal
$35<x\le 50$	10	Gold medal

(b) IQR $=23-4=19$

18C.12 HKCEE MA 2004 – I – 11

(a) S.S. in Paper I $=\dfrac{54-46.1}{15.2}=0.520$
S.S. in Paper II $=\dfrac{66\quad 60.3}{11.6}=0.491<$ S.S. in Paper I
$\therefore$ NO.

(b) New mean $=50.1$ marks
New median $=50$ marks
New range $=91$ marks

18C.13 HKCEE MA 2005 – I – 15

(a) Mean = 122 marks, SD = 22 marks

(b) Top 20% = 4 students
Mary's score = 122 + 22 = 144 marks, which is not within the top 4 students.
∴ NO.

(c) (i) (Mean unchanged ⇒ Datum deleted is 122.)
Required p $= \frac{1}{20}$

(ii) (Mean unchanged ⇒ Sum of data deleted is 122×2)
Required p $= \frac{2}{C_2^{20}} = \frac{1}{95}$

18C.14 HKCEE MA 2006 – I – 14

(a) (i) Class A: IQR $= 39 - 18 = 21$ (marks)
Class B: IQR $= 25 - 11 = 14$ (marks)

(ii) ∵ IQR of B < IQR of A
∴ Class B is less dispersed.

18C.15 HKCEE MA 2007 – I – 4

Median = 67 kg
Range = 25 kg
SD = 7.65 kg

18C.16 HKCEE MA 2008 – I – 10

(a) $a = 4$
$b = 37 \quad 12 - 4 = 21$
$c = 50 - a - 12 \quad b - 10 = 3$

(b) Mean = 3.28 kg
SD = 0.299 kg

18C.17 HKCEE MA 2008 – I – 14

(a) (i) Required p $= \frac{9}{15} = \frac{3}{5}$

(ii) (1) Required p $= \frac{8 \times 15}{C_2^{36}} = \frac{4}{21}$

(2) Required p $= 1 - \frac{C_2^8}{C_2^{36}} \quad \frac{C_2^{15}}{C_2^{36}} - \frac{C_2^{13}}{C_2^{36}} = \frac{419}{630}$

(b) (i) Median = 5000 dollars
IQR $= 6400 \quad 4300 = 2100$ (dollars)

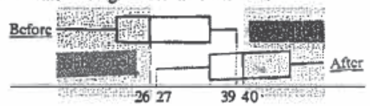
(ii) Extra $1000 to each salesgirl

18C.18 HKCEE MA 2009 – I – 10

(a) Median = 26 wpm
Range = 27 wpm
IQR $= 35 - 21 = 14$ (wpm)

(b) (i) *Method 1*
Range after training = 25 wpm < 27 wpm ⇒ NO
Method 2
IQR after training = 12 wpm < 14 wpm ⇒ NO

(ii) *Method 1*
Before the training, no speed was higher than 39 wpm. After the training, at least half of the speeds are 40 wpm or above. ⇒ YES.
Method 2
Before the training, at least half of the speeds were 26 wpm or below. After the training, their speeds become at least 27 wpm. ⇒ YES.
Remarks
To look for arguments against these claims, it is often helpful to provide yourself with a sketch of the box-and-whisker diagram for the other data.



18C.19 HKCEE MA 2010 – I – 11

(a) Mean = 25
Median = 26
Range = 13

(b) (i) Let x be the mean age of the 3 new players.
$(55 \times 22 \quad 31 \quad 31 + 3x) \div 23 = 25 \Rightarrow x = 29.$
∴ The required mean is 29.

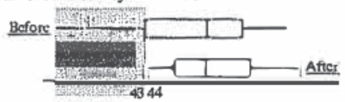
(ii) Median unchanged: If one new player is younger than the median, the other two has to be older – then the median will be the 12th datum; if two younger and one older than the mean, the median will be the 11th datum instead (which is still 26).
Range unchanged: New ages within 18 to 31
Possible ages: $\{25, 31, 31\}, \{26, 30, 31\}, \{27, 29, 31\}, \{28, 28, 31\}$

18C.20 HKCEE MA 2011 – I – 10

(a) Median = 57
Range $= 75 \quad 23 = 52$
IQR $= 65 - 43 = 22$

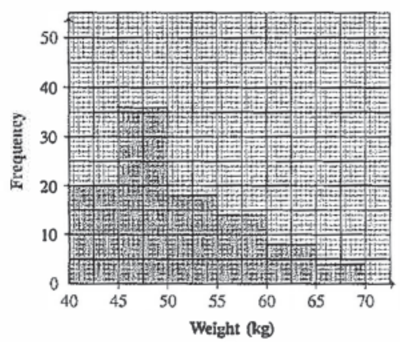
(b) (i) *Method 1*
IQR in 2nd survey $= 67 - 50 = 17 < 22$ ⇒ YES.
Method 2
Range in 2nd survey $= 78 - 44 = 34 < 52$ ⇒ YES.

(ii) At least 25% of the scores were 43 or below in the first survey. They have all improved to at least 44 in the second survey. ⇒ YES



18C.21 HKALE MS 1994 – 4

(a) **Weights of a group of 100 students**



(b) IQR $= 55 \quad 46 = 9$ (kg)

(c) Mean $= \frac{42.5 \times 20 + \cdot \quad + 67.5 \times 4}{100} = 50.8$ (kg)

18C.22 HKALE MS 1995 – 1

(a)

Stem (in 10)	Leaf (in 1)									
0	5	7	8							
1	0	1	2	2	5	8	8	8	8	9
2	0	1	2	3	5	5	5	6	9	
3	0	2								
4										
5	0									

(b) Mode = 18
Median = 19
IQR $= 25 - 12 = 13$

18C.23 HKALE MS 1996 – 1

(a) Mean = 59.4
Mode = 74
IQR $= 72 - 50 = 22$

(b) Mean becomes 57.4.
IQR becomes $72 - 49 = 23$

18C.24 HKALE MS 1997 – 2

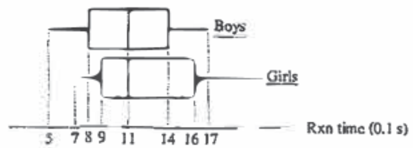
(a) $26 - 18 = 8$ (°C)

(b) (i) Median = 21.5 °C = 70.7 °F
IQR $= \left[\frac{9}{5}(22.5) + 32\right] \quad \left[\frac{9}{5}(20) + 32\right] = 4.5$ (°F)

(ii) Mean $= \frac{9}{5}(22) + 32 = 71.6$ (°F)
SD $= \frac{9}{5}(2) = 3.6$ (°F)

18C.25 HKALE MS 1999 – 3

(a)

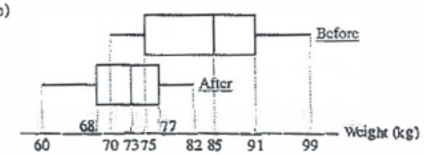


(b) Equal chance since both of the probabilities will be 0.5

18C.26 HKALE MS 2000 – 5

(a) Median = 85 kg
IQR $= 91 - 75 = 16$ (kg)

(b)

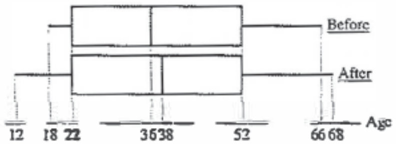


(c) No conclusion can be drawn as the diagrams show no individual difference.

18C.27 HKALE MS 2001 – 3

(a) $48 = 66 \quad (10 + a) \Rightarrow a = 8$
$30 + b = 36 \Rightarrow b = 6$
$c = 6 \quad l = 5$

(b)



18C.28 HKALE MS 2003 – 5

(a)

Stem (10 mins)	Leaf (1 min)				
1	2	3	4	5	5
2	1	5	9		
3	6	7	8		
4	1	7	9	9	9
5	2	4	7		

(b) (i) Revised median = 49 mins
Revised IQR $= 60 - 25 = 35$ (mins)

(ii) Both will become larger.

18C.29 HKALE MS 2004 – 5

(a)

(b) (ii) Since the distribution is not symmetrical, the normal distribution is not an appropriate model.

18C.30 HKALE MS 2005 – 4

(a) Change in mean = Change in sum ÷ 32
$= (3 \times 10) \div 32 = 0.9375$ (cm)

(b) Median unchanged

(c) Mode unchanged

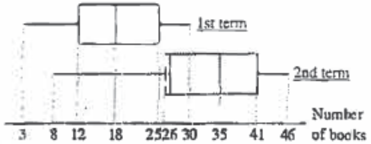
(d) Case 1: The 3 data were 145, 145 and 146.
Change in range = −1
Case 2: The 3 data were 145, 146 and 146.
Range unchanged

(e) Original IQR $= 168 - 154 = 14$
New IQR $= 168 - 155 \quad 13$
⇒ Change = −1

18C.31 HKALE MS 2006 – 4

(a) Median = 18
IQR = 25 − 12 = 13

(b) (i)



(ii) In 1st term, the maximum number was 30. In 2nd term, at least half of the numbers are 35 or above. Hence, at least 50% of students have read at least (35 − 30 =) 5 more books. Agreed.

18C.32 HKALE MS 2007 – 4

(a) (i) (k is the largest datum since 5.1 cannot be.)
$k = 1.2 + 5.3 = 6.5$

(ii)

Stem (1 hour)	Leaf (0.1 hour)
1	2 8 9
2	1 1 2 3 4 4 9
3	6 7 9
4	7
5	1
6	5

(iii) Mean = 3.05 hours
Median = 2.4 hours

(b) Mean will become larger.
Median will be unchanged.

18C.33 HKALE MS 2008 – 6

(a) k = Mean of the other 17 students = 74

(b) Required p $= \dfrac{C_1^3 C_2^{15}}{C_3^{18}} = \dfrac{105}{272}$

(c) (i) SD $= \sigma = 9.327$
Hence the interval is $(74 - 2\sigma + 2\sigma) = (55.3, 92.7)$
∴ 55 is the only outlier.

(ii) Median unchanged. SD will decrease.

18C.34 HKALE MS 2011 – 6

(a) $(40+b)19 = 743 + (30+a) + (40+b) \Rightarrow a = 18b - 53$
Since a and b are integers, $0 \le a \le 2$ and $0 \le b \le 3$,
$b = 3 \Rightarrow a = 1$

(b) 12.2 minutes

(c) Range unchanged: New data within 26 and 69
Mean unchanged: New data are $(43 - x, 43 + x)$
SD is smallest when both new data are 43.
⇒ Least possible SD = 11.6 mins
SD is greatest when the data are 26 and 60.
⇒ Greatest possible SD = 12.7 mins

18C.35 HKALE MS 2012 – 6

(a) $\dfrac{(30+a) + 52 + \cdots + 92 + (90+b)}{30} = 71$
$2120 + a + b = 2130$
$a + b = 10$
$(90+b) - (30+a) = 56 \Rightarrow a - b = 4$
Solving, $a = 7, b = 3$
$\Rightarrow \sigma = 12.7$

18C.36 HKDSE MA PP – I – 9

(a) Least possible IQR = 0
(when there are many many 2's or many many 5's)
Greatest possible IQR = 5 − 2 = 3

(b) $9 + 8 > 12 + s \Rightarrow s < 5$
∴ s = 1, 2, 3 or 4; i.e. 4 possible values of s

18C.37 HKDSE MA PP – I – 15

(a) SD $= (36 - 48) \div (-2) = 6$
∴ S.S. of John $= \dfrac{66 - 48}{6} = 3$

(b) Mean unchanged
SD increases (since 'more' data are 'far away' from mean)
∴ YES (decrease)

18C.38 HKDSE MA 2012 – I – 7

(a) $a = 18.1 - 6.8 = 11.3$
$b = 12.1 + 3.2 = 15.3$

(b) New longest time = 18.1 − 2.9 = 15.2 (s)
Before the program, at least 25% of students take 15.3 s or longer. After the program, they have shortened their time by at least 0.1 s. ⇒ YES.

18C.39 HKDSE MA 2012 – I – 15

(a) New SD = 10 × (1 + 20%) = 12

(b) Upon adjustment, the deviation of each score from the mean is increased by 20% while the SD is also increased by 20%. By the formula S.S. $= \dfrac{\text{Deviation}}{\text{SD}}$, there is no change in the standard score for each score

18C.40 HKDSE MA 2013 – I – 9

(a) Mean $= \dfrac{1 \times 4 + 2 \times 16 + \cdots + 7 \times 4}{4 + 16 + \cdots + 4} = 3.5$
IQR = 4 − 2 = 2
SD = 1.5

(b) New SD = 1.451456
∴ Change = 1.451456 − 1.5 = 0.0485

18C.41 HKDSE MA 2013 – I – 10

(a) Median = 31
Mode = 23

(b) (i) $(60+b) - (20+a) = 47 \Rightarrow b - a = 7$
∵ $0 \le a \le 5$ and $7 \le b \le 9$
∴ $(a, b) = (0, 7), (1, 8)$ or $(2, 9)$

(ii) Required p $= \dfrac{3+3+3+3+2+9+9}{20 \times 13} = \dfrac{8}{65}$

18C.42 HKDSE MA 2013 – I – 15

(a) Let $\bar{x}$ and σ be the mean and SD.
$\begin{cases} 90 = \bar{x} + 3\sigma \\ 65 = \bar{x} + 0.5\sigma \end{cases} \Rightarrow \begin{cases} \bar{x} = 60 \\ \sigma = 10 \end{cases}$

(b) Scores below the mean have negative standard scores. From the box-and-whisker diagram, at least half of students scored 55 or below. Hence they must have negative standard scores. ⇒ YES.

18C.43 HKDSE MA 2014 – I – 4

Median = 1
Mode = 2
SD = 0.889

18C.44 HKDSE MA 2014 – I – 11

(a) Range = 91 − 18 = 73 (000 dollars)
IQR = 63 − 42 = 21 (000 dollars)

(b) New mean $= (53 \times 33 - 32 - 34 - 58 - 59) \div 29$
= 54 (000 dollars)
New median = original median = 55 (000 dollars)

18C.45 HKDSE MA 2015 – I – 12

(a) Mean = 55 kg
Median = 52 kg
Range = 79 − 40 = 39 (kg)

(b) Let the new weights be a and b (kg).
$a + b + 55 \times 20 = 56 \times 22 \Rightarrow a + b = 132$
Since the range is increased by only 1,
If $a = 39$, then $b = 132 - 39 = 93$ (rejected)
If $b = 80$, then $a = 132 - 80 = 52$
Hence the only possibility is 52 kg and 80 kg.

18C.46 HKDSE MA 2015 – I – 15

(a) Score of David = 66 − 0.5(12) = 60

(b) S.S. in Science $= \dfrac{49 - 52}{10} = -0.3 >$ S.S. in Maths
∴ YES

18C.47 HKDSE MA 2016 – I – 16

SD $= (22 - 61) \div (-2.6) = 15$
⇒ Score of Mary = 61 + 1.4(15) = 82
∵ Range ≥ 82 − 22 = 60
∴ The claim is wrong.

18C.48 HKDSE MA 2017 – I – 11

(a) $(80+b) - 61 = 22 \Rightarrow b = 3$
$\dfrac{61 + \cdots + (70+a) + \cdots + 83}{15} = 70 \Rightarrow a = 2$
∴ Median = \$69, SD = \$7.33

(b) Required p $= \dfrac{6}{15} = \dfrac{2}{5}$

18C.49 HKDSE MA 2018 – I – 10

(a) $a = 27 + 21 = 48$
$b = 19 + 43 = 62$

(b) Least possible age in Team Y = 38 − 20 = 18
Since 18 < 19, the range of the new section would be larger than that of Team X. Disagreed.

18C.50 HKDSE MA 2019 – I – 12

(a) IQR $= 72 - (60 + c) = 8 \Rightarrow c = 4$

(b) (i) $(80+b) - (50+a) > 34 \Rightarrow b - a > 4$
$(50+a) + 60 + 60 + \cdots + 79 + (80+b) = 69 \times 20$
$\Rightarrow a + b = 7$
∴ $(a, b) = (0, 7)$ or $(1, 6)$

(ii) SD is smaller when the data are less dispersed.
∴ Least possible SD occurs when $(a, b) = (1, 6)$
By the calculator, Least possible SD = 7.34 (3 s.f.)

18C.51 HKDSE MA 2020 – I – 9

9a The mean is 5.4.
The median is 5.5.
The standard deviation is 0.917 (corr. to 3 sig. fig.).

b The new number of students = 8 + 12 + 16 + 4 + 1
= 41
Therefore, the median is the 21st smallest number of subjects taken.
Hence, the new median is 5.
The change in the median of the distribution = 5 − 5.5
= −0.5

18C.52 HKDSE MA 2020 – I – 11

11a The inter-quartile range $= \dfrac{38+38}{2} - \dfrac{23+23}{2}$
$= 15$
Since the range of the distribution is the triple of its inter-quartile range,
$(50 + w) - 11 = 15 \times 3$
$w = 6$

b The mode of the distribution is 38 g.
The required probability $= \dfrac{6}{20}$
$= \dfrac{3}{10}$