9 Arithmetic and Geometric Sequences

9A General terms and summations of sequences

9A.1 HKCEE MA 1980(1/1*/3) – I – 11

Let $k > 0$.

(a) (i) Find the common ratio of the geometric sequence $k, 10k, 100k$.
(ii) Find the sum of the first n terms of the geometric sequence $k, 10k, 100k, \ldots$.

(b) (i) Show that $\log_{10} k, \log_{10} 10k, \log_{10} 100k$ is an arithmetic sequence.
(ii) Find the sum of the first n terms of the arithmetic sequence $\log_{10} k, \log_{10} 10k, \log_{10} 100k, \ldots$. Also, if $n = 10$, what is the sum?

9A.2 HKCEE MA 1984(A/B) – I – 10

a and b are positive numbers. $a, -2, b$ is a geometric sequence and $2, b, a$ is an arithmetic sequence.

(a) Find the value of ab.
(b) Find the values of a and b.
(c) (i) Find the sum to infinity of the geometric sequence $a, -2, b, \ldots$.
(ii) Find the sum to infinity of all the terms that are positive in the geometric sequence $a, -2, b, \ldots$.

9A.3 HKCEE MA 1986(A/B I) – B – 9

$2, -1, -4, \ldots$ form an arithmetic sequence.

(a) Find
(i) the nth term,
(ii) the sum of the first n terms,
(iii) the sum of the sequence from the 21st term to the 30th term.

(b) If the sum of the first n terms of the sequence is less than -1000, find the least value of n.

9A.4 HKCEE MA 1989 – I – 9

The positive numbers $1, k, \frac{1}{2}, \ldots$ form a geometric sequence.

(a) Find the value of k, leaving your answer in surd form.
(b) Express the nth term $T(n)$ in terms of n.
(c) Find the sum to infinity, expressing your answer in the form $p + \sqrt{q}$, where p and q are integers.
(d) Express the product $T(1) \times T(3) \times T(5) \times \cdots \times T(2n-1)$ in terms of n.

9A.5 HKCEE MA 1995 – I – 3

(a) Find the sum of the first 20 terms of the arithmetic sequence $1, 5, 9, \ldots$.
(b) Find the sum to infinity of the geometric sequence $9, 3, 1, \ldots$.

9A.6 HKCEE MA 1996 – I – 3

The n-th term T_n of a sequence $T_1, T_2, T_3, \ldots$ is $7 - 3n$.

(a) Write down the first 4 terms of the sequence.
(b) Find the sum of the first 100 terms of the sequence.

9A.7 HKCEE MA 2003 – I – 7

Consider the arithmetic sequence $2, 5, 8, \ldots$. Find

(a) the 10th term of this sequence,
(b) the sum of the first 10 terms of this sequence.

9A.8 HKCEE MA 2005 – I – 7

The 1st term and the 2nd term of an arithmetic sequence are 5 and 8 respectively. If the sum of the first n terms of the sequence is 3925, find n.

9A.9 HKDSE MA 2015 – I – 17

For any positive integer n, let $A(n) = 4n - 5$ and $B(n) = 10^{4n-5}$.

(a) Express $A(1) + A(2) + A(3) + \cdots + A(n)$ in terms of n.
(b) Find the greatest value of n such that $\log(B(1)B(2)B(3)\ldots B(n)) \le 8000$.

9A.10 HKDSE MA 2016 – I – 17

The 1st term and the 38th term of an arithmetic sequence are 666 and 555 respectively. Find

(a) the common difference of the sequence,
(b) the greatest value of n such that the sum of the first n terms of the sequence is positive.

9A.11 HKDSE MA 2018 – I – 16

The 3rd term and the 4th term of a geometric sequence are 720 and 864 respectively.

(a) Find the 1st term of the sequence.
(b) Find the greatest value of n such that the sum of the $(n+1)$th term and the $(2n+1)$th term is less than 5×10^{14}.

9A.12 HKDSE MA 2019 – I – 16

Let α and β be real numbers such that $\begin{cases} \beta = 5\alpha - 18 \\ \beta = \alpha^2 - 13\alpha + 63 \end{cases}$.

(a) Find α and β.
(b) The 1st term and the 2nd term of an arithmetic sequence are $\log \alpha$ and $\log \beta$ respectively. Find the least value of n such that the sum of the first n terms of the sequence is greater than 888.

9A.13 HKDSE MA 2020 – I – 16

The 3rd term and the 6th term of a geometric sequence are 144 and 486 respectively.

(a) Find the 1st term of the sequence. (2 marks)

(b) Find the least value of n such that the sum of the first n terms of the sequence is greater than 8×10^{18}. (3 marks)

9B Applications

9B.1 HKCEE MA 1981(1/2/3) – I – 10

In Figure (1), B_1C_1CD is a square inscribed in the right angled triangle ABC. $\angle C = 90°$, $BC = a$, $AC = 2a$, $B_1C_1 = b$.

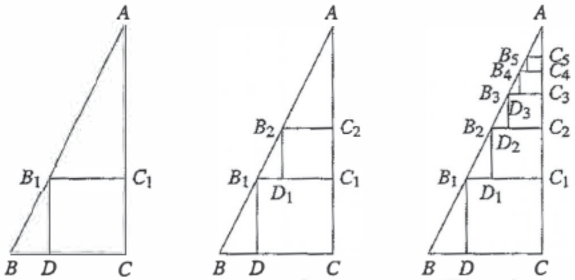


Figure (1) Figure (2) Figure (3)

(a) Express b in terms of a.

(b) $B_2C_2C_1D_1$ is a square inscribed in $\triangle AB_1C_1$ (see Figure (2)).

(i) Express B_2C_2 in terms of b.

(ii) Hence express B_2C_2 in terms of a.

(c) If squares $B_3C_3C_2D_2$, $B_4C_4C_3D_3$, $B_5C_5C_4D_4$, ... are drawn successively as indicated in Figure (3),

(i) write down the length of B_5C_5 in terms of a.

(ii) find, in terms of a, the sum of the areas of the infinitely many squares drawn in this way.

9B.2 HKCEE MA 1982(1/2/3) I 10

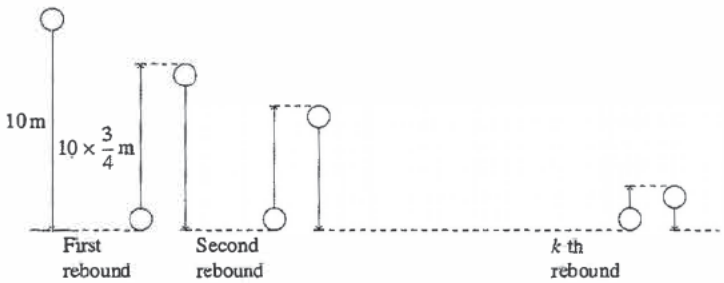
(a) (i) Find the sum of all the multiples of 3 from 1 to 1000.

(ii) Find the sum of all the multiples of 4 from 1 to 1000 (including 1000).

(b) Hence, or otherwise, find the sum of all the integers from 1 to 1000 (including 1 and 1000) which are neither multiples of 3 nor multiples of 4.

9B.3 HKCEE MA 1983(A/B) – I – 10

A ball is dropped vertically from a height of 10 m, and when it reaches the ground, it rebounds to a height of $10 \times \frac{3}{4}$ m. The ball continues to fall and rebound again, each time rebounding to $\frac{3}{4}$ of the height from which it previously fell (see the figure).



(a) Find the total distance travelled by the ball just before it makes its second rebound.

(b) Find, in terms of k, the total distance travelled by the ball just before it makes its $(k+1)$st rebound.

(c) Find the total distance travelled by the ball before it comes to rest.

9B.4 HKCEE MA 1985(A/B) – I 14

$\$P$ is deposited in a bank at the interest rate of $r\%$ per annum compounded annually. At the end of each year, $\frac{1}{3}$ of the amount in the account (including principal and interest) is drawn out and the remainder is redeposited at the same rate.

Let $\$Q_1$, $\$Q_2$, $\$Q_3$, ... denote respectively the sums of money drawn out at the end of the first year, second year, third year, ...

(a) (i) Express Q_1 and Q_2 in terms of P and r.

(ii) Show that $Q_3 = \frac{4}{27}P(1+r\%)^3$.

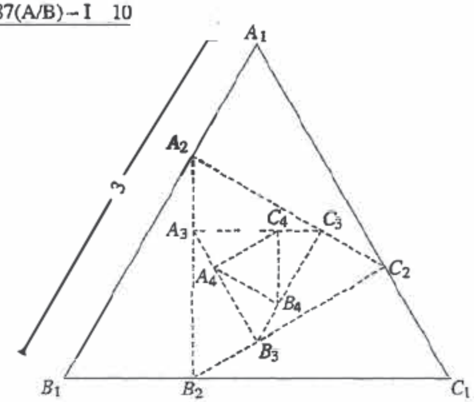
(b) $Q_1, Q_2, Q_3, \ldots$ form a geometric sequence. Find the common ratio in terms of r.

(c) Suppose $Q_3 = \frac{27}{128}P$.

(i) Find the value of r.

(ii) If $P = 10000$, find $Q_1 + Q_2 + Q_3 + \cdots + Q_{10}$. (Give your answer correct to the nearest integer.)

9B.5 HKCEE MA 1987(A/B) – I 10



In this quesiton you should leave your answers in surd form.

In the figure, $A_1B_1C_1$ is an equilateral triangle of side 3 and area T_1.

(a) Find T_1.

(b) The points A_2, B_2 and C_2 divide internally the line segments A_1B_1, B_1C_1 and C_1A_1 respectively in the same ratio 1 : 2. The area of $\triangle A_2B_2C_2$ is T_2.

(i) Find A_2B_2.

(ii) Find T_2.

(c) Triangles $A_3B_3C_3$, $A_4B_4C_4$, ... are constructed in a similar way. Their areas are $T_3, T_4, \ldots$, respectively. It is known that $T_1, T_2, T_3, T_4, \ldots$ form a geometric sequence.

(i) Find the common ratio.

(ii) Find T_n.

(iii) Find the value of $T_1 + T_2 + \cdots + T_n$.

(iv) Find the sum to infinity of the geometric sequence.

9B.6 HKCEE MA 1988 – I – 9

(a) Write down the smallest and the largest multiples of 7 between 100 and 999.

(b) How many multiples of 7 are there between 100 and 999? Find the sum of these multiples.

(c) Find the sum of all positive three digit integers which are **NOT** divisible by 7.

9B.7 HKCEE MA 1990 I – 14

The positive integers $1, 2, 3\ldots$ are divided into groups $G_1, G_2, G_3, \ldots$, so that the k^{th} group G_k consists of k consecutive integers as follows:

$$\begin{aligned} G_1 &: 1 \\ G_2 &: 2, 3 \\ G_3 &: 4, 5, 6 \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ G_{k-1} &: u_1, u_2, \ldots, u_{k-1} \\ G_k &: v_1, v_2, \ldots, v_{k-1}, v_k \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \end{aligned}$$

(a) (i) Write down all the integers in the 6^{th} group G_6.

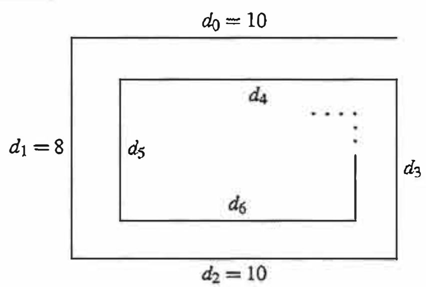
(ii) What is the total number of integers in the first 6 groups $G_1, G_2, \ldots, G_6$?

(b) Find, in terms of k,

(i) the last integer u_{k-1} in G_{k-1} and the first integer v_1 in G_k,

(ii) the sum of all the integers in G_k.

9B.8 HKCEE MA 1991 – I – 12



A maze is formed by line segments of lengths $d_0, d_1, d_2, \ldots, d_n, \ldots$, with adjacent line segments perpendicular to each other as shown in the figure. Let $d_0 = 10$, $d_1 = 8$, $d_2 = 10$ and $\dfrac{d_{n+2}}{d_n} = 0.9$ when $n \geq 1$,

i.e. $\dfrac{d_3}{d_1} = \dfrac{d_5}{d_3} = \cdots = 0.9$ and $\dfrac{d_4}{d_2} = \dfrac{d_6}{d_4} = \cdots = 0.9$.

(a) Find d_3 and d_5, and express d_{2n-1} in terms of n.

(b) Find d_6 and express d_{2n} in terms of n.

(c) Find, in terms of n, the sums

(i) $d_1 + d_3 + d_5 + \cdots + d_{2n-1}$,

(ii) $d_2 + d_4 + d_6 + \cdots + d_{2n}$.

(d) Find the value of the sum $d_0 + d_1 + d_2 + d_3 + \ldots$ to infinity.

9B.9 HKCEE MA 1992 – I – 14

(a) Given the geometric sequence $a^n, a^{n-1}b, a^{n-2}b^2, \ldots, a^2b^{n-2}, ab^{n-1}$, where a and b are unequal and non-zero real numbers, find the common ratio and the sum to n terms of the geometric sequence.

(b) A man joins a saving plan by depositing in his bank account a sum of money at the beginning of every year. At the beginning of the first year, he puts an initial deposit of $\$P$. Every year afterwards, he deposits 10% more than he does in the previous year. The bank pays interest at a rate of 8% p.a., compounded yearly.

(i) Find, in terms of P, an expression for the amount in his account at the end of

(1) the first year,

(2) the second year,

(3) the third year.

(Note: You need not simplify your expressions)

(ii) Using (a), or otherwise, show that the amount in his account at the end of the nth year is $\$54P(1.1^n - 1.08^n)$.

(c) A flat is worth $1 080 000 at the beginning of a certain year and at the same time, a man joins the saving plan in (b) with an initial deposit $\$P = \$20\,000$. Suppose the value of the flat grows by 15% every year. Show that at the end of the nth year, the value of the flat is greater than the amount in the man's account.

9B.10 HKCEE MA 1993 – I – 10

Consider the food production and population problems of a certain country. In the 1st year, the country's annual food production was 8 million tonnes. At the end of the 1st year its population was 2 million. It is assumed that the annual food production increases by 1 million tonnes each year and the population increases by 6% each year.

(a) Find, in million tonnes, the annual food production of the country in

(i) the 3rd year,

(ii) the nth year.

(b) Find, in million tonnes, the total food production in the first 25 years.

(c) Find the population of the country at the end of

(i) the 3rd year,

(ii) the nth year.

(d) Starting from the end of the first year, find the minimum number of years it will take for the population to be doubled.

(e) If the 'annual food production per capita' (i.e. $\dfrac{\text{annual food production in a certain year}}{\text{population at the end of that year}}$) is less than 0.2 tonne, the country will face a food shortage problem. Determine whether the country will face a food shortage problem or not at the end of the 100th year.

9B.11 HKCEE MA 1994 – I – 15

Suppose the number of babies born in Hong Kong in 1994 is 70 000 and in subsequent years, the number of babies born each year increased by 2% of that of the previous year.

(a) Find the number of babies born in Hong Kong

(i) in the first year after 1994;

(ii) in the nth year after 1994.

(b) In which year will the number of babies born in Hong Kong first exceed 90 000?

(c) Find the total number of babies born in Hong Kong from 1997 to 2046 inclusive.

(d) It is known that from 1901 to 2099, a year is a leap year if its number is divisible by 4.

(i) Find the number of leap years between 1997 and 2046.

(ii) Find the total number of babies born in Hong Kong in the leap years between 1997 and 2046.

9B.12 HKCEE MA 1997 – I – 10

Suppose the population of a town grows by 2% each year and its population at the end of 1996 was 300 000.

(a) Find the population at the end of 1998.

(b) At the end of which year will the population just exceed 330 000?

9B.13 HKCEE MA 1997 – I – 15

As shown below, figure A_1 is a square of side ℓ. To the middle of each of three sides of figure A_1, a square of side $\frac{\ell}{3}$ is added to give figure A_2.

Following the same pattern, squares of side $\frac{\ell}{9}$ are added to figure A_2 to give figure A_3. The process is repeated indefinitely to give figures $A_4, A_5, \ldots, A_n, \ldots$.

(a) (i) Table 1 shows the numbers and the lengths of sides of the squares added when producing A_2 from A_1, A_3 from A_2 and A_4 from A_3. Complete Table 1.

(ii) Find the total area of all the squares in A_4.

(iii) As n increases indefinitely, the total area of all the squares in A_n tends to a constant k. Express k in terms of ℓ.

(b) The overlapping line segments in figures $A_1, A_2, A_3, \ldots, A_n, \ldots$ are removed to form figures $B_1, B_2, B_3, \ldots, B_n, \ldots$ as shown.

(i) Complete Table 2.

(ii) Write down the perimeter of B_n.
What would the perimeter of B_n become if n increases indefinitely?

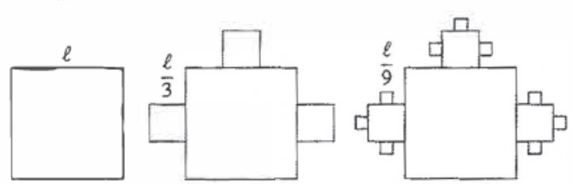


Table 1	$A_1 \to A_2$	$A_2 \to A_3$	$A_3 \to A_4$
Number of squares added	3	9	
Length of sides of the squares added	$\frac{\ell}{3}$	$\frac{\ell}{9}$	

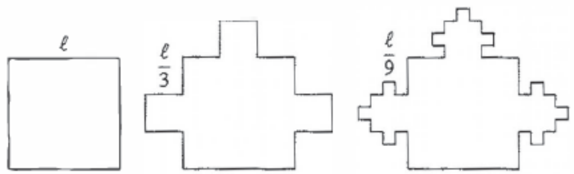


Table 2	B_1	B_2	B_3	B_4
Perimeter	4ℓ			

9B.14 HKCEE MA 1998 – I – 13

In Figure (1), $A_1B_1C_1D_1$ is a square of side 14 cm. A_2, B_2, C_2 and D_2 divide A_1B_1, B_1C_1, C_1D_1 and D_1A_1 respectively in the ratio 3 : 4 and form the square $A_2B_2C_2D_2$. Following the same pattern, A_3, B_3, C_3 and D_3 divide A_2B_2, B_2C_2, C_2D_2 and D_2A_2 respectively in the ratio 3 : 4 and form the square $A_3B_3C_3D_3$. The process is repeated indefinitely to give squares $A_4B_4C_4D_4$, $A_5B_5C_5D_5$, … , $A_nB_nC_nD_n$, ….

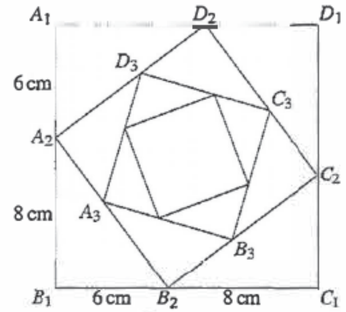


Figure (1)

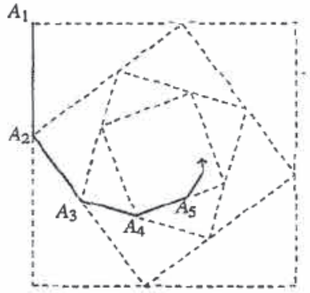


Figure (2)

(a) Find A_2B_2.

(b) Find $A_2A_3 : A_1A_2$.

(c) An ant starts at A_1 and crawls along the path $A_1A_2A_3 \ldots A_n \ldots$ as shown in Figure (2). Show that the total distance crawled by the ant cannot exceed 21 cm.

9B.15 HKCEE MA 1999 – I – 17

The manager of a factory estimated that in year 2000, the income of the factory will drop by r% each month from \$500000 in January to \$284 400 in December.

(a) Find r correct to the nearest integer.

(b) Suppose the factory's production cost is \$400000 in January 2000. The manager proposed to cut the cost by \$20 000 every month (i.e., the cost will be \$380000 in February and \$360 000 in March etc.) and claimed that it would not affect the monthly income.

(i) Using the value of r obtained in (a), show that the factory will still make a profit for the whole year.

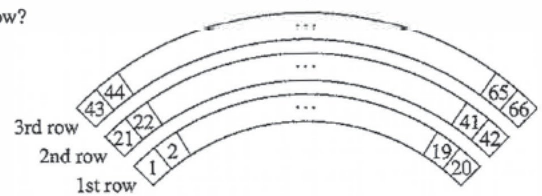
(ii) The factory will start a research project at the beginning of year 2000 on improving its production method. The cost of running the research project is \$300000 per month. The project will be stopped at the end of the k th month if the total cost spent in these k months on running the project exceeds the total production cost for the remaining months of the year.
Show that $k^2 - 71k + 348 < 0$. Hence determine how long the research project will last.

9B.16 HKCEE MA 2000 – I – 14

An auditorium has 50 rows of seats. All seats are numbered in numerical order from the first row to the last row, and from left to right, as shown in the figure. The first row has 20 seats. The second row has 22 seats. Each succeeding row has 2 more seats than the previous one.

(a) How many seats are there in the last row?

(b) Find the total number of seats in the first n rows. Hence determine in which row the seat numbered 2000 is located.



9B.17 HKCEE MA 2001 – I – 12

$F_1, F_2, F_3, \ldots, F_{40}$ as shown below are 40 similar figures. The perimeter of F_1 is 10 cm. The perimeter of each succeeding figure is 1 cm longer than that of the previous one.

(a) (i) Find the perimeter of F_{40}.

(ii) Find the sum of the perimeters of the 40 figures.

(b) It is known that the area of F_1 is $4\,\text{cm}^2$.

(i) Find the area of F_2.

(ii) Determine with justification whether the areas of $F_1, F_2, F_3, \ldots, F_{40}$ form an arithmetic sequence.

9B.18 HKCEE MA 2001 – I – 14

(a) *[Out of syllabus: The result* "The solution to the equation $x^5 \quad 6x+5=0$ is $x \approx 1.091$" *is obtained.]*

(b) From 1997 to 2000, Mr. Chan deposited \$1000 in a bank at the beginning of each year at an interest rate of $r\%$ per annum, compounded yearly. For the money deposited, the amount accumulated at the beginning of 2001 was \$5000. Using (a), find r correct to 1 decimal place.

9B.19 HKCEE MA 2002 – I – 13

A line segment AB of length 3 m is cut into three equal parts AC_1, C_1C_2 and C_2B as shown in Figure (1).

Figure (1)

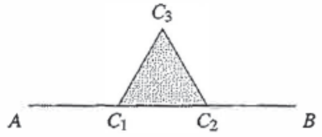


Figure (2)

On the middle part C_1C_2, an equilateral triangle $C_1C_2C_3$ is drawn as shown in Figure (2).

(a) Find, in surd form, the area of triangle $C_1C_2C_3$.

(b) Each of the line segments AC_1, C_1C_3, C_3C_2 and C_2B in Figure (2) is further divided into three equal parts. Similar to the previous process, four smaller equilateral triangles are drawn as shown in Figure (3). Find, in surd form, the total area of all the equilateral triangles.

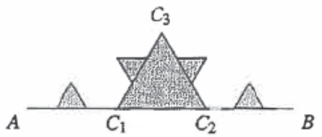


Figure (3)

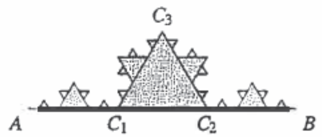


Figure (4)

(c) Figure (4) shows all the equilateral triangles so generated when the previous process is repeated again. What would the total area of all the equilateral triangles become if this process is repeated indefinitely? Give your answer in surd form.

9B.20 HKCEE MA 2003 I – 15

Figure (1) shows an equilateral triangle $A_0B_0C_0$ of side 1 m. Another triangle $A_1B_1C_1$ is inscribed in triangle $A_0B_0C_0$ such that $\frac{A_0A_1}{A_0B_0} = \frac{B_0B_1}{B_0C_0} = \frac{C_0C_1}{C_0A_0} = k$, where $0 < k < 1$. Let $A_1B_1 = x$ m.

(a) (i) Express the area of triangle $A_1B_0B_1$ in terms of k.

(ii) Express x in terms of k.

(iii) Explain why $A_1B_1C_1$ is an equilateral triangle.

(b) Another equilateral triangle $A_2B_2C_2$ in inscribed in triangle $A_1B_1C_1$ such that $\frac{A_1A_2}{A_1B_1} = \frac{B_1B_2}{B_1C_1} = \frac{C_1C_2}{C_1A_1} = k$ as shown in Figure (2).

(i) Prove that the triangles $A_1B_0B_1$ and $A_2B_1B_2$ are similar.

(ii) The above process of inscribing triangles is repeated indefinitely to generate equilateral triangles $A_3B_3C_3, A_4B_4C_4, A_5B_5C_5, \ldots$. Find the total area of the triangles $A_1B_0B_1, A_2B_1B_2, A_3B_2B_3, \ldots$.

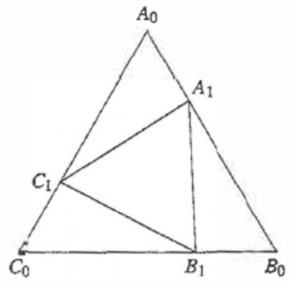


Figure (1)

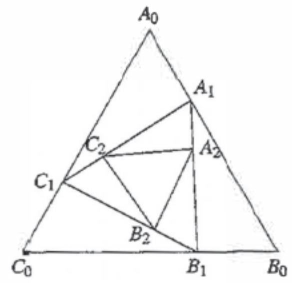


Figure (2)

9B.21 HKCEE MA 2004 – I – 15

In Figure (1), $F_1, F_2, F_3 \ldots$ are square frames. The perimeter of F_1 is 8 cm. Starting from F_2, the perimeter of each square frame is 4 cm longer than the perimeter of the previous frame.

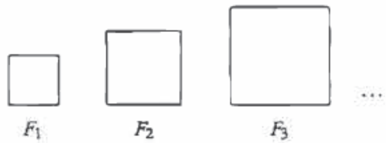


Figure (1)

(a) (i) Find the perimeter of F_{10}.

(ii) If a thin metal wire of length 1000 cm is cut into pieces and these pieces are then bent to form the above square frames, find the greatest number of distinct square frames that can be formed.

(b) Figure (2) shows three similar solid right pyramids S_1, S_2 and S_3. The total lengths of the four sides of the square bases of S_1, S_2 and S_3 are equal to the perimeters of F_1, F_2 and F_3 respectively.

(i) Do the volumes of S_1, S_2 and S_3 form a geometric sequence? Explain your answer.

(ii) When the length of the slant edge of S_1 is 5 cm, find the volume of S_3. Give the answer in surd form.

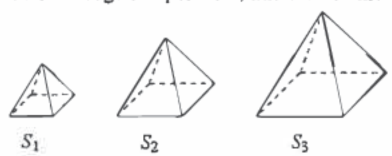


Figure (2)

9B.22 HKCEE MA 2005 – I – 16

Peter borrows a loan of \$200 000 from a bank at an interest rate of 6% per annum, compounded monthly. For each successive month after the day when the loan is taken, loan interest is calculated and then a monthly instalment of $\$x$ is immediately paid to the bank until the loan is fully repaid (the last instalment may be less than $\$x$), where $x < 200000$.

(a) (i) Find the loan interest for the 1st month.

(ii) Express, in terms of x, the amount that Peter still owes the bank after paying the 1st instalment.

(iii) Prove that if Peter has not yet fully repaid the loan after paying the nth instalment, he still owes the bank $\$\{200000(1.005)^n - 200x[(1.005)^n - 1]\}$.

(b) Suppose that Peter's monthly instalment is \$1 800 (the last instalment may be less than \$1 800).

(i) Find the number of months for Peter to fully repay the loan.

(ii) Peter wants to fully repay the loan with a smaller monthly instalment. He requests to pay a monthly instalment of \$900. However, the bank refuses his request. Why?

9B.23 HKCEE MA 2008 – I – 16

In the current financial year of a city, the amount of salaries tax charged for a citizen is calculated according to the following rules:

Net chargeable income (\$)	Rate
On the first 30 000	a%
On the next 30 000	10%
On the next 30 000	b%
Remainder	24%

The net chargeable income is equal to the net total income minus the sum of allowances. The salaries tax charged shall not exceed the standard rate of salaries tax applied to the net total income. The standard rate of salaries tax for the current financial year is 20%.

It is given that a, 10, b, 24 is an arithmetic sequence.

(a) Find a and b.

(b) Suppose that in the current financial year of the city, the sum of allowances of a citizen is \$172 000.

(i) Let $\$P$ be the net total income of the citizen. If the citizen has to pay salaries tax at the standard rate, express the amount of salaries tax charged for the citizen in terms of P.

(ii) Find the least net total income of the citizen so that the salaries tax is charged at the standard rate.

(c) Peter is a citizen in the city. In the current financial year, the net total income and the sum of allowances of Peter are \$1 400000 and \$172000 respectively. In order to pay his salaries tax, Peter begins to save money 12 months before the due day of paying salaries tax. A deposit of \$23000 is saved in a bank on the same day of each month at an interest rate of 3% per annum, compounded monthly. There are totally 12 deposits. Will Peter have enough money to pay his salaries tax on the due day? Explain your answer.

9B.24 HKCEE MA 2009 – I – 15

In a city, the taxi fare is charged according to the following table:

Distance travelled	Taxi fare
The first 2 km (under 2 km will be counted as 2 km)	\$30
Every 0.2km thereafter (under 0.2 km will be counted as 0.2 km)	\$2.4

Assume that there are no other extra fares.

(a) A hired taxi in the city travels a distance of x km, where $x \geq 2$.

(i) Suppose that x is a multiple of 0.2. Prove that the taxi fare is $\$(6 + 12x)$.

(ii) Suppose that x is not a multiple of 0.2. Is the taxi fare $\$(6+12x)$? Explain your answer.

(b) If a hired taxi in the city travels a distance of 3.1 km, find the taxi fare.

(c) In the city, a taxi is hired for 99 journeys. The 1st journey covers a distance of 3.1 km. Starting from the 2nd journey, the distance covered by each journey is 0.5 km longer than that covered by the previous journey. The taxi driver claims that the total taxi fare will not exceed \$33 000. Is the claim correct? Explain your answer.

9B.25 HKCEE MA 2010 – I – 17

Figure (1) shows the circle passing through the four vertices of the square $ABCD$. A rectangular coordinate system is introduced in Figure (1) so that the coordinates of A and B are $(0,0)$ and $(8,6)$ respectively.

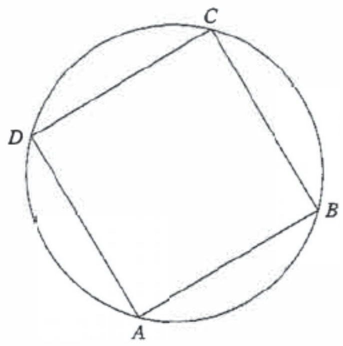


Figure (1)

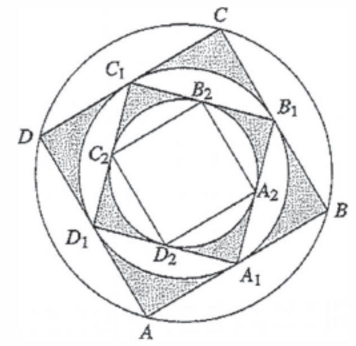


Figure (2)

(a) (i) Using a suitable transformation, or otherwise, write down the coordinates of D. Hence, or otherwise, find the coordinates of the centre of the circle $ABCD$.

(ii) Find the radius of the circle $ABCD$.

(b) A student uses the circle $ABCD$ of Figure (1) to design a logo the class association. The process of designing the logo starts by constructing the inscribed circle of the square $ABCD$ such that the inscribed circle touches AB, BC, CD and DA at A_1, B_1, C_1 and D_1 respectively. The region between the square $ABCD$ and its inscribed circle is shaded as shown in Figure (2). The inscribed circle of the square $A_1B_1C_1D_1$ is then constructed such that this inscribed circle touches A_1B_1, B_1C_1, C_1D_1 and D_1A_1 at A_2, B_2, C_2 and D_2 respectively. The region between the square $A_1B_1C_1D_1$ and its inscribed circle is also shaded. The process is carried in until the region between the square $A_9B_9C_9D_9$ and its inscribed circle is shaded.

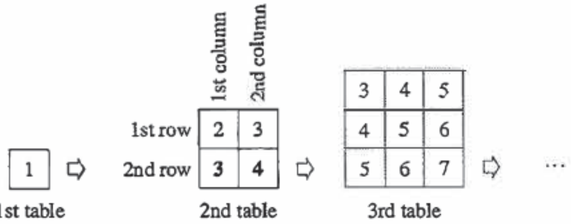
(i) Find the ratio of the area of the circle $A_1B_1C_1D_1$ to the area of the circle $ABCD$.

(ii) Suppose that the ratio of the total area of all the shaded regions to the area of the circle $ABCD$ is $p : 1$. The student thinks that the design of the logo is good when p lies between 0.2 and 0.3. According to the student, is the design of the logo good? Explain your answer.

9B.26 HKCEE MA 2011 – I – 15

The figure shows a sequence of tables filled with integers. The 1st table consists of 1 row and 1 column and 1 is assigned to the cell of the 1st table. For any integer $n > 1$, the nth table consists of n rows and n columns and the integers in the cells of the n table satisfy the following conditions:

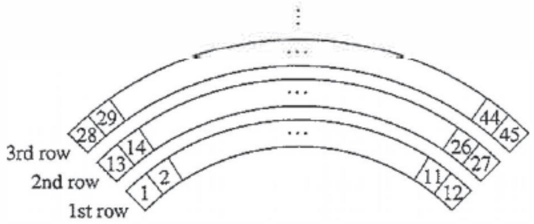
(1) The integer in the cell at the top left corner is n.
(2) In each row, the integer in the cell of the $(r+1)$th column is greater than that of the rth column by 1, where $1 \le r \le n$ 1.
(3) In each column, the integer in the cell of the $(r+1)$th row is greater than that of the rth row by 1, where $1 \le r \le n-1$.



(a) Construct and complete the 4th table.
(b) Find the sum of all integers in the 1st row of the 99th table.
(c) Find the sum of all integers in the 99th table.
(d) Is there an odd number k such that the sum of all integers in the kth table is an even number? Explain your answer.

9B.27 HKDSE MA SP – I – 15

The seats in a theatre are numbered in numerical order from the first row to the last row, and from left to right, as shown in the figure. The first row has 12 seats. Each succeeding row has 3 more seats than the previous one. If the theatre cannot accommodate more than 930 seats, what is the greatest number of rows in the theatre?



9B.28 HKDSE MA PP – I – 19

The amount of investment of a commercial firm in the 1st year is \$4000000. The amount of investment in each successive year is $r\%$ less than the previous year. The amount of investment in the 4th year is \$1048576.

(a) Find r.
(b) The revenue made by the firm in the 1st year is \$2000000. The revenue made in each successive year is 20% less than the previous year.
  (i) Find the least number of years needed for the total revenue made by the firm to exceed \$9000000.
  (ii) Will the total revenue made by the firm exceed \$10000000? Explain your answer.
  (iii) The manager of the firm claims that the total revenue made by the firm will exceed the total amount of investment. Do you agree? Explain your answer.

9B.29 HKDSE MA 2012 – I – 19

In a city, the air cargo terminal X of an airport handles goods of weight $A(n)$ tonnes in the nth year since the start of its operation, where n is a positive integer. It is given that $A(n) = ab^{2n}$, where a and b are positive constants. It is found that the weights of the goods handled by X in the 1st year and the 2nd year since the start of its operation are 254 100 tonnes and 307 461 tonnes respectively.

(a) (i) Find a and b. Hence find the weight of the goods handled by X in the 4th year since the start of its operation.
  (ii) Express, in terms of n, the total weight of the goods handled by X in the first n year since the start of its operation.
(b) The air cargo terminal Y starts to operate since X has been operated for 4 years. Let $B(m)$ tonnes be the weight of the goods handled by Y in the mth year since the start of its operation, where m is a positive integer. It is given that $B(m) = 2ab^m$.
  (i) The manager of the airport claims that after Y has been operated, the weight of the goods handled by Y is less than that handled by X in each year. Do you agree? Explain your answer.
  (ii) The supervisor of the airport thinks that when the total weight of the goods handled by X and Y since the start of the operation of X exceeds 20000000 tonnes, new facilities should be installed to maintain the efficiency of the air cargo terminals. According to the supervisor, in which year since the start of the operation of X should the new facilities be installed?

9B.30 HKDSE MA 2013 – I – 19

The development of public housing in a city is under study. It is given that the total floor area of all public housing flats at the end of the 1st year is 9×10^6 m^2 and in subsequent years, the total floor area of public housing flats built each year is $r\%$ of the total floor area of all public housing flats at the end of the previous year, where r is a constant, and the total floor area of public housing flats pulled down each year is 3×10^5 m^2. It is found that the total floor area of all public housing flats at the end of the 3rd year is 1.026×10^7 m^2.

(a) (i) Express, in terms of r, the total floor area of all public housing flats at the end of the 2nd year.
  (ii) Find r.
(b) (i) Express, in terms of n, the total floor area of all public housing flats at the end of the nth year.
  (ii) At the end of which year will the total floor area of all public housing flats first exceed 4×10^7 m^2?
(c) It is assumed that the total floor area of public housing flats needed at the end of the nth year is $(a(1.21)^n + b)$ m^2, where a and b are constants. Some research results reveal the following information:

n	The total floor area of public housing flats needed at the end of the nth year (m^2)
1	1×10^7
2	1.063×10^7

A research assistant claims that based on the above assumption, the total floor area of all public housing flats will be greater than the total floor area of public housing flats needed at the end of a certain year. Is the claim correct? Explain your answer.

9B.31 HKDSE MA 2014 – I – 16

In the figure, the 1st pattern consists of 3 dots. For any positive integer n, the $(n+1)$st pattern is formed by adding 2 dots to the nth pattern. Find the least value of m such that the total number of dots in the first m patterns exceeds 6 888.

9B.32 HKDSE MA 2017 – I – 16

A city adopts a plan to import water from another city. It is given that the volume of water imported in the 1st year since the start of the plan is 1.5×10^7 m^3 and in subsequent years, the volume of water imported each year is 10% less than the volume of water imported in the previous year.

(a) Find the total volume of water imported in the first 20 years since the start of the plan.

(b) Someone claims that the total volume of water imported since the start of the plan will not exceed 1.6×10^8 m^3. Do you agree? Explain your answer.

9 Arithmetic and Geometric Sequences

9A General terms and summations of sequences

9A.1 HKCEE MA 1980(1/1*/3) – I – 11

(a) (i) Common ratio $= \frac{10k}{k} = 10$

(ii) Sum $= \frac{k(10^n \quad 1)}{10-1} = \frac{k(10^n \quad 1)}{9}$

(b) (i) $\log 10k - \log k = \log \frac{10k}{k} = 1$

$\log 100k - \log 10k = \log \frac{100k}{10k} = 1$

Since there is a common difference, it is an A.S.

(ii) Sum $= \frac{n}{2}[2(\log k) + (n-1)(1)]$

$= n\log k + 2n - 2$

When $n \quad 10$,

Sum $= 10\log k + 20 - 2 = 10\log k + 18$

9A.2 HKCEE MA 1984(A/B) – I – 10

(a) $\because \frac{-2}{a} = \frac{b}{-2} =$ common ratio

$\therefore ab = (-2)^2 = 4$

(b) $a - b = b - (\quad 2) \Rightarrow a = 2b + 2$

Put into (a): $(2b+2)(b) = 4$

$b^2 + b - 2 = 0$

$b = -2$ (rejected) or 1

$\therefore a = 4 \div 1 = 4$

(c) (i) Common ratio $= \frac{-2}{4} = \frac{-1}{2}$

$\therefore$ Sum to $\infty = \frac{4}{1-(\frac{-1}{2})} = \frac{8}{3}$

(ii) The positive terms are the 1st, 3rd, 5th, …, ones.

$\therefore$ Common ratio $= \left(\frac{-1}{2}\right)^2 = \frac{1}{4}$

$\Rightarrow$ Sum to $\infty = \frac{4}{1-\frac{1}{4}} = \frac{16}{3}$

9A.3 HKCEE MA 1986(A/B I) – B – 9

(a) (i) Common difference $= \quad 1 - 2 = -3$

n-th term $= 2 + (n \quad 1)(-3) = 5 - 3n$

(ii) Sum $= \frac{n}{2}[2 + (5-3n)] = \frac{7n - 6n^2}{2}$

(iii) Required sum

$= \frac{7(30) - 6(30)^2 \quad 7(20) \quad 6(20)^2}{2 \qquad\qquad 2} = -1465$

(b) $\frac{7n - 6n^2}{2} < -1000$

$6n^2 \quad 7n - 2000 > 0$

$n < \frac{7 - \sqrt{48049}}{12}$ or $n > \frac{7 + \sqrt{48049}}{12}$

$n < -17.68$ or $n > 18.85$

$\therefore$ Least $n = 19$

9A.4 HKCEE MA 1989 – I – 9

(a) $\frac{k}{1} = \frac{\frac{1}{2}}{k} \Rightarrow k = \frac{1}{\sqrt{2}}$

(b) $T(n) = \left(\frac{1}{\sqrt{2}}\right)^{n-1} = 2^{\frac{1-n}{2}}$

(c) Sum to $\infty = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{1+\frac{1}{\sqrt{2}}}{(1)^2 - (\frac{1}{\sqrt{2}})^2} = 2 + \sqrt{2}$

(d) $T(1) \times T(3) \times T(5) \times \cdots \times T(2n-1)$

$= 2^{\frac{1-1}{2}} \cdot 2^{\frac{1-3}{2}} \cdot 2^{\frac{1-5}{2}} \cdot \ldots \cdot 2^{\frac{1 \quad (2n-1)}{2}}$

$= 2^0 \cdot 2^{-1} \cdot 2^{-2} \cdot \ldots \cdot 2^{-(n-1)}$

$= 2^{-(1+2+\cdots+(n-1))} = 2^{\frac{-n(n-1)}{2}}$

9A.5 HKCEE MA 1995 – I – 3

(a) Sum $= \frac{20}{2}[2(1) + (20-1)(5-1)] = 780$

(b) Sum to $\infty = \frac{9}{1-(\frac{5}{9})} = \frac{27}{2}$

9A.6 HKCEE MA 1996 – I – 3

(a) 4, 1, −2, −5

(b) Sum $= \frac{100}{2}[2(4) + (100-1)(1-4)] = \quad 14450$

9A.7 HKCEE MA 2003 – I – 7

(a) 10th term $= 2 + (10-1)(5-2) = 29$

(b) Sum $= \frac{(2+29)(10)}{2} = 155$

9A.8 HKCEE MA 2005 – I – 7

$\frac{n}{2}[2(5) + (n \quad 1)(8 \quad 5)] = 3925$

$3n^2 + 7n - 7850 = 0$

$n = 50$ or $\frac{-157}{3}$ (rejected)

9A.9 HKDSE MA 2015 – I – 17

(a) Common difference $= 4$

Sum $= \frac{n}{2}[2(4-5) + (n-1)(4)] = 2n^2 - 3n$

(b) Note that $\log B(n) = A(n)$. Hence

$\log(B(1)B(2)B(3). \quad B(n)) \le 8000$

$A(1) + A(2) + A(3) + \cdots + A(n) \le 8000$

$2n^2 - 3n \le 8000$

$2n^2 - 3n - 8000 \le 0$

$-64 \le n \le 62.5$

$\therefore$ Greatest $n = 62$

9A.10 HKDSE MA 2016 – I – 17

(a) Common difference $= \frac{555 - 666}{38 - 1} = \quad 3$

(b) $\frac{n}{2}[2(666) + (n-1)(\quad 3)] > 0$

$n(1335 - 3n) > 0$

$0 < n < 445$

$\therefore$ Greatest $n = 444$

9A.11 HKDSE MA 2018 – I – 16

(a) Common ratio $= \frac{864}{720} = 1.2$

$\therefore$ 1st term $= 720 \div (1.2)^2 = 500$

(b) $500(1.2)^n + 500(1.2)^{2n} < 5 \times 10^{14}$

$(1.2^n)^2 + (1.2^n) - 1 \times 10^{12} < 0$

$-1000000.5 < 1.2^n < 999999.5$

$n < \frac{\log 999999.5}{\log 1.2} = 75.78$

$\therefore$ Least value of n is 75

9A.12 HKDSE MA 2019 – I – 16

(a) $5\alpha - 18 = \alpha^2 - 13\alpha + 63$

$\Rightarrow \alpha^2 - 18\alpha + 81 = 0$

$\Rightarrow \alpha = 9$ (repeated) $\Rightarrow \beta = 27$

(b) First term $= \log 9$

Common difference $= \log 27 - \log 9 = \log 3$

$\therefore \frac{n}{2}[2\log 9 + (n-1)\log 3] > 888$

$4n\log 3 + n^2\log 3 - n\log 3 > 1776$

$(\log 3)n^2 + (3\log 3)n - 1776 > 0$

$n < -62.53$ or $n > 59.53$

$\therefore$ The least n is 60.

9A.13 HKDSE MA 2020 – I – 16

16a Let a and r be the first term and the common ratio of the sequence respectively.

$\begin{cases} ar^{3-1} = 144 \\ ar^{6-1} = 486 \end{cases}$

$\begin{cases} ar^2 = 144 \text{------(1)} \\ ar^5 = 486 \text{------(2)} \end{cases}$

$(1)^3 \div (2)^2$:

$a^1 = 262144$

$a = 64$

Therefore, the 1[st] term of the sequence is 64.

b Sub. $a = 64$ into (2),

$64r^5 = 486$

$r^5 = \frac{243}{32}$

$r = \frac{3}{2}$

$\frac{64\left[\left(\frac{3}{2}\right)^n - 1\right]}{\frac{3}{2} - 1} > 8\times 10^{13}$

$\left(\frac{3}{2}\right)^n > 6.25\times 10^{11} + 1$

$n > \log_{\frac{3}{2}}(6.25\times 10^{11} + 1)$ $\left\{\because \left(\frac{3}{2}\right)^n \text{ is strictly increasing}\right\}$

$n > 95.38167941$

Therefore, the least value of n is 96.

9B Applications

9B.1 HKCEE MA 1981(1/2/3) – I – 10

(a) By similar triangles, $\frac{b}{a} = \frac{2a - b}{2a}$

$\frac{b}{a} = 1 - \frac{1}{2}\left(\frac{b}{a}\right)$

$\frac{3}{2}\cdot\frac{b}{a} = 1 \Rightarrow b = \frac{2}{3}a$

(b) (i) $B_2C_2 = \frac{2}{3}b$

(ii) $B_2C_2 = \frac{2}{3}\left(\frac{2}{3}a\right) = \frac{4}{9}a$

(c) (i) $B_5C_5 = \left(\frac{2}{3}\right)^5 a = \frac{32}{243}a$

(ii) $\text{Sum} = \frac{(\frac{2}{3}a)^2}{1 - (\frac{4}{9})} = \frac{4}{5}a^2$

9B.2 HKCEE MA 1982(1/2/3) – I – 10

(a) (i) $999 = 3(333)$

$\therefore$ Sum of all multiples of 3

$= 3(1) + 3(2) + 3(3) + \cdots + 3(333)$

$= \frac{(3 + 999)(333)}{2} = 166833$

(ii) Sum of all multiples of 4

$= 4(1) + 4(2) + \cdots + 4(250)$

$= \frac{(4 + 1000)(250)}{2} = 125500$

(b) Required sum

= Sum of all integers – Sum in (a)

– Sum in (b) + Sum of all multiples of 12

$= \frac{(1 + 1000)(1000)}{2} - 166833 - 125500 + \frac{(12 + 996)(83)}{2}$

$= 249999$

9B.3 HKCEE MA 1983(A/B) – I – 10

(a) Required distance $= 10 + 2\times\left(10\times\frac{3}{4}\right) = 25$ (m)

(b) Required distance

$= 10 + 2\left(10\times\frac{3}{4}\right) + 2\left(10\times\left(\frac{3}{4}\right)^2\right) + \cdots + 2\left(10\times\left(\frac{3}{4}\right)^k\right)$

$= 10 + \frac{2(10\times\frac{3}{4})[1 - (\frac{3}{4})^k]}{1 - \frac{3}{4}}$

$= 10 + 60\left[1 - \left(\frac{3}{4}\right)^k\right] = 70 - 60\left(\frac{3}{4}\right)^k$ (m)

(c) Sum to $\infty = 70$ m

9B.4 HKCEE MA 1985(A/B) – I – 14

(a) (i) $Q_1 = P(1 + r\%)\times\frac{1}{3} = \frac{1}{3}P(1 + r\%)$

$Q_2 = P(1 + r\%)\times\frac{2}{3}\times(1 + r\%)\times\frac{1}{3}$

$= \frac{2}{9}P(1 + r\%)^2$

(ii) $Q_3 = P(1 + r\%)\times\frac{2}{3}\times(1 + r\%)\times\frac{2}{3}\times(1 + r\%)\times\frac{1}{3}$

$= \frac{4}{27}P(1 + r\%)^3$

(b) Common ratio $= \frac{2}{3}(1 + r\%)$

(c) (i) $\frac{27}{128}P = \frac{4}{27}P(1 + r\%)^3$

$\frac{729}{512} = (1 + r\%)^3 \Rightarrow 1 + r\% = \frac{9}{8} \Rightarrow r = 12.5$

(ii) $Q_1 + Q_2 + Q_3 + \cdots + Q_{10}$

$= \frac{\frac{1}{3}(10000)(1 + 12.5\%)\left(1 - \left[\frac{2}{3}(1 + 12.5\%)\right]^{10}\right)}{1 - \frac{2}{3}(1 + 12.5\%)}$

$= \frac{\frac{1}{3}(10000)(\frac{9}{8})(1 - 0.75^{10})}{1 - 0.75} = (\$)14155$ (nrst int)

9B.5 HKCEE MA 1987(A/B) – I – 10

(a) $T_1 = \frac{1}{2}(3)(3)\sin 60^\circ = \frac{9\sqrt{3}}{4}$

(b) (i) $A_2B_1 = 3\times\frac{2}{3} = 2$, $B_1B_2 = 3\times\frac{1}{3} = 1$

$\therefore A_2B_2 = \sqrt{2^2 + 1^2 - 2(2)(1)\cos 60^\circ} = \sqrt{3}$

(ii) Ratio in length $= \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$

$\Rightarrow$ Ratio in area $= \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$

$\therefore T_2 = \frac{1}{3}T_1 = \frac{3\sqrt{3}}{4}$

(c) (i) $\frac{1}{3}$

(ii) $T_n = \frac{9\sqrt{3}}{4}\cdot\left(\frac{1}{3}\right)^{n-1} = \frac{\sqrt{3}}{4\cdot 3^{n-3}}$

(iii) $T_1 + T_2 + \cdots + T_n = \frac{\frac{9\sqrt{3}}{4}\left(1 - \frac{1}{3^n}\right)}{1 - \frac{1}{3}} = \frac{27\sqrt{3}}{8}\left(1 - \frac{1}{3^n}\right)$

(iv) Sum to $\infty = \frac{27\sqrt{3}}{8}$

9B.6 HKCEE MA 1988 – I – 9

(a) Smallest: 105, Largest: 994

(b) 128 multiples

$\text{Sum} = \frac{128}{2}(105 + 994) = 70336$

(c) $\text{Sum} = \frac{900}{2}(100 + 999) - 70336 = 424214$

9B.7 HKCEE MA 1990 – I – 14

(a) (i) G_6: 16, 17, 18, 19, 20, 21

(ii) Total number of integers $= 1 + 2 + 3 + 4 + 5 + 6 = 21$

(b) (i) $u_{k-1} = 1 + 2 + 3 + \cdots + (k - 1) = \frac{k(k-1)}{2}$

$v_1 = \frac{k(k-1)}{2} + 1$

(ii) $\text{Sum} = \frac{\left[\left(\frac{k(k-1)}{2} + 1\right) + \left(\frac{k(k-1)}{2} + k\right)\right](k)}{2}$

$= \frac{[k(k-1) + 1 + k](k)}{2} = \frac{k(k^2 + 1)}{2}$

9B.8 HKCEE MA 1991 – I – 12

(a) $d_3 = 0.9d_1 = 7.2$ $d_5 = 0.9d_3 = 6.48$

$\therefore d_{2n-1} = 0.9^{n-1}d_1 = 8\cdot 0.9^{n-1}$

(b) $d_6 = 0.9d_4 = 0.9^2d_2 = 8.1$

$\therefore d_{2n} = 0.9^{n-1}d_2 = 10\cdot 0.9^{n-1}$

(c) (i) $d_1 + d_3 + \cdots + d_{2n-1} = \frac{8(1 - 0.9^n)}{1 - 0.9} = 80(1 - 0.9^n)$

(ii) $d_2 + d_4 + \cdots + d_{2n} = \frac{10(1 - 0.9^n)}{1 - 0.9} = 100(1 - 0.9^n)$

(d) $d_0 + d_1 + \cdots = d_0 + (80) + (100) = 190$

9B.9 HKCEE MA 1992 – I – 14

(a) Common ratio $= \frac{a^{n-1}b}{a^n} = \frac{b}{a}$

$\therefore \text{Sum} = \frac{a^n[1 - (\frac{b}{a})^n]}{1 - \frac{b}{a}}$

$= a^n\left(\frac{a^n - b^n}{a^n}\right)\cdot\frac{a}{a - b} = \frac{a(a^n - b^n)}{a - b}$

(b) (i) (1) $P(1 + 8\%) = 1.08P$

(2) $(1.08P + 1.1P)(1.08) = [(1.08)^2 + (1.1)(1.08)]P$

(3) $\{[(1.08)^2 + (1.1)(1.08)]P + (1.1)^2P\}(1.08)$

$= [(1.08)^3 + (1.1)(1.08)^2 + (1.1)^2(1.08)]P$

(ii) Take $a = 1.08$ and $b = 1.1$.

$\Rightarrow$ Amount

$[(1.08)^n + (1.1)(1.08)^{n-1} + (1.1)^2(1.08)^{n-2} + \cdots + (1.1)^{n-1}(1.08)]P$

$= \frac{1.08(1.08^n - 1.1^n)}{1.08 - 1.1}P$

$= (\$)54(1.1^n - 1.08^n)P$

(c) Value of flat at the end of the nth year $= \$1080000(1.15)^n$

Amount in account $= \$54(20000)(1.1^n - 1.08^n)$

$= \$1080000(1.1^n - 1.08^n)$

$< \$1080000(1.1^n)$

$< \$1080000(1.15^n) =$ Value of flat

9B.10 HKCEE MA 1993 – I – 10

(a) (i) Food pdtn $= 8 + 2(1) = 10$ (mil. tonnes)

(ii) Food pdtn $= 8 + (n - 1)(1) = 7 + n$ (mil. tonnes)

(b) $\text{Total} = \frac{25}{2}[2(8) + (25 - 1)(1)] = 500$ (mil. tonnes)

(c) (i) Popln $= 2(1 + 6\%)^2 = 2.2472$ (mil.)

(ii) Popln $= 2(1 + 6\%)^{n-1} = 2(1.06)^{n-1}$ (mil.)

(d) Let it take n years.

$(1.06)^n = 2 \Rightarrow n = \frac{\log 2}{\log 1.06} = 11.896$

$\therefore$ At least 12 years

(e) At the end of the 100th year,

Anl food pdtn per capita $= \frac{7 + 100}{2(1.06)^{100-1}} = 0.167 < 0.2$

$\therefore$ YES.

9B.11 HKCEE MA 1994 – I – 15

(a) (i) No. of babies $= 70000(1 + 2\%) = 71400$

(ii) No. of babies $= 70000(1 + 2\%)^n = 70000(1.02)^n$

(b) Let it happen in the kth year after 1994.

$70000(1.02)^k > 90000$

$1.02^k > \frac{9}{7} \Rightarrow k > \frac{\log\frac{9}{7}}{\log 1.02} = 12.69$

$\therefore$ It happens in 2007.

(c) No. of years = 50

First term $= 70000(1.02)^3$

$\therefore \text{Total} = \frac{70000(1.02)^3(1.02^{50} - 1)}{1.02 - 1} \approx 6282944$ (nrst int.)

(d) (i) Leap years: 2000, 2004, 2008, …, 2044

$\Rightarrow$ No. of leap years $= \frac{2044 - 2000}{4} + 1 = 12$

(ii) First term $= 70000(1.02)^6$

Common ratio $= 1.02^4$

$\therefore \text{Total} = \frac{70000(1.02)^6[(1.02^4)^{12} - 1]}{(1.02^4) - 1}$

$= 1517744$ (nearest integer)

9B.12 HKCEE MA 1997 – I – 10

(a) Population $= 300000 \times (1+2\%)^2 = 312120$

(b) Let it take n years.

$300000(1+2\%)^n > 330000$

$1.02^n > 1.1$

$n\log 1.02 > \log 1.1$

$n > \dfrac{\log 1.1}{\log 1.02} = 4.81$

$\therefore$ After 5 years, i.e. at the end of 2001.

9B.13 HKCEE MA 1997 – I – 15

(a) (i) Table 1

3	9	27
$\frac{\ell}{3}$	$\frac{\ell}{9}$	$\frac{\ell}{27}$

(ii) Total area $= \ell^2 + 3\left(\dfrac{\ell}{3}\right)^2 + 9\left(\dfrac{\ell}{9}\right)^2 + 27\left(\dfrac{\ell}{27}\right)^2$

$= \dfrac{820}{729}\ell^2$

(iii) $k = \ell^2 + 3\left(\dfrac{\ell}{3}\right)^2 + 9\left(\dfrac{\ell}{9}\right)^2 + 27\left(\dfrac{\ell}{27}\right)^2 + \ldots$

$= \ell^2 + \dfrac{\ell^2}{3} + \dfrac{\ell^2}{9} + \dfrac{\ell^2}{27} + \ldots$

$= \dfrac{\ell^2}{1-\frac{1}{3}} = \dfrac{3}{2}\ell^2$

(b) (i) Table 2

4ℓ	6ℓ	8ℓ	10ℓ

(ii) Perimeter of $B_n = 4\ell + (n-1)(2\ell) = 2\ell + 2\ell n$, which becomes infinitely large!

9B.14 HKCEE MA 1998 – I – 13

(a) $A_2B_2 = \sqrt{8^2+6^2} = 10$ (cm)

(b) $A_2A_3 = \dfrac{3}{3+4}(10) = \dfrac{30}{7}$ (cm)

$\therefore A_2A_3 : A_1A_2 = \dfrac{30}{7} : 6 = 5 : 7$

(c) Total dist. $= A_1A_2 + A_2A_3 + A_3A_4 + \ldots$

$< \dfrac{6}{1-\frac{5}{7}} = 21$ (cm)

9B.15 HKCEE MA 1999 – I – 17

(a) $500000(1-r\%)^{11} = 254400$

$1 - r\% = 0.949999986 \Rightarrow r = 5$ (nrst int.)

(b) (i) Total income

$= 500000 + 500000(1-5\%) + 500000(1-5\%)^2 + \cdots + 500000(1-5\%)^{11}$

$= \dfrac{500000(1-0.95^{12})}{1-0.95} = (\$)4596399$

Total cost

$= \dfrac{12}{2}[2(400000) + (12-1)(-20000)]$

$= (\$)3480000 < (\$)4596399$

Hence, there is still a profit.

(ii) $300000k > 3480000 - \dfrac{k}{2}[2(400000) + (k-1)(-20000)] + 10000k^2$

$300000k > 3480000 - 410000k + 10000k^2$

$0 > k^2 - 71k + 348$

$5.2965 < k < 65.7035$

The project will last for 5 months.

9B.16 HKCEE MA 2000 – I – 14

(a) Number of seats $= 20 + 49(2) = 118$

(b) Total number of seats in the first n rows

$= \dfrac{n}{2}[2(20) + (n-1)(2)] = 19n + n^2$

$\therefore 19n + n^2 \geq 2000$

$n^2 + 19n - 2000 \geq 0$

$n \leq -14.28$ or $n \geq 36.22$

$\therefore$ Seat 2000 is in the 37th row.

9B.17 HKCEE MA 2001 – I – 12

(a) (i) Perimeter $= 10 + 39(1) = 49$ (cm)

(ii) Sum $= \dfrac{(10+49)(40)}{2} = 1180$ (cm)

(b) (i) $\dfrac{\text{Area of } F_2}{\text{Area of } F_1} = \left(\dfrac{\text{Perimeter of } F_2}{\text{Perimeter of } F_1}\right)^2$

Area of $F_2 = 4 \times \left(\dfrac{11}{10}\right)^2 = 4.84$ (cm^2)

(ii) Area of $F_3 = 4 \times \left(\dfrac{12}{10}\right)^2 = 5.76$ (cm^2)

$\because 4.84 - 4 = 0.84$

$5.76 - 4.84 = 0.92 \neq 0.84$

$\therefore$ They do not form an A.S.

9B.18 HKCEE MA 2001 – I – 14

(b) $1000(1+r\%)^4 + 1000(1+r\%)^3 + 1000(1+r\%)^2 + 1000(1+r\%) = 5000$

$\dfrac{1000(1+r\%)[(1+r\%)^4 - 1]}{(1+r\%)-1} = 5000$

$(1+r\%)^5 - (1+r\%) = 5(1+r\%) - 5$

$(1+r\%)^5 - 6(1+r\%) + 5 = 0$

By (a), $\quad 1 + r\% = 1.091$

$r = 9.1$

9B.19 HKCEE MA 2002 – I – 13

(a) Area $= \dfrac{1}{2}(1)(1)\sin 60° = \dfrac{\sqrt{3}}{4}$ (m^2)

(b) Area of small $\triangle = \dfrac{\sqrt{3}}{4} \times \left(\dfrac{1}{3}\right)^2 = \dfrac{\sqrt{3}}{4} \cdot \dfrac{1}{9}$

$\therefore$ Total area $= \dfrac{\sqrt{3}}{4} + \dfrac{\sqrt{3}}{4} \cdot \dfrac{1}{9}$

$= \dfrac{\sqrt{3}}{4} \cdot \dfrac{10}{9} = \dfrac{5\sqrt{3}}{18}$ (m^2)

(c) Total area $= \dfrac{\sqrt{3}}{4} + \dfrac{\sqrt{3}}{4} \cdot \dfrac{1}{9} + \left(\dfrac{\sqrt{3}}{4} \cdot \dfrac{1}{9}\right) \cdot \dfrac{1}{9} + \ldots$

$= \dfrac{\frac{\sqrt{3}}{4}}{1-\frac{1}{9}} = \dfrac{9\sqrt{3}}{32}$ (m^2)

9B.20 HKCEE MA 2003 – I – 15

(a) (i) Area $= \dfrac{1}{2}(k)(1-k)\sin 60° = \dfrac{\sqrt{3}}{4}k(1-k)$ (m^2)

(ii) $x = \sqrt{k^2 + (1-k)^2 - 2(k)(1-k)\cos 60°}$

$= \sqrt{1 - 2k + 2k^2 - (k - k^2)} = \sqrt{1 - 3k + k^2}$

(iii) $\because \triangle A_1B_0B_1 \cong \triangle B_1C_0C_1 \cong \triangle C_1A_0A_1$

$\therefore A_1B_1 = B_1C_1 = C_1A_1$

(b) (i) In $\triangle A_1B_0B_1$ and $\triangle A_2B_1B_2$,

$\dfrac{A_1B_0}{B_0B_1} = \dfrac{1-k}{k}$ (given)

$\dfrac{A_2B_1}{B_1B_2} = \dfrac{1-k}{k}$ (given)

$\angle B_0 = \angle B_1 \quad 60°$ (property of equil. $\triangle$)

$\therefore \triangle A_1B_0B_1 \sim \triangle A_2B_1B_2$ (ratio of 2 sides, inc. $\angle$)

(ii) $\dfrac{\text{Area of } A_1B_1C_1}{\text{Area of } A_0B_0C_0} = \left(\dfrac{x}{1}\right)^2 = 1 - 3k + k^2$

$\therefore$ Total area $= \dfrac{\frac{\sqrt{3}}{4}k(1-k)}{1-(1-3k+k^2)}$

$= \dfrac{\sqrt{3}k(1-k)}{4k(3-k)} = \dfrac{\sqrt{3}(1-k)}{4(3-k)}$

9B.21 HKCEE MA 2004 – I – 15

(a) (i) Perimeter $= 8 + (10-1)(4) = 44$ (cm)

(ii) Let n frames can be formed.

$\dfrac{n}{2}[2(8) + (n-1)(4)] \leq 1000$

$10n + 2n^2 \leq 1000$

$n^2 + 5n - 500 \leq 0$

$-25 \leq n \leq 20$

$\therefore$ 20 frames can be formed.

(b) (i) Vol of S_1 : Vol of S_2 : Vol of S_3

$=$ (Peri of S_1 : Peri of S_2 : Peri of S_3)^3

$= (8 : 12 : 16)^3 = 8 : 27 : 81$

Since $8 : 27 \neq 27 : 81$, the volumes do not form a G.S.

(ii) For S_1, Diag of base $= \sqrt{2^2+2^2} = \sqrt{8}$ (cm)

Height $= \sqrt{5^2 - \left(\dfrac{\sqrt{8}}{2}\right)^2} = \sqrt{23}$ (cm)

Volume $= \dfrac{1}{3}(2)^2(\sqrt{23}) = \dfrac{4\sqrt{23}}{3}$ (cm^3)

$\therefore$ Vol of $S_3 = \dfrac{4\sqrt{23}}{3} \cdot \dfrac{81}{8} = \dfrac{27\sqrt{23}}{2}$ (cm^3)

9B.22 HKCEE MA 2005 – I – 16

(a) (i) Interest $= 200000\left(1 + \dfrac{6\%}{12}\right) - 200000$

$= 200000(1.005 - 1) = (\$)1000$

(ii) Amt owed $= \$(201000 - x)$

(iii) Amount owed after 2nd instalment

$= [200000(1.005) - x](1.005) - x$

$= 200000(1.005)^2 - x(1.005 + 1)$

Amount owed after 3rd instalment

$= [200000(1.005)^2 - x(1.005+1)](1.005) - x$

$= 200000(1.005)^3 - x(1.005^2 + 1.005 + 1)$

$\therefore$ Amount owed after nth instalment

$= 200000(1.005)^n - x(1.005^{n-1} + 1.005^{n-2} + \cdots + 1.005 + 1)$

$= 200000(1.005)^n - x\left(\dfrac{1.005^n - 1}{1.005 - 1}\right)$

$= (\$)200000(1.005)^n - 200x[(1.005)^n - 1]$

(b) (i) Let the last instalment be the $(n+1)$st one.

$200000(1.005)^n - 200(1800)(1.005^n - 1) < 1800$

$2000(1.005)^n - 3600(1.005)^n + 3600 < 18$

$1600(1.005)^n > 3582$

$1.005^n > 2.23875$

$n > \dfrac{\log 2.23875}{\log 1.005}$

$= 161.586$

$\therefore$ The last instalment is the 162nd one.

(ii) $200000(1.005)^n - 200(900)(1.005^n - 1) < 900$

$200(1.005)^n < -1791$

which has no solution.

i.e. Peter cannot fully repay the loan with $x = 900$.

9B.23 HKCEE MA 2008 – I – 16

(a) Common difference $= \dfrac{24-10}{2} = 7$

$\therefore a = 10 - 7 = 3,\ b = 10 + 7 = 17$

(b) (i) Tax $= (P - 172000) \times 20\% = (\$)0.2P - 34400$

(ii) $0.2P - 34400 = 30000 \times 3\% + 30000 \times 10\% + 30000 \times 17\% + (P - 172000) \times 24\%$

$= 9000 + 0.24P - 62880$

$\Rightarrow 19480 = 0.04P \Rightarrow P = 487000$

Hence, the least net total income is \$487000.

(c) Total amount in bank $= \dfrac{23000(1+\frac{3\%}{12})[(1+\frac{3\%}{12})^{12} - 1]}{(1+\frac{3\%}{12}) - 1}$

$= (\$)280526.37$

Tax payable $= (1400000 - 172000) \times 20\%$

$= (\$)245600 < (\$)280526.37$

$\therefore$ He will have enough.

9B.24 HKCEE MA 2009 – I – 15

(a) (i) Fare $= 30 + \dfrac{x-2}{0.2} \times 2.4 = (\$)6 + 12x$

(ii) The fare will be $6 + 2y$, where y is the least multiple of 0.2 which is larger than x.

$\therefore$ NO.

(b) Fare $= 6 + 12(3.2) = (\$)44.4$

(c) In the city, a taxi is hired for 99 journeys. The 1st journey covers a distance of 3.1 km. Starting from the 2nd journey, the distance covered by each journey is 0.5 km longer than that covered by the previous journey. The taxi driver claims that the total taxi fare will not exceed \$33000. Is the claim correct? Explain your answer.

9B.25 HKCEE MA 2010 – I – 17

(a) (i) Rotate B about A anticlockwise through 90°

$\Rightarrow D = (-6, 8)$

Centre $=$ mid-pt of $BD = \left(\dfrac{-6+8}{2}, \dfrac{8+6}{2}\right) = (1, 7)$

(ii) Radius $= \sqrt{(8-1)^2 + (6-7)^2} = \sqrt{50}$

(b) (i) Radius of circle $A_1B_1C_1D_1 = \dfrac{1}{2}AB = \dfrac{\sqrt{8^2+6^2}}{2} = 5$

$\therefore \dfrac{\text{Area of circle } A_1B_1C_1D_1}{\text{Area of circle } ABCD} = \left(\dfrac{\text{Radius of circle } A_1B_1C_1D_1}{\text{Radius of circle } ABCD}\right)^2 = \left(\dfrac{5}{\sqrt{50}}\right)^2 = \dfrac{1}{2}$

(ii) Shaded area between sq. $ABCD$ and cl. $A_1B_1C_1D_1$

$= 10^2 - \pi(5)^2 = 100 - 25\pi$

$\therefore$ Total shaded area

$= (100 - 25\pi) + \dfrac{100 - 25\pi}{2} + \dfrac{100 - 25\pi}{2^2} + \cdots + \dfrac{100 - 25\pi}{2^9}$

$= \dfrac{(100 - 25\pi)[1 - (\frac{1}{2})^{10}]}{1 - \frac{1}{2}} = 42.87845$

$\therefore p = \dfrac{42.87845}{\pi(\sqrt{50})^2} = 0.27297$

which is indeed between 0.2 and 0.3.

Hence the design is good.

9B.26 HKCEE MA 2011 – I – 15

(a)

4	5	6	7
5	6	7	8
6	7	8	9
7	8	9	10

(b) The 1st row contains: 99, 100, ... (99 integers) ....

$\Rightarrow \text{Sum} = \frac{99}{2}[2(99)+98\times 1] = 14652$

(c) Sum of all integers in the 2nd row

$= $ Sum of all integers in the 1st row $+ 99$

Sum of all integers in the 3rd row

$= $ Sum of all integers in the 1st row $+ 99 \times 2$

Similarly, sum of all integers in the ith row

$= $ Sum of all integers in the 1st row $+ 99 \times (i-1)$

$\therefore$ Sum of all integers

$= $ Sum of all integers in the 1st row $\times 99$

$+(99+99\times 2+\cdots+99\times 98)$

$= 14652 \times 99 + 99 \times \frac{(1+98)(98)}{2}$

$= 1930797$

(d) In the kth table, 1st row: $k, k+1, \ldots, k+(k-1)$

$\Rightarrow \text{Sum} = \frac{[k+(2k-1)](k)}{2} = \frac{(3k-1)k}{2}$

$\therefore$ Sum of all integers

$= \frac{(3k-1)(k)}{2} \times k + [k+2k+3k+\cdots+(k-1)k]$

$= \frac{(3k-1)k^2}{2} + k \times \frac{[1+(k-1)](k-1)}{2}$

$= \frac{(3k-1)k^2}{2} + \frac{k^2(k-1)}{2}$

$= \frac{k^2(3k-1+k-1)}{2} = k^2(2k-1)$, which must be odd.

$\therefore$ NO.

9B.27 HKDSE MA SP – I – 15

Let there be n rows.

$\frac{n}{2}[2(12)+(n-1)(3)] \le 930$

$n(21+3n) \le 930 \times 2$

$n^2+7n-620 \le 0$

$-28.64 \le n \le 21.64$

$\therefore$ Greatest number of rows is 21.

9B.28 HKDSE MA PP – I – 19

(a) $4000000(1-r\%)^3 = 1048576$

$1-r\% = 0.64 \Rightarrow r = 36$

(b) (i) Let n be the number of years.

$2000000 + 2000000(0.8) +$

$+\cdots+2000000(0.8)^{n-1} > 9000000$

$\frac{1-0.8^n}{1-0.8} > \frac{9000000}{2000000}$

$0.8^n > 0.1$

$n\log 0.8 > \log 0.1$

$n > \frac{\log 0.1}{\log 0.8} = 10.319$

$\therefore$ The least number of years is 11.

(ii) Total revenue $< \frac{2000000}{1-0.8} = 10000000$

$\therefore$ No.

(iii) In n years, total revenue $= \frac{2000000(1-0.8^n)}{1-0.8}$

$= 10000000(1-0.8^n)$

Total investment $= \frac{4000000(1-0.64^n)}{1-0.64}$

~~$\frac{100000000}{9}(1-0.64^n)$~~

$\therefore$ Total revenue – Total investment

$= \frac{10000000}{9}[9(1-0.8^n)-10(1-0.64^n)]$

$= \frac{10000000}{9}[10(0.8^2)^n - 9(0.8^n) - 1]$

$= \frac{10000000}{9}[10(0.8^n)^2 - 9(0.8^n) - 1]$

$= \frac{10000000}{9}[10(0.8^n)+1][(0.8^n)-1]$

$< 0 \quad (\because 0.8^n < 1 \text{ for any } n > 0)$

Hence, Total revenue < Total investment

Thus the claim is disagreed.

9B.29 HKDSE MA 2012 – I – 19

(a) (i) $\begin{cases} ab^2 = 254100 \\ ab^4 = 307461 \end{cases}$

$\Rightarrow b^2 = \frac{307461}{254100} \Rightarrow b = 1.1 \Rightarrow a = 210000$

$\therefore$ Required weight $= (210000)(1.1)^{2(4)}$

$= 450000$ (tonnes, 3 s.f.)

(ii) Total weight $= \frac{210000(1.1)^2[(1.1^2)^n - 1]}{1.1^2-1}$

$= 1210000(1.21^n - 1)$ (tonnes)

(b) (i) In the mth year, $n = m+4$.

Then, $A(m+4) = ab^{2(m+4)}$ and $B(m) = 2ab^{2n}$

$\Rightarrow \frac{A(m+4)}{B(m)} = \frac{ab^{2m}b^8}{2ab^m}$

$= \frac{b^8}{2}b^m$

$= 1.072(1.1)^m > 1$

$\therefore A(m+4) > B(m)$, and the claim is agreed.

(ii) Total weight by Y in the first $n-4$ years

$= \frac{2(210000)(1.1)(1.1^{n-4}-1)}{1.1-1}$

$= 4620000(1.1^{n-4}-1)$

$1210000(1.21^n-1)$

$+4620000(1.1^{n-4}-1) > 20000000$

$121[(1.1^n)^2-1] + 462\left(\frac{1.1^n}{1.1^4}-1\right) > 2000$

$177.1561[(1.1^n)^2-1] + 462(1.1^n - 1.4641) > 2928.2$

$177.1561(1.1^n)^2 + 462(1.1^n) - 3781.7703 > 0$

$1.1^n < -6.1047$ (rejected) or $1.1^n > 3.4968$

$\therefore n > \frac{\log 3.4968}{\log 1.1} = 13.13$

$\therefore$ The 14th year since the start of X.

9B.30 HKDSE MA 2013 – I – 19

(a) (i) Total floor area $= 9\times 10^6(1+r\%) - 3\times 10^5$

$= 9\times 10^6 + 9r\times 10^4 - 3\times 10^5$

$= (870+9r)\times 10^4$ (m^2)

(ii) $[9\times 10^6(1+r\%) - 3\times 10^5](1+r\%)$

$-3\times 10^5 = 1.026\times 10^7$

$150(1+r\%)^2 - 5(1+r\%) - 176 = 0$

$1+r\% = \frac{11}{10}$ or $\frac{-16}{15}$ (rej)

$r = 10$

(b) (i) Required area

$= 9\times 10^6(1.1)^{n-1} - 3\times 10^5(1.1)^{n-2}$

$-3\times 10^5(1.1)^{n-3} - \cdots - 3\times 10^5$

$= 9\times 10^6(1.1)^{n-1} - 3\times 10^5 \cdot \frac{(1.1)^{n-1}-1}{1.1-1}$

$= 9\times 10^6(1.1)^{n-1} - 3\times 10^6(1.1^{n-1}-1)$

$= [6(1.1)^{n-1}+3]\times 10^6$ (m^2)

(ii) $[6(1.1)^{n-1}+3]\times 10^6 > 4\times 10^7$

$1.1^{n-1} > \frac{37}{6}$

$n-1 > \frac{\log\frac{37}{6}}{\log 1.1} \Rightarrow n > 20.0867$

$\therefore$ At the end of the 21st year.

(c) $\begin{cases} a(1.21)^1 + b = 1\times 10^7 \\ a(1.21)^2 + b = 1.063\times 10^7 \end{cases}$

$\Rightarrow (1.4641-1.21)a = (1.063-1)\times 10^7$

$\Rightarrow a = \frac{3\times 10^8}{121} \Rightarrow b = 7\times 10^6$

If the claim happens at the end of the nth year,

$[6(1.1)^{n-1}+3]\times 10^6 > \frac{3\times 10^8}{121}(1.21)^n + 7\times 10^6$

$\frac{6(1.1^n)}{1.1} + 3 > \frac{300}{121}(1.1^n)^2 + 7$

$300(1.1^n)^2 - 660(1.1^n) + 484 < 0$

Since the inequality has no solution, the claim is wrong.

9B.31 HKDSE MA 2014 – I – 16

$\frac{m}{2}[2(3)+(m-1)(2)] > 6888$

$m(2+m) > 6888$

$m^2+2m-6888 > 0$

$(m+84)(m-82) > 0$

$m < -84$ (rejected) or $m > 82$

$\therefore$ Least value of m is 83.

9B.32 HKDSE MA 2017 – I – 16

(a) Total volume $= \frac{1.5\times 10^7(1-0.9^{20})}{1-0.9} = 131763501.8$ (m^3)

(b) Total volume $< \frac{1.5\times 10^7}{1-0.9}$

$= 1.5\times 10^7 < 1.6\times 10^8$

$\therefore$ The claim is agreed.