6 Identities, Equations and the Number System

6A Simple equations

6A.1 HKCEE MA 1980(1*/3) – I 13(b)

Solve the equation $1-2x=\sqrt{2-x}$.

6A.2 HKCEE MA 1982(2/3) I – 7

Solve $x-\sqrt{x+1}=5$.

6A.3 HKCEE MA 1984(A) – I – 3

Expand $(1+\sqrt{2})^4$ and express your answer in the form $a+b\sqrt{2}$ where a and b are integers.

6A.4 HKCEE MA 1984(A/B) I – 6

Solve $x-5\sqrt{x}-6=0$.

6A.5 HKCEE MA 2003 – I – 6

There are only two kinds of tickets for a cruise: first-class tickets and economy class tickets. A total of 600 tickets are sold. The number of economy-class tickets sold is three times that of first class tickets sold. If the price of a first class ticket is \$850 and that of an economy class ticket is \$500, find the sum of money for the tickets sold.

6A.6 HKCEE MA 2004 I 7

The prices of an orange and an apple are \$2 and \$3 respectively. A sum of \$46 is spent buying some oranges and apples. If the total number of oranges and apples bought is 20, find the number of oranges bought.

6A.7 HKCEE MA 2007 – I – 7

The consultation fees charged to an elderly patient and a non elderly patient by a doctor are \$120 and \$160 respectively. On a certain day, there were 67 patients consulted the doctor and the total consultation fee charged was \$9000. How many elderly patients consulted the doctor on that day?

6A.8 HKCEE MA 2008 – I – 3

(a) Write down all positive integers m such that $m+2n=5$, where n is an integer.

(b) Write down all values of k such that $2x^2+5x+k\equiv(2x+m)(x+n)$, where m and n are positive integers.

6A.9 HKCEE MA 2009 – I – 6

The total number of stamps owned by John and Mary is 300. If Mary buys 20 stamps from a post office, the number of stamps owned by her will be 4 times that owned by John. Find the number of stamps owned by John.

6A.10 HKCEE MA 2010 I 6

The cost of a bottle of orange juice is the same as the cost of 2 bottles of milk. The total cost of 3 bottles of orange juice and 5 bottles of milk is \$66. Find the cost of a bottle of milk.

6A.11 HKDSE MA SP – I – 5

In a football league, each team gains 3 points for a win, 1 point for a draw and 0 point for a loss. The champion of the league plays 36 games and gains a total of 84 points. Given that the champion does not lose any games, find the number of games that the champion wins.

6A.12 HKDSE MA 2012 – I – 5

There are 132 guards in an exhibition centre consisting of 6 zones. Each zone has the same number of guards. In each zone, there are 4 more female guards than male guards. Find the number of male guards in the exhibition centre.

6A.13 HKDSE MA 2013 – I – 4

The price of 7 pears and 3 oranges is \$47 while the price of 5 pears and 6 oranges is \$49. Find the price of a pear.

6A.14 HKDSE MA 2015 – I – 7

The number of apples owned by Ada is 4 times that owned by Billy. If Ada gives 12 of her apples to Billy, they will have the same number of apples. Find the total number of apples owned by Ada and Billy.

6A.15 HKDSE MA 2017 – I – 4

There are only two kinds of admission tickets for a theatre: regular tickets and concessionary tickets. The prices of a regular ticket and a concessionary ticket are \$126 and \$78 respectively. On a certain day, the number of regular tickets sold is 5 times the number of concessionary tickets sold and the sum of money for the admission tickets sold is \$50 976. Find the total number of admission tickets sold that day.

6A.16 HKDSE MA 2019 – I – 3

The length and the breadth of a rectangle are 24 cm and $(13+r)$ cm respectively. If the length of a diagonal of the rectangle is $(17-3r)$ cm, find r.

6B Nature of roots of quadratic equations

6B.1 HKCEE MA 1988 – I – 4

The quadratic equation $9x^2-(k+1)x+1=0$ ....(*) has equal roots.

(a) Find the two possible values of the constant k.

(b) If k takes the negative value obtained, solve equation (*).

6B.2 HKCEE MA 2007 – I – 5

Let k be a constant. If the quadratic equation $x^2+14x+k=0$ has no real roots, find the range of values of k.

6B.3 HKCEE AM 1980 – I – 1

Find the range of values of k for which the equation $2x^2+x+5=k(x+1)^2$ has no real roots.

6B.4 HKCEE AM 1998 – I – 3

The quadratic equations $x^2-6x+2k=0$ and $x^2-5x+k=0$ have a common root α. (i.e. α is a root of both equations.)
Show that $\alpha=k$ and hence find the value(s) of k.

6C Roots and coefficients of quadratic equations

6C.1 HKCEE MA 1980(1/1*/3) – I – 3

What is the product of the roots of the quadratic equation $2x^2+kx-5=0$?
If one of the roots is 5, find the other root and the value of k.

6C.2 HKCEE MA 1982(2/3) – I – 1

If $a-b=10$ and $ab=k$, express a^2+b^2 in terms of k.

6C.3 HKCEE MA 1983(B) – I – 14

(To continue as **10C.1**.)

α and β are the roots of the quadratic equation $x^2-2mx+n=0$, where m and n are real numbers.

(a) Find, in terms of m and n,
  (i) $(m-\alpha)+(m-\beta)$,
  (ii) $(m-\alpha)(m-\beta)$.

(b) Find, in terms of m and n, the quadratic equation having roots $m-\alpha$ and $m-\beta$.

6C.4 HKCEE MA 1985(A/B) – I – 5

Let α and β be the roots of $x^2+kx+1=0$, where k is a constant.

(a) Find, in terms of k,
  (i) $(\alpha+2)+(\beta+2)$,
  (ii) $(\alpha+2)(\beta+2)$.

(b) Suppose $\alpha+2$ and $\beta+2$ are the roots of $x^2+px+q=0$, where p and q are constants. Find p and q in terms of k.

6C.5 HKCEE MA 1986(A/B) – I – 7

If $\dfrac{1}{m}+\dfrac{1}{n}=\dfrac{1}{a}$ and $m+n=b$, express the following in terms of a and b

(a) mn,

(b) m^2+n^2.

6C.6 HKCEE MA 1987(A/B) – I – 5

α and β are the roots of the quadratic equation $kx^2-4x+2k=0$, where k ($k\neq 0$) is a constant. Express the following in terms of k:

(a) $\alpha^2+\beta^2$,

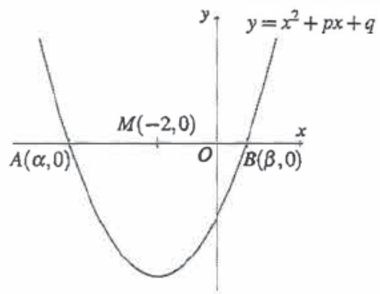
(b) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$.

6C.7 HKCEE MA 1990 – I – 6

In the figure, the curve $y=x^2+px+q$ cuts the x axis at the two points $A(\alpha,0)$ and $B(\beta,0)$. $M(-2,0)$ is the mid point of AB.

(a) Express $\alpha+\beta$ in terms of p. Hence find the value of p.

(b) If $\alpha^2+\beta^2=26$, find the value of q.



6C.8 HKCEE MA 1991 – I – 7 (Also as **3B.5.**)

Let α and β be the roots of the equation $10x^2+20x+1=0$. Without solving the equation, find the values of

(a) $4^{\alpha}\times 4^{\beta}$,

(b) $\log_{10}\alpha+\log_{10}\beta$.

6C.9 HKCEE MA 1993 – I – 2(f)

If $(x-1)(x+2)=x^2+rx+s$, find r and s.

6C.10 HKCEE MA 1993 – I – 6

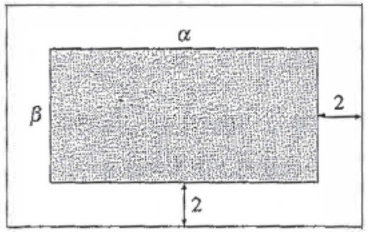
The length α and the breadth β of a rectangular photograph are the roots of the equation $2x^2-mx+500=0$. The photo graph is mounted on a piece of rectangular cardboard, leaving a uniform border of width 2 as shown in the figure.

(a) Find the area of the photograph.

(b) Find, in terms of m,

(i) the perimeter of the photograph,

(ii) the area of the border.



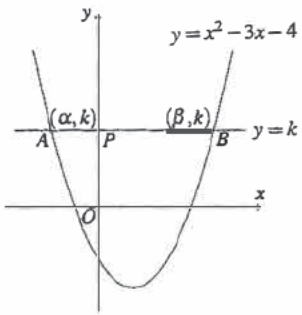
6C.11 HKCEE MA 1995 – I – 8

In the figure, the line $y=k\ (k>0)$ cuts the curve $y=x^2-3x-4$ at the points $A(\alpha,k)$ and $B(\beta,k)$.

(a) (i) Find the value of $\alpha+\beta$.

(ii) Express $\alpha\beta$ in terms of k.

(b) If the line AB cuts the y-axis at P and $BP=2PA$, find the value of k.



6C.12 HKCEE MA 1997 – I – 8

The roots of the equation $2x^2-7x+4=0$ are α and β.

(a) Write down the values of $\alpha+\beta$ and $\alpha\beta$.

(b) Find the quadratic equation whose roots are $\alpha+2$ and $\beta+2$.

6C.13 (HKCEE AM 1984 – I – 5)

Let α and β be the roots of the equation $x^2-2x-(m^2-m+1)=0$, where m is a real number.

(a) Show that $(\alpha-\beta)^2>0$ for any value of m.

(b) Find the minimum value of $\sqrt{(\alpha-\beta)^2}$.

6C.14 HKCEE AM 1987 – I – 5

The equation $x^2+4x+p=0$, where p is a real constant, has distinct real roots α and β.

(a) Find the range of values of p.

(b) If $\alpha^2+\beta^2+\alpha^2\beta^2+3(\alpha+\beta)-19=0$, find the value of p.

6C.15 HKCEE AM 1989 – I – 11 ***[Difficult]***

(a) Let α,β be the roots of the equation $x^2+px+q=0$ ......(*), where p and q are real constants. Find, in terms of p and q,

(i) $\alpha^2+\beta^2$,

(ii) $\alpha^3+\beta^3$,

(iii) $(\alpha^2-\beta-1)(\beta^2-\alpha-1)$.

(b) If the square of one root of (*) minus the other root equals 1, use (a), or otherwise, to show that $q^2-3(p-1)q+(p-1)^2(p+1)=0$ ....(**).

(c) Find the range of values of p such that the quadratic equation (**) in q has real roots.

(d) Suppose k is a real constant. If the square of one root of $4x^2+5x+k=0$ minus the other root equals 1, use the result in (b), or otherwise, to find the value of k.

6C.16 HKCEE AM 1990 – I – 4

α,β are the roots of the quadratic equation $x^2-(k+2)x+k=0$.

(a) Find $\alpha+\beta$ and $\alpha\beta$ in terms of k.

(b) If $(\alpha+1)(\beta+2)=4$, show that $\alpha=-2k$. Hence find the two values of k.

6C.17 HKCEE AM 1991 – I – 7 (To continue as **10C.10.**)

p, q and k are real numbers satisfying the following conditions: $\begin{cases} p+q+k=2, \\ pq+qk+kp=1. \end{cases}$

(a) Express pq in terms of k.

(b) Find a quadratic equation, with coefficients in terms of k, whose roots are p and q.

6C.18 HKCEE AM 1992 – I – 9

α,β are the roots of the quadratic equation $x^2+(p+1)x+(p-1)=0$, where p is a real number.

(a) Show that α,β are real and distinct.

(b) Express $(\alpha-2)(\beta-2)$ in terms of p.

(c) Given $\beta<2<\alpha$.

(i) Using the result of (b), show that $p<-\frac{5}{3}$.

(ii) If $(\alpha-\beta)^2<24$, find the range of possible values of p. Hence write down the possible integral value(s) of p.

6C.19 HKCEE AM 1993 – I – 3

α,β are the roots of the equation $x^2+px+q=0$ and $\alpha+3$, $\beta+3$ are the roots of the equations $x^2+qx+p=0$. Find the values of p and q.

6C.20 (HKCEE AM 1995 – I – 10) ***[Difficult]*** (To continue as **10C.13.**)

Let $f(x)=12x^2+2px-q$ and $g(x)=12x^2+2qx-p$, where p, q are distinct real numbers. α,β are the roots of the equation $f(x)=0$ and α,γ are the roots of the equation $g(x)=0$.

(a) Using the fact that $f(\alpha)=g(\alpha)$, find the value of α. Hence show that $p+q=3$.

(b) Express β and γ in terms of p.

6C.21 HKCEE AM 1998 – I – 2

α,β are the roots of the quadratic equation $x^2-2x+7=0$. Find the quadratic equation whose roots are $\alpha+2$ and $\beta+2$.

6C.22 HKCEE AM 2000 – I – 7

α and β are the roots of the quadratic equation $x^2+(p-2)x+p=0$, where p is real.

(a) Express $\alpha+\beta$ and $\alpha\beta$ in terms of p.

(b) If α and β are real such that $\alpha^2+\beta^2=11$, find the value(s) of p.

6C.23 (HKCEE AM 2011 – I – 7)

Let α and β be the roots of the quadratic equation $x^2+(k+2)x+k=0$, where k is real.

(a) Prove that α and β are real and distinct.

(b) If $\alpha=\sqrt{\beta^2}$, find the value of k.

6C.24 HKDSE MA PP – I – 17 (Continued from **6D.1**.)

(a) Express $\frac{1}{1+2i}$ in the form of $a+bi$, where a and b are real numbers.

(b) The roots of the quadratic equation $x^2+px+q=0$ are $\frac{10}{1+2i}$ and $\frac{10}{1-2i}$. Find

(i) p and q,

(ii) the range of values of r such that the quadratic equation $x^2+px+q=r$ has real roots.

6D Complex numbers

6D.1 HKDSE MA PP – I – 17 (To continue as **6C.24**.)

(a) Express $\frac{1}{1+2i}$ in the form of $a+bi$, where a and b are real numbers.

6A Simple equations

6A.1 HKCEE MA 1980(1*/3) – I – 13(b)

(b) $(1-2x)^2 = 2-x$

$4x^2-3x-1=0$

$(4x+1)(x-1)=0 \Rightarrow x = -\frac{1}{4}$ or 1 (rejected)

6A.2 HKCEE MA 1982(2/3) – I – 7

$x-5 \quad \sqrt{x+1}$

$(x-5)^2 = x+1$

$x^2-11x+24=0 \Rightarrow x=8$ or 3 (rejected)

6A.3 HKCEE MA 1984(A) – I – 3

$(1+\sqrt{2})^4 = \left[(1+\sqrt{2})^2\right]^2 = (1+2\sqrt{2}+2)^2$

$= (3+2\sqrt{2})^2$

$= 9+12\sqrt{2}+8 = 17+12\sqrt{2}$

6A.4 HKCEE MA 1984(A/B) – I – 6

Let $\sqrt{x}=u \Rightarrow u^2-5u \quad 6=0$

$u=6$ or -1

$\sqrt{x}=6$ or -1 (rejected) $\Rightarrow x=36$

6A.5 HKCEE MA 2003 – I – 6

Let x and y first- and economy-class tickets be sold respectively.

$\begin{cases} x+y=600 \\ y=3x \end{cases} \Rightarrow \begin{cases} x=150 \\ y=450 \end{cases}$

∴ Sum of money $= 150 \times \$850 + 450 \times \$500 =$ \$352500

6A.6 HKCEE MA 2004 – I – 7

Let x oranges and y apples be bought.

$\begin{cases} 2x+3y=46 \\ x+y=20 \end{cases} \Rightarrow \begin{cases} x=14 \\ y=6 \end{cases}$

∴ 14 oranges were bought.

6A.7 HKCEE MA 2007 – I – 7

Let there be x elderly patients.

Then there were $67-x$ non-elderly patients.

$120x+160(67-x)=9000$

$10720-40x=9000$

$x=(10720-9000)\div 6=43$

∴ There were 43 elderly patients.

6A.8 HKCEE MA 2008 – I – 3

(a) $m=1$ or 3 (corresponding $n=2$ or 1)

(b) $2x^2+5x+k \equiv 2x^2+(m+2n)x+mn$

Comparing coefficients of like terms, $\begin{cases} 5=m+2n \\ k=mn \end{cases}$

∴ Possible values of k are $(1)(2)=2$ and $(3)(1)=3$ only

6A.9 HKCEE MA 2009 – I – 6

Let John own x stamps.

Then Mary owns $300-x$ stamps.

$(300-x)+20=4x$

$320=5x \Rightarrow x=64$

∴ John owns 64 stamps.

6A.10 HKCEE MA 2010 I 6

Let \$2x and \$x be the costs of 1 orange juice and 1 bottle of milk respectively.

$3(2x)+5(x)=66$

$11x=66 \Rightarrow x=6$

∴ The cost of a bottle of milk is \$6.

6A.11 HKDSE MA SP – I – 5

Let the champion win x games.

Then it has $36-x$ draws.

$3(x)+1(36-x)=84$

$2x=48 \Rightarrow x=24$

∴ The champion wins 24 games.

6A.12 HKDSE MA 2012 – I – 5

Let there be x male guards.

Then there are $132 \quad x$ female guards.

$\frac{132 \quad x}{6} = \frac{x}{6}+4$

$132-x=x+24 \Rightarrow x=54$

∴ There are 54 male guards.

6A.13 HKDSE MA 2013 – I – 4

Let the prices of a pear and an orange be \$x and \$y respectively.

$\begin{cases} 7x+3y=47 & (1) \\ 5x+6y=49 & (2) \end{cases}$

$2(1)-(2): \quad 9x=45 \Rightarrow x=5$

∴ The price of a pear is \$5.

6A.14 HKDSE MA 2015 – I – 7

Let Ada and Billy own $4x$ and x apples.

$4x \quad 12=x+12$

$3x=24 \Rightarrow x=8$

∴ Billy owns 8 apples and Ada $4(8)=32$ apples.

6A.15 HKDSE MA 2017 – I – 4

Let x regular and y concessionary tickets be sold that day.

$\begin{cases} x=5y \\ 126x+78y=50976 \end{cases} \Rightarrow \begin{cases} y=72 \\ x=5(72)=360 \end{cases}$

∴ $360+72=432$ tickets were sold that day

6A.16 HKDSE MA 2019 – I – 3

$(17-3r)^2 = 24^2+(13+r)^2$

$289-102r+9r^2 = 576+169+26r+r^2$

$8r^2-128r-456=0 \Rightarrow r=-3$ or 19 (rejected)

6B Nature of roots of quadratic equations

6B. 1 HKCEE MA 1988 – I – 4

(a) $\Delta = 0$

$(k+1)^2 \quad 36 = 0$

$k+1 = \pm 6 \Rightarrow k = 5$ or -7

(b) When $k = -7$, (*) becomes

$9x^2 + 6x + 1 = 0$

$(3x+1)^2 = 0 \Rightarrow x = -\frac{1}{3}$ (repeated)

6B.2 HKCEE MA 2007 – I – 5

$\Delta < 0$

$14^2 - 4k < 0$

$4k > 196 \Rightarrow k > 49$

6B. 3 HKCEE AM 1980 – I – 1

$2x^2+x+5 = k(x+1)^2 \Rightarrow (2 \quad k)x^2 + (1 \quad 2k)x + (5 \quad k) = 0$

No real roots $\Rightarrow \Delta < 0$

$(1-2k)^2 - 4(2-k)(5 \quad k) < 0$

24k 39 < 0 ⇒ k < 39/24

6B.4 HKCEE AM 1998 – I – 3

$\begin{cases} \alpha^2 - 6\alpha + 2k = 0 & (1) \\ \alpha^2 - 5\alpha + k = 0 & (2) \end{cases}$

$(1)-(2) \Rightarrow -\alpha + k = 0 \Rightarrow \alpha = k$

Hence the equation becomes

$k^2 - 6k + 2k = 0$

$k^2 \quad 4k = 0 \Rightarrow k = 0$ or 4

6C Roots and coefficients of quadratic equations

6C. 1 HKCEE MA 1980(1/1*/3) – I – 3

product of rt=-5/2, k=-9

6C. 2 HKCEE MA 1982(2/3) – I – 1

$a^2 + b^2 = (a \quad b)^2 \quad 2ab = (10)^2 - 2(k) = 100 - 2k$

6C. 3HKCEE MA 1983(B) – I – 14

(a) $\begin{cases} \alpha + \beta = 2m \\ \alpha\beta = n \end{cases}$

(i) $(m \quad \alpha) + (m \quad \beta) = 2m \quad (\alpha+\beta) = 2m \quad (2m) = 0$

(ii) $(m \quad \alpha)(m \quad \beta) = m^2 \quad (\alpha+\beta)m + \alpha\beta$

$= m^2 - (2m)m + (n) = n - m^2$

(b) By (a), the equation is

$x^2 \quad (\text{sum})x + (\text{product}) = 0$

$x^2 - (0)x + (n - m^2) = 0 \Rightarrow x^2 + n - m^2 = 0$

6C. 4 HKCEE MA 1985(A/B) – I – 5

$\begin{cases} \alpha + \beta = \quad k \\ \alpha\beta = 1 \end{cases}$

(a) (i) $(\alpha+2) + (\beta+2) = (\alpha+\beta) + 4 = 4 \quad k$

(ii) $(\alpha+2)(\beta+2) = \alpha\beta + 2(\alpha+\beta) + 4 = 5 \quad 2k$

(b) $p = -(\text{sum of roots}) = -(4-k) = k-4$

$q = \text{product of roots} = 5 \quad 2k$

6C. 5 HKCEE MA 1986(A/B) – I – 7

(a) $\frac{1}{a} = \frac{1}{m} + \frac{1}{n} = \frac{m+n}{mn} = \frac{b}{mn} \Rightarrow mn = \frac{b}{a}$

(b) $m^2 + n^2 = (m+n)^2 - 2mn = (b)^2 - 2\left(\frac{b}{a}\right) = b^2 - \frac{2b}{a}$

6C. 6 HKCEE MA 1987(A/B) I – 5

$\begin{cases} \alpha + \beta = \frac{4}{k} \\ \alpha\beta = 2 \end{cases}$

(a) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \left(\frac{4}{k}\right)^2 - 2(2) = \frac{16}{k^2} - 4$

(b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{\frac{16}{k^2} \quad 4}{2} = \frac{8}{k^2} \quad 2$

6C. 7 HKCEE MA 1990 – I – 6

(a) $\alpha + \beta = -p \Rightarrow -2 = \frac{\alpha+\beta}{2} = \frac{-p}{2} \Rightarrow p = 4$

(b) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 26$

$4^2 \quad 2(q) = 26 \Rightarrow q = -5$

6C. 8 HKCEE MA 1991 – I – 7

$\begin{cases} \alpha + \beta = \quad \frac{20}{10} = \quad 2 \\ \alpha\beta = \frac{1}{10} \end{cases}$

(a) $4^\alpha \times 4^\beta = 4^{\alpha+\beta} = 4^{-2} = \frac{1}{16}$

(b) $\log_{10}\alpha + \log_{10}\beta = \log_{10}\alpha\beta = \log_{10}\frac{1}{10} = \quad 1$

6C.9 HKCEE MA 1993 – I – 2(f)

$r = \quad (\text{sum of roots}) \qquad s = \text{product of roots}$

$= -[1+(-2)] = 1 \qquad = (1)(-2) = -2$

6C. 10 HKCEE MA 1993 – I – 6

(a) From the equation, $\alpha\beta = \frac{500}{2} = 250$

$\therefore$ Area of photograph $= 250$

(b) (i) Perimeter $= 2(\alpha+\beta) = 2\left(\frac{m}{2}\right) = m$

(ii) Area of border $= (\alpha+4)(\beta+4) \quad \alpha\beta$

$= 4(\alpha+\beta) + 16 = 4m + 16$

6C. 11 HKCEE MA 1995 – I – 8

(a) α and β are the roots of the equation $\quad (k) = x^2 - 3x - 4$

$\Rightarrow x^2 \quad 3x \quad 4 \quad k = 0$

(i) $\alpha + \beta = 3$

(ii) $\alpha\beta = \quad 4 \quad k$

(b) $BP = 2PA \Rightarrow \beta = 2(-\alpha) = -2\alpha$

Hence, $\alpha + \beta = 4 \Rightarrow \alpha + (\quad 2\alpha) = 3$

$\alpha = \quad 3 \Rightarrow \beta = 6$

$\therefore (-3)(6) = \alpha\beta = \quad 4 \quad k \Rightarrow k = 14$

6C. 12 HKCEE MA 1997 – I – 8

(a) $\alpha + \beta = \frac{7}{2}, \quad \alpha\beta = \frac{4}{2} = 2$

(b) Sum of roots $= (\alpha+2) + (\beta+2)$

$= (\alpha+\beta) + 4 = \left(\frac{7}{2}\right) + 4 = \frac{15}{2}$

Product of roots $= (\alpha+2)(\beta+2)$

$= \alpha\beta + 2(\alpha+\beta) + 4$

$= (2) + 2\left(\frac{7}{2}\right) + 4 = 13$

Hence, required equation is $x^2 - \frac{15}{2}x + 13 = 0$

$\Rightarrow 2x^2 \quad 15x + 26 = 0$

6C. 13 (HKCEE AM 1984 – I – 5)

(a) $\begin{cases} \alpha + \beta = 2 \\ \alpha\beta = -(m^2 - m + 1) \end{cases}$

$(\alpha \quad \beta)^2 = (\alpha+\beta)^2 \quad 4\alpha\beta = (2)^2 + 4(m^2 \quad m + 1)$

= 4m²-4m+8

=(x-1/2)²+7/4

=>7/4>0

for any value of m.

(b) From (a), minimum of $(\alpha - \beta)^2 = $ 7/4

$\therefore$ minimu m of $\sqrt{(\alpha \quad \beta)^2} = $ 7/4

6C. 14 HKCEE AM 1987 – I – 5

(a) $\Delta > 0 \Rightarrow 16 - 4p > 0 \Rightarrow p < 4$

(b) $\begin{cases} \alpha + \beta = \quad 4 \\ \alpha\beta = p \end{cases}$

$0 = \alpha^2 + \beta^2 + \alpha^2\beta^2 + 3(\alpha+\beta) \quad 19$

$= (\alpha+\beta)^2 - 2\alpha\beta + (\alpha\beta)^2 + 3(\alpha+\beta) - 19$

$= (\quad 4)^2 - 2(p) + (p)^2 + 3(-4) - 19$

$= p^2 - 2p - 15 = (p-5)(p+3)$

$\Rightarrow p = 5$ (rejected) or -3

6C. 15 HKCEE AM 1989 – I – 11

(a) $\begin{cases} \alpha + \beta = -p \\ \alpha\beta = q \end{cases}$

(i) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (\quad p)^2 \quad 2q$

$= p^2 - 2q$

(ii) $\alpha^3 + \beta^3 = (\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)$

$= (\alpha+\beta)[(\alpha+\beta)^2 - 3\alpha\beta]$

$= (-p)[(-p)^2 \quad 3(q)] = 3pq \quad p^3$

(iii) $(\alpha^2 - \beta - 1)(\beta^2 - \alpha - 1)$

$= \alpha^2\beta^2 - (\alpha^3 + \beta^3) \quad (\alpha^2 + \beta^2) + \alpha\beta + (\alpha+\beta) + 1$

$= (q)^2 - (3pq - p^3) - (p^2 - 2q) + (q) + (-p) + 1$

$p^3 \quad p^2 + q^2 - 3pq + 3q - p + 1$

(b) The given informa tionmeans either

$\alpha^2 - \beta = 1$ or $\beta^2 \quad \alpha = 1$

$\Rightarrow (\alpha^2 - \beta - 1)(\beta^2 - \alpha - 1) = 0$

$p^3 \quad p^2 + q^2 \quad 3pq + 3q \quad p + 1 = 0$

$q^2 \quad 3(p \quad 1)q + p^3 \quad p^2 \quad p + 1 = 0$

$q^2 - 3(p \quad 1)q + p^2(p \quad 1) - (p-1) = 0$

$q^2 - 3(p \quad 1)q + (p-1)(p^2 \quad 1) = 0$

$q^2 - 3(p-1)q + (p-1)^2(p+1) = 0$

(c) $\Delta \geq 0$

$9(p \quad 1)^2 \quad 4(p \quad 1)^2(p+1) \geq 0$

$(p-1)^2[9 - 4(p+1)] \geq 0$

$(p-1)^2(5-4p) \geq 0$

Since $(p-1)^2 \geq 0$, $5 - 4p \geq 0 \Rightarrow p < \frac{5}{4}$

(d) $4x^2 + 5x + k = 0 \Leftrightarrow x^2 + \frac{5}{4}x + \frac{k}{4} = 0$

Put $p = \frac{5}{4}$ and $q = \frac{k}{4}$ into (b):

$\left(\frac{k}{4}\right)^2 - 3\left(\frac{1}{4}\right)\left(\frac{k}{4}\right) + \left(\frac{1}{4}\right)^2\left(\frac{9}{4}\right) = 0$

$4k^2 - 12k + 9 = 0 \Rightarrow k = \frac{3}{2}$

6C. 16 HKCEE AM 1990 – I – 4

(a) $\alpha + \beta = k + 2, \quad \alpha\beta = k$

(b) $(\alpha+1)(\beta+2) = 4$

$\alpha\beta + 2\alpha + \beta + 2 = 4$

$\alpha\beta + (\alpha+\beta) + \alpha + 2 = 4$

$(k+2) + (k) + \alpha + 2 = 4 \Rightarrow \alpha = \quad 2k$

Hence, putting $\alpha = \quad 2k$ into the equation:

$(-2k)^2 - (k+2)(-2k) + k = 0$

$6k^2 - 3k = 0 \Rightarrow k = 0$ or $\frac{1}{2}$

6C. 17 HKCEE AM 1991 – I – 7

(a) From the first equation, $p + q = 2 \quad k$

From the second equation, $pq + k(p+q) = 1$

$pq = 1 \quad k(2 \quad k)$

$= (k+1)^2$

(b) Sum of roots $= p + q = 2 \quad k$

Product of roots $= (k+1)^2$

$\therefore$ Required equation: $x^2 - (2-k)x + (k+1)^2 = 0$

6C.18 HKCEE AM 1992 – I – 9

(a) $\Delta = (p+1)^2 - 4(p-1) = p^2 - 2p + 5$
$= (p+1)^2 + 4 \ge 4 > 0$

Hence, the two roots are real and distinct.

(b) $\begin{cases} \alpha+\beta = -(p+1) \\ \alpha\beta = p-1 \end{cases}$

$\therefore (\alpha-2)(\beta-2) = \alpha\beta - 2(\alpha+\beta) + 4$
$= (p-1) + 2(p+1) + 4 = 3p+5$

(c) (i) $\beta < 2 < \alpha \Rightarrow \alpha - 2 > 0$ and $\beta - 2 < 0$

$\therefore (\alpha-2)(\beta-2) < 0$

$3p + 5 < 0 \Rightarrow p < -\frac{5}{3}$

(ii) $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta < 24$
$(p+1)^2 - 4(p-1) < 24$
$(p-1)^2 < 20$
$1 - \sqrt{20} < p < 1 + \sqrt{20}$

Together with (c)(i), $1 - \sqrt{20} < p < -\frac{5}{3}$

$\therefore$ Possible integral values $= -3$ and -2

6C.19 HKCEE AM 1993 – I – 3

$\begin{cases} \alpha+\beta = -p \\ \alpha\beta = q \end{cases}$ and $\begin{cases} (\alpha+3)+(\beta+3) = -q \\ (\alpha+3)(\beta+3) = p \end{cases}$

$\Rightarrow \begin{cases} \alpha+\beta = -q-6 \\ \alpha\beta = p - 3(\alpha+\beta) - 9 = 4p - 9 \end{cases}$

$\therefore \begin{cases} -p = -q-6 \\ q = 4p - 9 \end{cases} \Rightarrow \begin{cases} p = 1 \\ q = -5 \end{cases}$

6C.20 (HKCEE AM 1995 – I – 10)

(a)
$f(\alpha) = g(\alpha)$
$12\alpha^2 + 2p\alpha - q = 12\alpha^2 + 2q\alpha - p$
$2\alpha\cancel{(p-q)} = -\cancel{(p-q)}$ ($\because$ p, q are distinct)
$2\alpha = -1 \Rightarrow \alpha = -\frac{1}{2}$

(b) $\alpha + \beta = -\frac{2p}{12} \Rightarrow \beta = \frac{-p}{6} + \frac{1}{2}$

$\alpha\gamma = \frac{-p}{12} \Rightarrow \gamma = \frac{-p}{12} \div \frac{-1}{2} = \frac{p}{6}$

6C.21 HKCEE AM 1998 – I – 2

$\begin{cases} \alpha+\beta = 2 \\ \alpha\beta = 7 \end{cases}$

Sum of roots $= (\alpha+2) + (\beta+2) = (\alpha+\beta) + 4$
$= (2) + 4 = 6$

Product of roots $= (\alpha+2)(\beta+2) = \alpha\beta + 2(\alpha+\beta) + 4$
$= (7) + 2(2) + 4 = 15$

Required equation: $x^2 - 6x + 15 = 0$

6C.22 HKCEE AM 2000 – I – 7

(a) $\alpha + \beta = 2 - p$, $\alpha\beta = p$

(b)
$\alpha^2 + \beta^2 = 11$
$(\alpha+\beta)^2 - 2\alpha\beta = 11$
$(2-p)^2 - 2(p) = 11$
$p^2 - 6p - 7 = 0 \Rightarrow p = 7$ or -1

6C.23 (HKCEE AM 2011 – I – 7)

(a) $\Delta = (k+2)^2 - 4k = k^2 + 4$
$\ge 0 + 4 > 0$

$\therefore$ The roots are real and distinct.

(b) If $\alpha = \sqrt{\beta^2}$ and $\alpha \ne \beta$ (from (a)),
then $\alpha = -\beta$.
$\therefore \alpha + \beta = 0 \Rightarrow k = -2$

6C.24 HKDSE MA PP – I – 17

(a) $\frac{1}{1+2i} = \frac{1(1-2i)}{(1+2i)(1-2i)} = \frac{1-2i}{1^2+2^2} = \frac{1}{5} - \frac{2}{5}i$

(b) (i) By (a), the roots are $10\left(\frac{1}{5} - \frac{2}{5}i\right) = 2 - 4i$ and $2 + 4i$.

$\therefore \begin{cases} p = -(\text{sum of roots}) = -4 \\ q = \text{product of roots} = 2^2 + 4^2 = 20 \end{cases}$

(ii) The equation becomes $x^2 - 4x + (20 - r) = 0$.
$\Delta \ge 0$
$16 - 4(20 - r) \ge 0 \Rightarrow r \ge 16$

6D Complex numbers

6D.1 HKDSE MA PP – I – 17

(a) $\frac{1}{1+2i} = \frac{1(1-2i)}{(1+2i)(1-2i)} = \frac{1-2i}{1^2+2^2} = \frac{1}{5} - \frac{2}{5}i$