# 3 Indices and Logarithms

## 3A Laws of indices

**3A.1** HKCEE MA 1987(A) – I – 3(a)

Simplify $\sqrt{\dfrac{3^{5k+2}}{27^k}}$.

**3A.2** HKCEE MA 1990 – I – 2(a)

Simplify $\dfrac{a}{\sqrt{a}}$, expressing your answer in index form.

**3A.3** HKCEE MA 1993 – I – 5(b)

Simplify and express with positive indices $x\left(\dfrac{x^{-1}}{y^2}\right)^{-3}$.

**3A.4** HKCEE MA 1994 – I – 7(a)

Simplify $\dfrac{(a^4b^{-2})^2}{ab}$ and express your answer with positive indices.

**3A.5** HKCEE MA 1996 – I – 2

Simplify $\dfrac{a^{\frac{5}{4}}\sqrt[4]{a^3}}{a^{-2}}$.

**3A.6** HKCEE MA 1997 – I – 2(a)

Simplify $\dfrac{x^3y^2}{x^{-3}y}$ and express your answer with positive indices.

**3A.7** HKCEE MA 1998 – I – 4

Simplify $\dfrac{a^3a^4}{b^{-2}}$ and express your answer with positive indices.

**3A.8** HKCEE MA 1999 – I – 1

Simplify $\dfrac{(a^{-3})^2}{a}$ and express your answer with positive indices

**3A.9** HKCEE MA 2000 – I – 2

Simplify $\dfrac{x^{-3}y}{x^2}$ and express your answer with positive indices.

**3A.10** HKCEE MA 2001 – I – 1

Simplify $\dfrac{m^3}{(mn)^2}$ and express your answer with positive indices.





**3A.11** HKCEE MA 2002 – I – 1

Simplify $\dfrac{(ab^2)^2}{a^5}$ and express your answer with positive indices.

**3A.12** HKCEE MA 2003 – I – 4

Solve the equation $4^{x+1} = 8$.

**3A.13** HKCEE MA 2004 – I – 1

Simplify $\dfrac{(a^{-1}b)^3}{b^2}$ and express your answer with positive indices.

**3A.14** HKCEE MA 2005 – I – 2

Simplify $\dfrac{(x^3y)^2}{y^5}$ and express your answer with positive indices.

**3A.15** HKCEE MA 2006 – I – 1

Simplify $\dfrac{(a^3)^5}{a^{-6}}$ and express your answer with positive indices.

**3A.16** HKCEE MA 2007 – I – 2

Simplify $\dfrac{m^6}{m^9n^{-5}}$ and express your answer with positive indices

**3A.17** HKCEE MA 2008 – I – 1

Simplify $\dfrac{(ab)^3}{a^2}$ and express your answer with positive indices.

**3A.18** HKCEE MA 2009 – I – 2

Simplify $\dfrac{x^2}{(x^{-7}y)^3}$ and express your answer with positive indices.

**3A.19** HKCEE MA 2010 – I – 1

Simplify $a^{14}\left(\dfrac{b^3}{a^2}\right)^5$ and express your answer with positive indices.

**3A.20** HKCEE MA 2011 – I – 2

Simplify $\dfrac{x^{65}}{(x^4y^3)^2}$ and express your answer with positive indices.

**3A.21** HKDSE MA SP – I – 1

Simplify $\dfrac{(xy)^2}{x^{-5}y^6}$ and express your answer with positive indices.

**3A.22** HKDSE MA PP – I – 1

Simplify $\dfrac{(m^5n^{-2})^6}{m^4n^{-3}}$ and express your answer with positive indices.

**3A.23** HKDSE MA 2012 – I – 1

Simplify $\dfrac{m^{-12}n^{8}}{n^{3}}$ and express your answer with positive indices.

**3A.24** HKDSE MA 2013 – I – 1

Simplify $\dfrac{x^{20}y^{13}}{(x^{5}y)^{6}}$ and express your answer with positive indices.

**3A.25** HKDSE MA 2014 – I – 1

Simplify $\dfrac{(xy^{-2})^{3}}{y^{4}}$ and express your answer with positive indices.

**3A.26** HKDSE MA 2015 – I – 1

Simplify $\dfrac{m^{9}}{(m^{3}n^{-7})^{5}}$ and express your answer with positive indices.

**3A.27** HKDSE MA 2016 – I – 1

Simplify $\dfrac{(x^{8}y^{7})^{2}}{x^{5}y^{-6}}$ and express your answer with positive indices.

**3A.28** HKDSE MA 2017 – I – 2

Simplify $\dfrac{(m^{4}n^{-1})^{3}}{(m^{-2})^{5}}$ and express your answer with positive indices.

**3A.29** HKDSE MA 2018 – I – 2

Simplify $\dfrac{xy^{7}}{(x^{-2}y^{3})^{4}}$ and express your answer with positive indices.

**3A.30** HKDSE MA 2020 – I – 1

Simplify $\dfrac{(mn^{-2})^{5}}{m^{-4}}$ and express your answer with positive indices.





## 3B Logarithms

**3B.1** HKCEE MA 1986(A) – I – 5(a)

Evaluate $\log_2 8 + \log_2 \dfrac{1}{16}$.

**3B.2** HKCEE MA 1987(A) – I – 3(b)

Simplify $\dfrac{\log a^3 b^2 - \log ab^2}{\log \sqrt{a}}$.

**3B.3** HKCEE MA 1988 – I – 6

Give that $\log 2 = r$ and $\log 3 = s$, express the following in terms of $r$ and $s$:

(a) $\log 18$,

(b) $\log 15$.

**3B.4** HKCEE MA 1990 – I – 2(b)

Simplify $\dfrac{\log(a^2) + \log(b^4)}{\log(ab^2)}$, where $a, b > 0$.

**3B.5** HKCEE MA 1991 – I – 7 (Also as **6C.8**.)

Let $\alpha$ and $\beta$ be the roots of the equation $10x^2 + 20x + 1 = 0$. Without solving the equation, find the values of

(a) $4^{\alpha} \times 4^{\beta}$,

(b) $\log_{10} \alpha + \log_{10} \beta$.

**3B.6** HKCEE MA 1992 – I – 2(a)

If $\log x = p$ and $\log y = q$, express $\log xy$ in terms of $p$ and $q$.

**3B.7** HKCEE MA 1994 – I – 7(b)

If $\log 2 = x$ and $\log 3 = y$, express $\log \sqrt{12}$ in terms of $x$ and $y$.

**3B.8** HKCEE MA 1997 – I – 2(b)

Simplify $\dfrac{\log 8 + \log 4}{\log 16}$.

**3B.9** HKDSE MA SP – I – 17

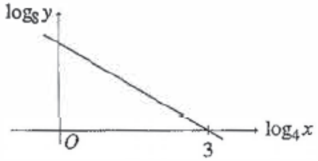
A researcher defined Scale $A$ and Scale $B$ to represent the magnitude of an explosion as shown in the table:

| Scale | Formula |
| --- | --- |
| $A$ | $M = \log_4 E$ |
| $B$ | $N = \log_8 E$ |

It is given that $M$ and $N$ are the magnitudes of an explosion on Scale $A$ and Scale $B$ respectively, while $E$ is the relative energy released by the explosion. If the magnitude of an explosion is 6.4 on Scale $B$, find the magnitude of the explosion on Scale $A$.

**3B.10** HKDSE MA 2014 – I – 15

The graph in the figure shows the linear relation between $\log_4 x$ and $\log_8 y$. The slope and the intercept on the horizontal axis of the graph are $\frac{-1}{3}$ and 3 respectively. Express the relation between $x$ and $y$ in the form $y = Ax^k$, where $A$ and $k$ are constants.




**3B.11** HKDSE MA 2017 I 15

Let $a$ and $b$ be constants. Denote the graph of $y = a + \log_b x$ by $G$. The $x$ intercept of $G$ is 9 and $G$ passes through the point (243, 3). Express $x$ in terms of $y$.





## 3C Exponential and logarithmic equations

**3C.1** HKCEE MA 1980(3) – I 7

Find $x$ if $\log_3(x-3) + \log_3(x+3) = 3$.

**3C.2** HKCEE MA 1981(1) I 5 & HKCEE MA 1981(2) – I – 6

Solve $4^x = 10 \quad 4^{x+1}$.

**3C.3** HKCEE MA 1982(1/2) I 2

If $\begin{cases} 4^{x-y} = 4 \\ 4^{x+y} = 16 \end{cases}$, solve for $x$ and $y$.

**3C.4** HKCEE MA 1985(B) I – 3

Solve $2^{2x} - 3(2^x) \quad 4 = 0$.

**3C.5** HKCEE MA 1986(A) I 5(b)

If $2\log_{10} x - \log_{10} y = 0$, show that $y = x^2$.

**3C.6** HKCEE MA 1987(B) I – 3

Solve the equation $3^{2x} + 3^x - 2 = 0$.

**3C.7** HKCEE MA 1993 I 5(a)

If $9^x = \sqrt{3}$, find $x$.

**3C.8** HKCEE MA 1995 I – 7

Solve the following equations without using a calculator:

(a) $3^x = \frac{1}{\sqrt{27}}$;

(b) $\log x + 2\log 4 = \log 48$.

# 3 Indices and Logarithms

## 3A Laws of indices

**3A.1** HKCEE MA 1987(A) – I – 3(a)

$$\sqrt{\frac{3^{5k+2}}{27^k}} = \left(\frac{3^{5k+2}}{3^{3k}}\right)^{\frac{1}{2}} = (3^{2k+2})^{\frac{1}{2}} = 3^{k+1}$$

**3A.2** HKCEE MA 1990 – I – 2(a)

$$\frac{a}{\sqrt{a}} = a^{1-\frac{1}{2}} = a^{\frac{1}{2}}$$

**3A.3** HKCEE MA 1993 – I – 5(b)

$$x\left(\frac{x^{-1}}{y^2}\right)^{-3} = x\left(\frac{x^{+3}}{y^{-6}}\right) = x^4y^6$$

**3A.4** HKCEE MA 1994 – I – 7(a)

$$\frac{(a^4b^{-2})^2}{ab} = \frac{a^8b^{-4}}{ab} = \frac{a^{8-1}}{b^{1+4}} = \frac{a^7}{b^5}$$

**3A.5** HKCEE MA 1996 – I – 2

$$\frac{a^{\frac{5}{4}}\sqrt[4]{a^3}}{a^{-2}} = \frac{a^{\frac{5}{4}}a^{\frac{3}{4}}}{a^{-2}} = a^{\frac{5}{4}+\frac{3}{4}-(-2)} = a^4$$

**3A.6** HKCEE MA 1997 – I – 2(a)

$$\frac{x^3y^2}{x^{-3}y} = x^{3-(-3)}y^{2-1} = x^6y$$

**3A.7** HKCEE MA 1998 – I – 4

$$\frac{a^3a^4}{b^{-2}} = a^{3+4}b^2 = a^7b^2$$

**3A.8** HKCEE MA 1999 – I – 1

$$\frac{(a^{-3})^2}{a}\frac{a^{-6}}{a} = \frac{1}{a^{1+6}} = \frac{1}{a^7}$$

**3A.9** HKCEE MA 2000 – I – 2

$$\frac{x^{-3}y}{x^2} = \frac{y}{x^{2+3}} = \frac{y}{x^5}$$

**3A.10** HKCEE MA 2001 – I – 1

$$\frac{m^3}{(mn)^2} = \frac{m^3}{m^2n^2} = \frac{m}{n^2}$$

**3A.11** HKCEE MA 2002 – I – 1

$$\frac{(ab^2)^2}{a^5} = \frac{a^2b^4}{a^5} = \frac{b^4}{a^{5-2}} = \frac{b^4}{a^3}$$

**3A.12** HKCEE MA 2003 – I – 4

$$2^{2(x+1)} = 2^3 \Rightarrow 2x+2=3 \Rightarrow x=\frac{1}{2}$$

**3A.13** HKCEE MA 2004 – I – 1

$$\frac{(a^{-1}b)^3}{b^2} = \frac{a^{-3}b^3}{b^2} = \frac{b^{3-2}}{a^3} = \frac{b}{a^3}$$

**3A.14** HKCEE MA 2005 – I – 2

$$\frac{(x^3y)^2}{y^5} = \frac{x^6y^2}{y^5} = \frac{x^6}{y^3}$$

**3A.15** HKCEE MA 2006 – I – 1

$$\frac{(a^3)^5}{a^{-6}} = \frac{a^{15}}{a^{-6}} = a^{15-(-6)} = a^{21}$$

**3A.16** HKCEE MA 2007 – I – 2

$$\frac{m^6}{m^9n^{-5}} = \frac{n^5}{m^{9-6}} = \frac{n^5}{m^3}$$

**3A.17** HKCEE MA 2008 – I – 1

$$\frac{(ab)^3}{a^2} = \frac{a^3b^3}{a^2} = ab^3$$

**3A.18** HKCEE MA 2009 – I – 2

$$\frac{x^2}{(x^{-7}y)^3} = \frac{x^2}{x^{-21}y^3} = \frac{x^{2+21}}{y^3} = \frac{x^{23}}{y^3}$$

**3A.19** HKCEE MA 2010 – I – 1

$$a^{14}\left(\frac{b^3}{a^2}\right)^5 = a^{14}\cdot\frac{b^{15}}{a^{10}} = a^4b^{15}$$

**3A.20** HKCEE MA 2011 – I – 2

$$\frac{x^{65}}{(x^4y^3)^2} = \frac{x^{65}}{x^8y^6} = \frac{x^{57}}{y^6}$$

**3A.21** HKDSE MA SP – I – 1

$$\frac{(xy)^2}{x^{-5}y^6} = \frac{x^2y^2}{x^{-5}y^6} = \frac{x^{2+5}}{y^{6-2}} = \frac{x^7}{y^4}$$

**3A.22** HKDSE MA PP – I – 1

$$\frac{(m^5n^{-2})^6}{m^4n^{-3}}\frac{m^{30}n^{-12}}{m^4n^{-3}} = \frac{m^{30-4}}{n^{-3+12}} = \frac{m^{26}}{n^9}$$

**3A.23** HKDSE MA 2012 – I – 1

$$\frac{m^{-12}n^8}{n^3} = \frac{n^{8-3}}{m^{12}} = \frac{n^5}{m^{12}}$$

**3A.24** HKDSE MA 2013 – I – 1

$$\frac{x^{20}y^{13}}{(x^5y)^6} = \frac{x^{20}y^{13}}{x^{30}y^6} = \frac{y^7}{x^{10}}$$

**3A.25** HKDSE MA 2014 – I – 1

$$\frac{(xy^{-2})^3}{y^4} = \frac{x^3y^{-6}}{y^4} = \frac{x^3}{y^{4+6}} = \frac{x^3}{y^{10}}$$

**3A.26** HKDSE MA 2015 – I – 1

$$\frac{m^9}{(m^3n^{-7})^5} = \frac{m^9}{m^{15}n^{-35}} = \frac{n^{35}}{m^6}$$

**3A.27** HKDSE MA 2016 – I – 1

$$\frac{(x^8y^7)^2}{x^5y^{-6}} = \frac{x^{16}y^{14}}{x^5y^{-6}} = x^{16-5}y^{14-(-6)} = x^{11}y^{20}$$

### 3A.28 HKDSE MA 2017 – I – 2

$$\frac{(m^4n^{-1})^3}{(m^{-2})^5}=\frac{m^{12}n^{-3}}{m^{-10}}=\frac{m^{12-(\ \ 10)}}{n^{3}}=\frac{m^{22}}{n^3}$$

### 3A.29 HKDSE MA 2018 – I – 2

$$\frac{xy^7}{(x^{-2}y^3)^4}=\frac{xy^7}{x^{-8}y^{12}}=\frac{x^{1+8}}{y^{12-7}}=\frac{x^9}{y^5}$$

### 3A.30 HKDSE MA 2020 – I – 1

$$\frac{\left(mn^{-2}\right)^5}{m^{-4}}=m^{5-(-4)}n^{-2\times5}$$
$$=m^9n^{-10}$$
$$=\frac{m^9}{n^{10}}$$

## 3B Logarithms

### 3B.1 HKCEE MA 1986(A) – I – 5(a)

$$\log_2 8+\log_2\frac{1}{16}=\log_2 2^3+\log_2 2^{-4}=3+(-4)=-1$$

### 3B.2 HKCEE MA 1987(A) – I – 3(b)

$$\frac{\log a^3b^2-\log ab^2}{\log\sqrt{a}}=\frac{\log\frac{a^3b^2}{ab^2}}{\frac{1}{2}\log a}=\frac{\log a^2}{\frac{1}{2}\log a}=\frac{2\log a}{\frac{1}{2}\log a}=4$$

### 3B.3 HKCEE MA 1988 – I – 6

(a) $\log 18=\log 2\cdot 3^2=\log 2+2\log 3=r+2s$

(b) $\log 15=\log\frac{3\times 10}{2}=\log 3+1-\log 2=s+1-r$

### 3B.4 HKCEE MA 1990 – I – 2(b)

$$\frac{\log(a^2)+\log(b^4)}{\log(ab^2)}=\frac{\log a^2b^4}{\log ab^2}=\frac{\log(ab^2)^2}{\log ab^2}=\frac{2\log ab^2}{\log ab^2}=2$$

### 3B.5 HKCEE MA 1991 – I – 7

$$\begin{cases}\alpha+\beta=\ \ 2\\ \alpha\beta=\frac{1}{10}\end{cases}$$

(a) $4^\alpha\times4^\beta=4^{\alpha+\beta}=4^{-2}=\frac{1}{16}$

(b) $\log_{10}\alpha+\log_{10}\beta=\log_{10}\alpha\beta=\log_{10}\frac{1}{10}=-1$

### 3B.6 HKCEE MA 1992 – I – 2(a)

$$\log xy=\log x+\log y=p+q$$

### 3B.7 HKCEE MA 1994 – I – 7(b)

$$\log\sqrt{12}=\frac{1}{2}\log 2^2\cdot 3=\frac{1}{2}(2\log 2+\log 3)=\frac{2x+y}{2}$$

### 3B.8 HKCEE MA 1997 – I – 2(b)

$$\frac{\log 8+\log 4}{\log 16}\quad\frac{3\log 2+2\log 2}{4\log 2}=\frac{5\log 2}{4\log 2}=\frac{5}{4}$$

### 3B.9 HKDSE MA SP – I – 17

*Method 1*

$6.4=\log_8 E\ \Rightarrow\ E=8^{6.4}$

$$\therefore\ M=\log_4 E=\log_4(8^{6.4})=\frac{\log_2 8^{6.4}}{\log_2 4}$$
$$=\frac{\log_2 2^{3(6.4)}}{\log_2 2^2}=\frac{19.2}{2}=9.6$$

*Method 2*

$$\begin{cases}M=\log_4 E\\ N=\log_8 E\end{cases}\Rightarrow\begin{cases}E=4^M\\ E=8^N\end{cases}\Rightarrow\ 4^M=8^N$$
$$2^{2M}=2^{3N}$$
$$M=\frac{3}{2}N=\frac{3}{2}(6.4)=9.6$$



### 3B.10 HKDSE MA 2014 – I – 15

*Method 1*

From the graph, $(\log_4 x,\log_8 y)=(3,0)$ and Slope $=\frac{-1}{3}$.

Using point-slope form, the equation is:

$$\log_8 y-0=\frac{-1}{3}(\log_4 x-3)$$
$$\log_8 y=\frac{-1}{3}\log_4 x+1$$
$$=\log_4\left(x^{\frac{-1}{3}}\cdot 4\right)$$
$$\frac{\log_2 y}{\log_2 8}=\frac{\log_2 4x^{\frac{-1}{3}}}{\log_2 4}$$
$$\frac{\log_2 y}{3}=\frac{\log_2 4x^{\frac{-1}{3}}}{2}$$
$$\log_2 y=\frac{3}{2}\log_2 4x^{\frac{-1}{3}}$$
$$=\log_2\left(4x^{\frac{-1}{3}}\right)^{\frac{3}{2}}=\log_2 8x^{\frac{-1}{2}}$$
$$\Rightarrow\ y=8x^{\frac{-1}{2}}$$

*Method 2*

$(\log_4 x,\log_8 y)=(3,0)\ \Rightarrow\ (x,y)=(64,1)$

Let the point of the line cutting the vertical axis be $(0,b)$.

$\frac{b-0}{0-3}=\frac{-1}{3}\ \Rightarrow\ b=1$

$\therefore\ (\log_4 x,\log_8 y)=(0,1)\ \Rightarrow\ (x,y)=(1,8)$

Putting into $y=Ax^k$, $\begin{cases}8=A\\ 1=A(64)^k\ \Rightarrow\ 1=8^{1+2k}\ \Rightarrow\ k=\frac{-1}{2}\end{cases}$

Hence, $y=8x^{\frac{-1}{2}}$.

*Method 3*

$$y=Ax^k\ \Rightarrow\ \log_2 y=\log_2 Ax^k=\log_2 A+k\log_2 x$$
$$\frac{\log_8 y}{\log_8 2}=\log_2 A+k\frac{\log_4 x}{\log_4 2}$$
$$3\log_8 y=\log_2 A+2k\log_4 x$$
$$\log_8 y=\frac{2k}{3}\log_4 x+\frac{1}{3}\log_2 A$$

From theory of straight lines,

$$\begin{cases}\frac{-1}{3}=\text{Slope}=\frac{2k}{3}\ \Rightarrow\ k=\frac{-1}{2}\\ 3=x\text{-intercept}=-\frac{\frac{1}{3}\log_2 A}{\frac{2k}{3}}=\frac{-1}{2k}\log_2 A\ \Rightarrow\ A=2^3=8\end{cases}$$

Hence, $y=8x^{\frac{-1}{2}}$.

### 3B.11 HKDSE MA 2017 – I – 15

$G$ passes through $(9,0)$ and $(243,3)$

$$\Rightarrow\ \begin{cases}0=a+\log_b 9\\ 3=a+\log_b 243\end{cases}\ \Rightarrow\ 3=\log_b 243-\log_b 9=\log_b\frac{243}{9}$$

$\Rightarrow\ b^3=27\ \Rightarrow\ b=9\ \Rightarrow\ a=-\log_b 9=-2$

$\therefore\ y=-2+\log_3 x\ \Rightarrow\ \log_3 x=y+2\ \Rightarrow\ x=3^{y+2}$

## 3C Exponential and logarithmic equations

### 3C.1 HKCEE MA 1980(3) – I – 7

$$\log_3(x-3)+\log_3(x+3)=3$$
$$\log_3(x-3)(x+3)=3$$
$$x^2-9=27$$
$$x=6\text{ or }\ \ 6\text{ (rejected)}$$

### 3C.2 HKCEE MA 1981(1) – I – 5 & 1981(2) – I – 6

$$4^x=10-4^{x+1}$$
$$4^x=10-4^x\cdot 4$$
$$(1+4)4^x=10$$
$$4^x=2\ \Rightarrow\ x=\frac{1}{2}$$

### 3C.3 HKCEE MA 1982(1/2) – I – 2

$$\begin{cases}4^{x-y}=4\ \Rightarrow\ x\ \ \ y=1\\ 4^{x+y}=16\ \Rightarrow\ x+y=2\end{cases}\ \Rightarrow\ \begin{cases}x=\frac{3}{2}\\ y=\frac{1}{2}\end{cases}$$

### 3C.4 HKCEE MA 1985(B) – I – 3

$$2^{2x}\ \ \ 3(2^x)\ \ \ 4=0$$
$$(2^x)^2\ \ \ 3(2^x)\ \ \ 4=0$$
$$(2^x-4)(2^x+1)=0$$
$$2^x=4\text{ or }-1\text{ (rejected)}\ \Rightarrow\ x=2$$

### 3C.5 HKCEE MA 1986(A) – I – 5(b)

$$2\log_{10}x-\log_{10}y=0$$
$$\log_{10}x^2=\log_{10}y$$
$$x^2=y$$

### 3C.6 HKCEE MA 1987(B) – I – 3

$$3^{2x}+3^x-2=0$$
$$(3^x)^2+(3^x)-2=0$$
$$(3^x+2)(3^x-1)=0$$
$$3^x=-2\text{ (rejected) or }1\ \Rightarrow\ x=0$$

### 3C.7 HKCEE MA 1993 – I – 5(a)

$$9^x=\sqrt{3}$$
$$3^{2x}=3^{\frac{1}{2}}\ \Rightarrow\ 2x=\frac{1}{2}\ \Rightarrow\ x=\frac{1}{4}$$

### 3C.8 HKCEE MA 1995 – I – 7

(a) $3^x=\frac{1}{\sqrt{27}}=27^{\frac{-1}{2}}=(3^3)^{\frac{-1}{2}}$

$x=\frac{-3}{2}$

(b) $\log x+2\log 4=\log 48$

$\log x+\log 4^2=\log 48$

$\log 16x=\log 48\ \Rightarrow\ 16x=48\ \Rightarrow\ x=3$