16 Coordinate Geometry

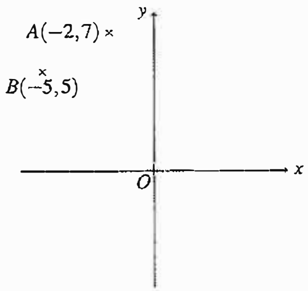
16A Transformation in the rectangular coordinate plane

16A.1 HKCEE MA 2006 – I – 7

In the figure, the coordinates of the points A and B are $(-2,7)$ and $(-5,5)$ respectively. A is rotated clockwise about the origin O through 90° to A'. B' is the reflection image of B with respect to the y-axis.

(a) Write down the coordinates of A' and B'.

(b) Are the lengths of AB and $A'B'$ equal? Explain your answer.

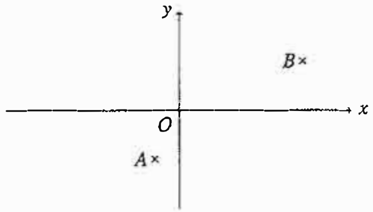


16A.2 HKCEE MA 2009 – I – 9

In the figure, the coordinates of the points A and B are $(-1,-2)$ and $(5,2)$ respectively. A is translated vertically upward by 6 units to A'. B' is the reflection image of B with respect to the y axis.

(a) Write down the coordinates of A' and B'.

(b) Is AB parallel to $A'B'$? Explain your answer.



16A.3 HKCEE MA 2011 I 8

The coordinates of the point A are $(-4,6)$. A is rotated anticlockwise about the origin O through 90° to B. M is the mid-point of AB.

(a) Find the coordinates of M.

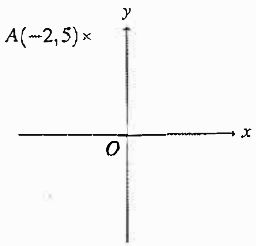
(b) Is OM perpendicular to AB? Explain your answer.

16A.4 HKDSE MA SP – I – 8

In the figure, the coordinates of the point A are $(-2,5)$. A is rotated clockwise about the origin O through 90° to A'. A'' is the reflection image of A with respect to the y-axis.

(a) Write down the coordinates of A' and A''.

(b) Is OA'' perpendicular to AA'? Explain your answer.



16A.5 HKDSE MA 2014 I – 8

The coordinates of the points P and Q are $(-3,5)$ and $(2,-7)$ respectively. P is rotated anticlockwise about the origin O through 270° to P'. Q is translated leftwards by 21 units to Q'.

(a) Write down the coordinates of P' and Q'.

(b) Prove that PQ is perpendicular to $P'Q'$.

16A.6 HKDSE MA 2017 – I – 6

The coordinates of the points A and B are $(\ 3,4)$ and $(9, \ 9)$ respectively. A is rotated anticlockwise about the origin through 90° to A'. B' is the reflection image of B with respect to the x-axis.

(a) Write down the coordinates of A' and B'.

(b) Prove that AB is perpendicular to $A'B'$.

16B Straight lines in the rectangular coordinate plane

16B.1 HKCEE MA 1992 – I 5

L_1 is the line passing through the point $A(10,5)$ and perpendicular to the line $L_2: x-2y+5=0$.

(a) Find the equation of L_1.

(b) Find the intersection point of L_1 and L_2.

16B.2 HKCEE MA 1998 I – 8

$A(0,4)$ and $B(-2,1)$ are two points.

(a) Find the slope of AB.

(b) Find the equation of the line passing through $(1,3)$ and perpendicular to AB.

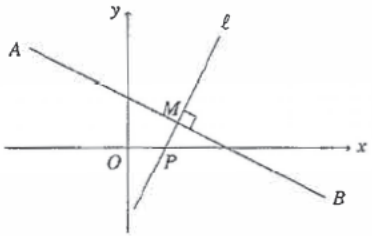
16B.3 HKCEE MA 1999 I – 10

In the figure, $A(-8,8)$ and $B(16,\ \ 4)$ are two points. The perpendicular bisector ℓ of the line segment AB cuts AB at M and the x axis at P.

(a) Find the equation of ℓ.

(b) Find the length of BP.

(c) If N is the mid point of AP, find the length of MN.



16B.4 HKCEE MA 2000 – I – 9

Let L be the straight line passing through $(\ \ 4,4)$ and $(6,0)$.

(a) Find the slope of L.

(b) Find the equation of L.

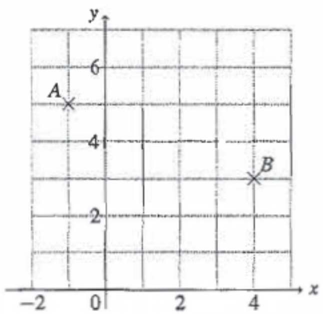
(c) If L intersects the y axis at C, find the coordinates of C.

16B.5 HKCEE MA 2001 – I 7

Two points A and B are marked in the figure.

(a) Write down the coordinates of A and B.

(b) Find the equation of the straight line joining A and B.

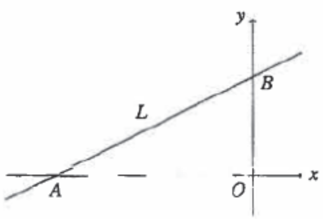


16B.6 HKCEE MA 2002 – I – 8

In the figure, the straight line $L: x\ \ 2y+8=0$ cuts the coordinate axes at A and B.

(a) Find the coordinates of A and B.

(b) Find the coordinates of the mid-point of AB.



16B.7 HKCEE MA 2003 – I 12

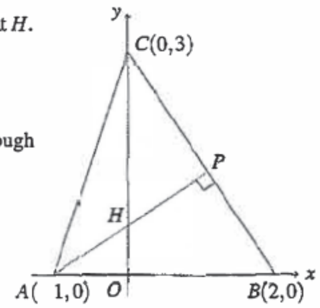
In the figure, AP is an altitude of the triangle ABC. It cuts the y-axis at H.

(a) Find the slope of BC.

(b) Find the equation of AP.

(c) (i) Find the coordinates of H.

(ii) Prove that the three altitudes of the triangle ABC pass through the same point.



16B.8 HKCEE MA 2004 – I – 13

In the figure, $ABCD$ is a rhombus. The diagonals AC and BD cut at E.

(a) Find

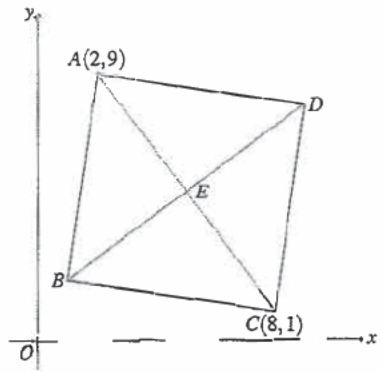
(i) the coordinates of E,

(ii) the equation of BD.

(b) It is given that the equation of AD is $x+7y-65=0$. Find

(i) the equation of BC,

(ii) the length of AB.



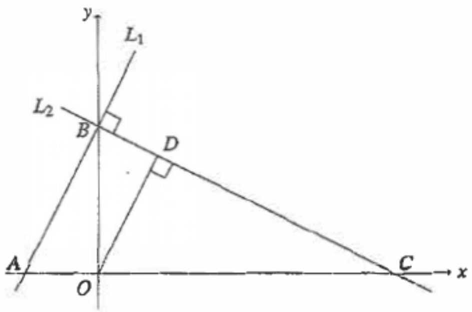
16B.9 HKCEE MA 2005 I – 13

In the figure, the straight line $L_1: 2x-y+4=0$ cuts the x-axis and the y axis at A and B respectively. The straight line L_2, passing through B and perpendicular to L_1, cuts the x-axis at C. From the origin O, a straight line perpendicular to L_2 is drawn to meet L_2 at D.

(a) Write down the coordinates of A and B.

(b) Find the equation of L_2.

(c) Find the ratio of the area of $\triangle ODC$ to the area of quadrilateral $OABD$.



16B.10 HKCEE MA 2006 I – 12

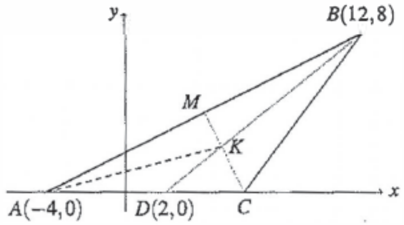
In the figure, CM is the perpendicular bisector of AB, where C and M are points lying on the x axis and AB respectively. BD and CM intersect at K.

(a) Write down the coordinates of M.

(b) Find the equation of CM. Hence, or otherwise, find the coordinates of C.

(c) (i) Find the equation of BD.

(ii) Using the result of (c)(i), find the coordinates of K. Hence find the ratio of the area of $\triangle AMC$ to the area of $\triangle AKC$.



16B.11 HKCEE MA 2007 – I – 13

In the figure, the perpendicular from B to AC meets AC at D.

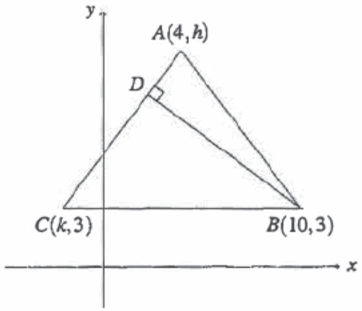
It is given that $AB = AC$ and the slope of AB is $\frac{-4}{3}$.

(a) Find the equation of AB.

(b) Find the value of h.

(c) (i) Write down the value of k

(ii) Find the area of $\triangle ABC$. Hence, or otherwise, find the length of BD.



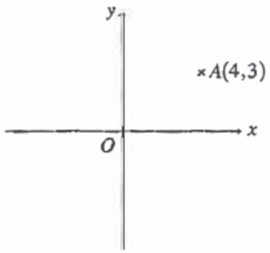
16B.12 HKCEE MA 2008 – I – 12

In the figure, the coordinates of the point A are $(4,3)$. A is rotated anticlockwise about the origin O through 90° to B. C is the reflection image of A with respect to the x axis.

(a) Write down the coordinates of B and C.

(b) Are O, B and C collinear? Explain your answer.

(c) A is translated horizontally to D such that $\angle BCD = 90°$. Find the equation of the straight line passing through C and D. Hence, or otherwise, find the coordinates of D.



16B.13 HKCEE MA 2010 I – 12

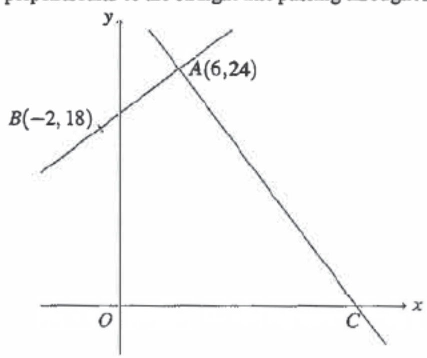
In the figure, the straight line passing through A and B is perpendicular to the straight line passing through A and C, where C is a point lying on the x-axis.

(a) Find the equation of the straight line passing through A and B.

(b) Find the coordinates of C.

(c) Find the area of $\triangle ABC$.

(d) A straight line passing through A cuts the line segment BC at D such that the area of $\triangle ABD$ is 90 square units. Let $BD : DC = r : 1$. Find the value of r.



16B.14 HKCEE AM 1982 II 2

Find the ratio in which the line segment joining $A(3,-1)$ and $B(-1,1)$ is divided by the straight line $x-y-1=0$.

16B.15 HKCEE AM 1982 – II – 10

(a) The lines $3x \quad 2y-8=0$ and $x \quad y-2=0$ meet at a point P. L_1 and L_2 are lines passing through P and having slopes $\frac{1}{2}$ and 2 respectively. Find their equations.

(b) ***[Out of syllabus]***

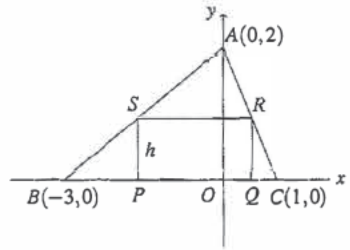
16B.16 (HKCEE AM 1985 II – 10)

$A(0,2)$, $B(-3,0)$ and $C(1,0)$ are the vertices of a triangle. $PQRS$ is a variable rectangle inscribed in the triangle with PQ on the x-axis, R on AC and S on AB, as shown in the figure. Let the length of PS be h.

(a) Find the coordinates of S and R in terms of h.

(b) Let A_1 be the area of $PQRS$ when it is a square, A_2 be the maximum possible area of rectangle $PQRS$, and A_3 be the area of $\triangle ABC$. Find the ratios $A_1 : A_2 : A_3$.

(c) The centre of $PQRS$ is the point $M(x,y)$. Express x and y in terms of h. Hence show that M lies on the line $x-y+1=0$.



16B.17 (HKCEE AM 1984 II – 4)

The area of the triangle bounded by the two lines $L_1 : x+y=4$ and $L_2 : x-y=2p$ and the y-axis is 9.

(a) Find the coordinates of the point of intersection of L_1 and L_2 in terms of p.

(b) Hence, find the possible value(s) of p.

16B.18 HKCEE AM 1988 – II – 2

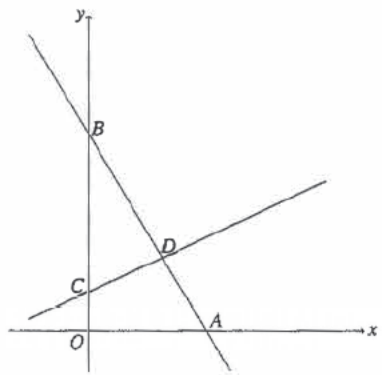
A and B are the points $(1,2)$ and $(7,4)$ respectively. P is a point on the line segment AB such that $\frac{AP}{PB} = k$.

(a) Write down the coordinates of P in terms of k.

(b) Hence find the ratio in which the line $7x-3y \quad 28=0$ divides the line segment AB.

16B.19 HKCEE AM 1990 – II – 7

In the figure, $A(3,0)$, $B(0,5)$ and $C(0,1)$ are three points and O is the origin. D is a point on AB such that the area of $\triangle BCD$ equals half of the area of $\triangle OAB$. Find the equation of the line CD.



16B.20 (HKCEE AM 1996 II 8)

Given two straight lines $L_1 : 2x-y-4=0$ and $L_2 : x-2y+4=0$. Find the equation of the straight line passing through the origin and the point of intersection of L_1 and L_2.

16B.21 (HKCEE AM 1998 – II – 5)

Two lines $L_1 : 2x+y-3=0$ and $L_2 : x-3y+1=0$ intersect at a point P.

(a) Find the coordinates of P.

(b) L is a line passing through P and the origin. Find the equation of L.

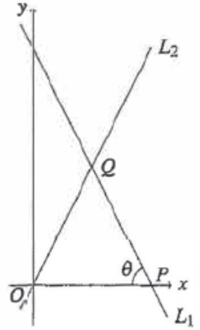
16B.22 HKCEE AM 2005 6

The figure shows the line $L_1 : 2x+y-6=0$ intersecting the x axis at point P.

(a) Let θ be the acute angle between L_1 and the x axis. Find $\tan\theta$.

(b) L_2 is a line with positive slope passing through the origin O. If L_1 intersects L_2 at a point Q such that $OP = OQ$, find the equation of L_2.

(Candidates can use the formula $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.)



16B.23 (HKCEE AM 2009 3)

Given two straight lines $L_1 : x-3y+7=0$ and $L_2 : 3x-y-11=0$. Find the equation of the straight line passing through the point $(2, 1)$ and the point of intersection of L_1 and L_2.

16B.24 HKCEE AM 2010 6

Two straight lines $L_1 : x-2y+3=0$ and $L_2 : 2x-y-1=0$ intersect at a point P. If L is a straight line passing through P and with equal positive intercepts, find the equation of L.

16C Circles in the rectangular coordinate plane

16C.1 HKCEE MA 1980(1/3 I) – B – 15

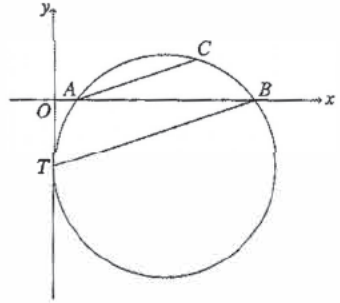
The circle $x^2+y^2-10x+8y+16=0$ cuts the x axis at A and B and touches the y-axis at T as shown in the figure.

(a) Find the coordinates of A, B and T.

(b) C is a point on the circle such that $AC//TB$.

(i) Find the equation of AC.

(ii) Find the coordinates of C by solving simultaneously the equation of AC and the equation of the given circle.



16C.2 HKCEE MA 1981(1/3) I – 13

Figure (1) shows a circle of radius 15 with centre at the origin O. The line TP, of slope $\frac{3}{4}$ ($=\tan\theta$), touches the circle at T and cuts the x axis at P.

(a) Find the equation of the circle.

(b) Calculate the length of OP.

(c) Find the equation of the line TP.

Another circle, with centre C and radius 15, is drawn to touch TP at P (see Figure (2)).

(d) Find the equation of the line OC.

(e) Find the equation of the circle with centre C.

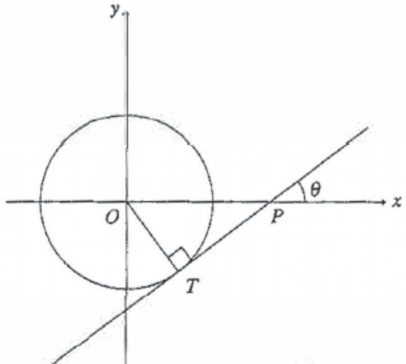


Figure (1)

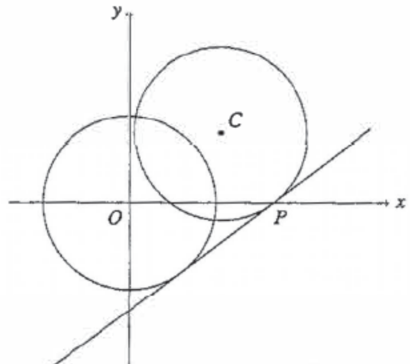


Figure (2)

16C.3 HKCEE MA 1982(1) – I – 13

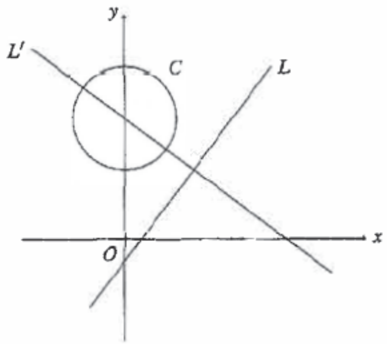
In the figure, C is the circle $x^2+y^2-14y+40=0$ and L is the line $4x-3y-4=0$.

(a) Find the radius and the coordinates of the centre of the circle C.

(b) The line L' passes through the centre of the circle C and is perpendicular to the given line L. Find the equation of the line L'.

(c) Find the coordinates of the point of intersection of the line L and the line L'.

(d) Hence, or otherwise, find the shortest distance between the circle C and the line L.



16C.4 HKCEE MA 1983(A/B) – I – 9

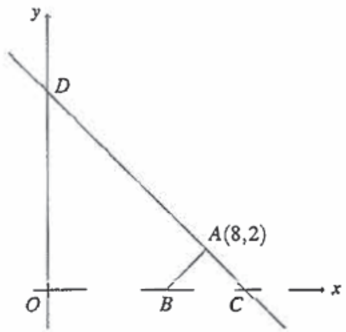
In the figure, O is the origin and A is the point $(8,2)$.

(a) B is a point on the x-axis such that the slope of AB is 1. Find the coordinates of B.

(b) C is another point on the x-axis such that $AB=AC$. Find the coordinates of C.

(c) Find the equation of the straight line AC. If the line AC cuts the y-axis at D, find the coordinates of D.

(d) Find the equation of the circle passing through the points O, B and D. Show that this circle passes through A.



16C.5 HKCEE MA 1984(A/B) – I – 9

Let L be the line $y=k-x$ (k being a constant) and C be the circle $x^2+y^2=4$.

(a) If L meets C at exactly one point, find the two values of k.

(b) If L intersects C at the points $A(2,0)$ and B,

 (i) find the value of k and the coordinates of B;

 (ii) find the equation of the circle with AB as diameter.

16C.6 HKCEE MA 1985(A/B) – I – 9

In the figure, $A(2,0)$ and $B(7,5)$ are the end-points of a diameter of the circle. P is a point on AB such that $\frac{AP}{PB}=\frac{1}{4}$.

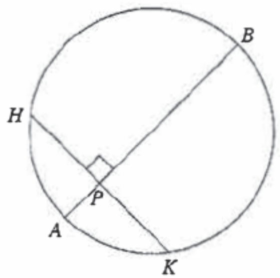
(a) Find the equation of the circle.

(b) Find the coordinates of P.

(c) The chord HPK is perpendicular to AB.

 (i) Find the equation of HPK.

 (ii) Find the coordinates of H and K.



16C.7 HKCEE MA 1986(A/B) – I – 8

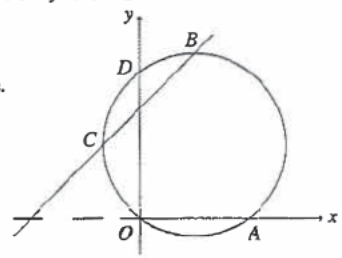
The line $y-x-6=0$ cuts the circle $x^2+y^2-6x-8y=0$ at the points B and C as shown in the figure. The circle cuts the x-axis at the origin O and the point A; it also cuts the y axis at D.

(a) Find the coordinates of B and C.

(b) Find the coordinates of A and D.

(c) Find $\angle ADO$, $\angle ABO$ and $\angle ACO$, correct to the nearest degree.

(d) Find the area of $\triangle ACO$.



16C.8 HKCEE MA 1987(A/B) – I – 8

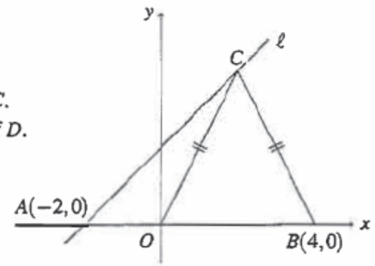
In the figure, O is the origin. A and B are the points $(-2,0)$ and $(4,0)$ respectively. ℓ is a straight line through A with slope 1. C is a point on ℓ such that $CO=CB$.

(a) Find the equation of ℓ.

(b) Find the coordinates of C.

(c) Find the equation of the circle passing through O, B and C.

(d) If the circle OBC cuts ℓ again at D, find the coordinates of D.

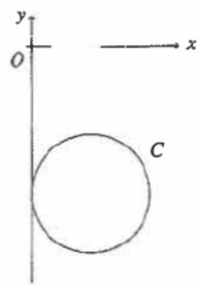


16C.9 HKCEE MA 1988 – I – 7

In the figure, the circle C has equation $x^2+y^2-4x+10y+k=0$, where k is a constant.

(a) Find the coordinates of the centre of C.

(b) If C touches the y axis, find the radius of C and the value of k.



16C.10 HKCEE MA 1989 – I – 8

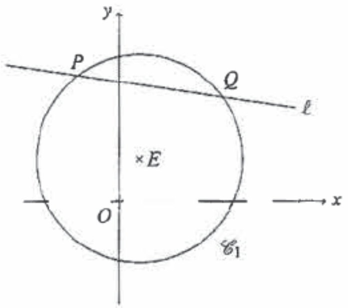
Let E be the centre of the circle $\mathscr{C}_1 : x^2+y^2-2x-4y-20=0$. The line $\ell : x+7y-40=0$ cuts $\mathscr{C}_1$ at the points P and Q as shown in the figure.

(a) Find the coordinates of E.

(b) Find the coordinates of P and Q.

(c) Find the equation of the circle $\mathscr{C}_2$ with PQ as diameter.

(d) Show that $\mathscr{C}_2$ passes through E. Hence, or otherwise, find $\angle EPQ$.



16C.11 HKCEE MA 1990 – I – 8

Let (C_1) be the circle $x^2+y^2-2x+6y+1=0$ and A be the point $(5,0)$.

(a) Find the coordinates of the centre and the radius of (C_1).

(b) Find the distance between the centre of (C_1) and A.
Hence determine whether A lies inside, outside or on (C_1).

(c) Let s be the shortest distance from A to (C_1).
  (i) Find s.
  (ii) Another circle (C_2) has centre A and radius s. Find its equation.

(d) A line touches the above two circles (C_1) and (C_2) at two distinct points E and F respectively.
Draw a rough diagram to show this information.
Find the length of EF.

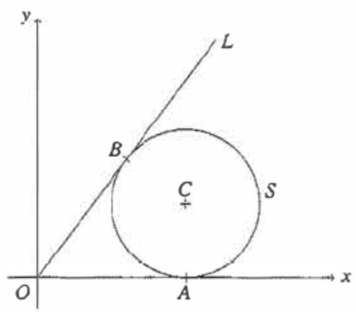
16C.12 HKCEE MA 1991 – I – 9

In the figure, the circle $S : x^2+y^2-4x-2y+4=0$ with centre C touches the x axis at A. The line $L : y=mx$, where m is a non-zero constant, passes through the origin O and touches S at B.

(a) Find the coordinates of C and A.

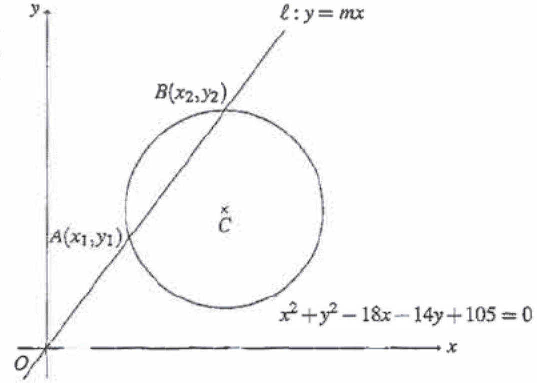
(b) Show that $m=\frac{4}{3}$.

(c) (i) Explain why the four points O, A, C, B are concyclic.
  (ii) Find the equation of the circle passing through these four points.



16C.13 HKCEE MA 1992 – I – 13

In the figure, the line $\ell : y=mx$ passes through the origin and intersects the circle $x^2+y^2-18x-14y+105=0$ at two distinct points $A(x_1,y_1)$ and $B(x_2,y_2)$.



(a) Find the coordinates of the centre C and the radius of the circle.

(b) By substituting $y=mx$ into $x^2+y^2-18x-14y+105=0$, show that $x_1x_2=\frac{105}{1+m^2}$.

(c) Express the length of OA in terms of m and x_1 and the length of OB in terms of m and x_2.
Hence find the value of the product of OA and OB.

(d) If the perpendicular distance between the line ℓ and the centre C is 3, find the lengths of AB and OA.

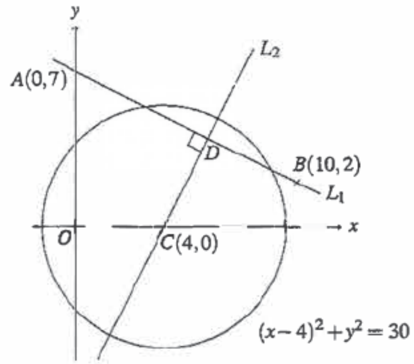
16C.14 HKCEE MA 1993 – I – 8

In the figure, L_1 is the line passing through $A(0,7)$ and $B(10,2)$; L_2 is the line passing through $C(4,0)$ and perpendicular to L_1; L_1 and L_2 meet at D.

(a) Find the equation of L_1.

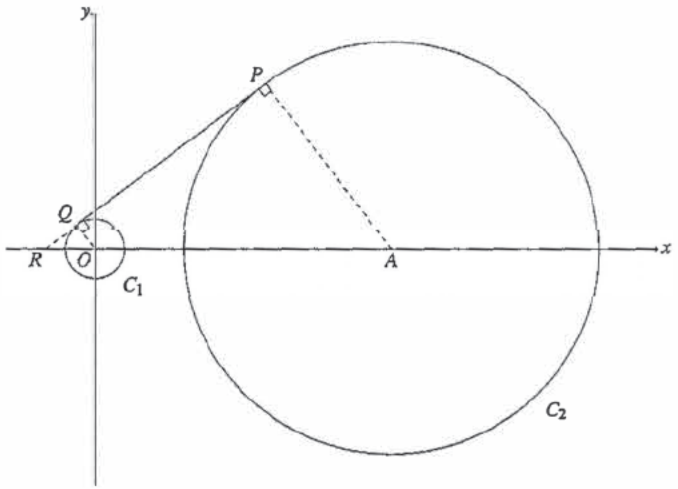
(b) Find the equation of L_2 and the coordinates of D.

(c) P is a point on the line segment AB such that $AP : PB = k : 1$. Find the coordinates of P in terms of k.
If P lies on the circle $(x-4)^2+y^2=30$, show that $2k^2-16k+7=0$ ....(*).
Find the roots of equation (*).
Furthermore, if P lies between A and D, find the value of $\frac{AP}{PB}$.



16C.15 HKCEE MA 1994 – I – 12

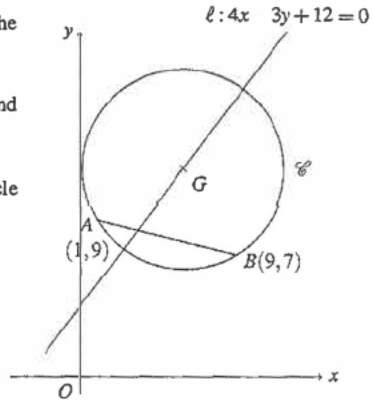
The figure shows two circles $C_1: x^2+y^2=1$, $C_2: (x-10)^2+y^2=49$.
O is the origin and A is the centre of C_2. QP is an external common tangent to C_1 and C_2 with points of contact Q and P respectively. The slope of QP is positive.



(a) Write down the coordinates of A and the radius of C_2.
(b) PQ is produced to cut the x axis at R. Find the x-coordinate of R by considering similar triangles.
(c) Using the result in (b), find the slope of QP.
(d) Using the results of (b) and (c), find the equation of the external common tangent QP.
(e) Find the equation of the other external common tangent to C_1 and C_2.

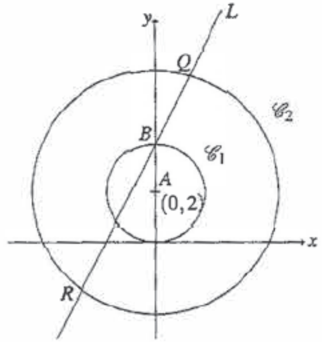
16C.16 HKCEE MA 1995 – I – 10

In the figure, $A(1,9)$ and $B(9,7)$ are points on a circle $\mathscr{C}$. The centre G of the circle lies on the line $\ell: 4x-3y+12=0$.
(a) Find the equation of the line AB.
(b) Find the equation of the perpendicular bisector of AB, and hence the coordinates of G.
(c) Find the equation of the circle $\mathscr{C}$.
(d) If DE (not shows in the figure) is another chord of the circle $\mathscr{C}$ such that AB and DE are equal and parallel, find
 (i) the coordinates of the mid-point of DE, and
 (ii) the equation of the line DE.



16C.17 HKCEE MA 1996 – I – 11

$\mathscr{C}_1$ is the circle with centre $A(0,2)$ and radius 2. It cuts the y-axis at the origin O and the point B. $\mathscr{C}_2$ is another circle with equation $x^2+(y-2)^2=25$. The line L passing through B with slope 2 cuts $\mathscr{C}_2$ at the points Q and R as shown in the figure.
(a) Find
 (i) the equation of $\mathscr{C}_1$;
 (ii) the equation of L.
(b) Find the coordinates of Q and R.
(c) Find the coordinates of
 (i) the point on L which is nearest to A;
 (ii) the point on $\mathscr{C}_1$ which is nearest to Q.



16C.18 HKCEE MA 1997 – I – 16

(Continued from **12B.14**.)

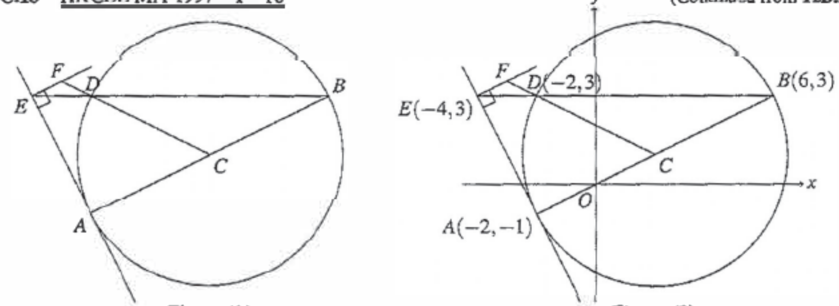
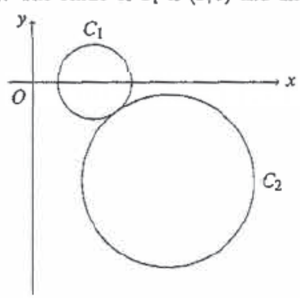


Figure (1) Figure (2)

(a) In Figure (1), D is a point on the circle with AB as diameter and C as the centre. The tangent to the circle at A meets BD produced at E. The perpendicular to this tangent through E meets CD produced at F.
 (i) Prove that $AB//EF$.
 (ii) Prove that $FD=FE$.
 (iii) Explain why F is the centre of the circle passing through D and touching AE at E.
(b) A rectangular coordinate system is introduced in Figure (1) so that the coordinates of A and B are $(-2,-1)$ and $(6,3)$ respectively. It is found that the coordinates of D and E are $(-2,3)$ and $(-4,3)$ respectively as shown in Figure (2). Find the coordinates of F.

16C.19 HKCEE MA 1998 – I – 15

The figure shows two circles C_1 and C_2 touching each other externally. The centre of C_1 is $(5,0)$ and the equation of C_2 is $(x-11)^2+(y+8)^2=49$.
(a) Find the equation of C_1.
(b) Find the equations of the two tangents to C_1 from the origin.
(c) One of the tangents in (b) cuts C_2 at two distinct points A and B. Find the coordinates of the mid-point of AB.



16C.20 HKCEE MA 1999 – I – 16 (Continued from 12A.17.)

(a) In Figure (1), ABC is a triangle right-angled at B. D is a point on AB. A circle is drawn with DB as a diameter. The line through D and parallel to AC cuts the circle at E. CE is produced to cut the circle at F.

  (i) Prove that A, F, B and C are concyclic.

  (ii) If M is the mid-point of AC, explain why $MB = MF$.

(b) In Figure (2), the equation of circle RST is $x^2 + y^2 + 10x - 6y + 9 = 0$. QST is a straight line. The coordinates of P, Q, R, S are $(-17, 0)$, $(0, 17)$, $(-9, 0)$ and $(-2, 7)$ respectively.

  (i) Prove that $PQ // RS$.

  (ii) Find the coordinates of T.

  (iii) Are the points P, Q, O and T concyclic? Explain your answer.

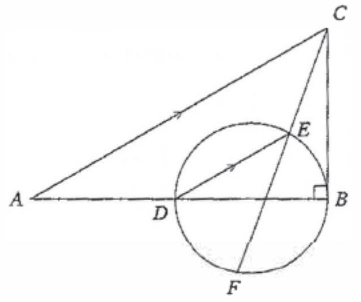


Figure (1)

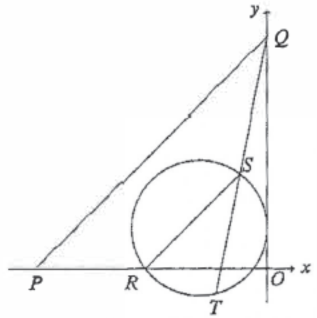


Figure (2)

16C.21 HKCEE MA 2000 – I – 16 (Continued from 12B.15.)

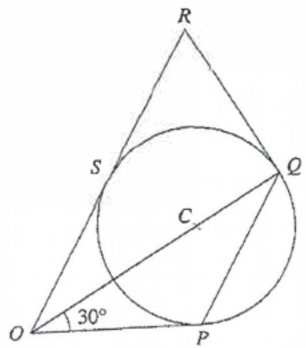
In the figure, C is the centre of the circle PQS. OR and OP are tangent to the circle at S and P respectively. OCQ is a straight line and $\angle QOP = 30°$.

(a) Show that $\angle PQO = 30°$.

(b) Suppose $OPQR$ is a cyclic quadrilateral.

  (i) Show that RQ is tangent to circle PQS at Q.

  (ii) A rectangular coordinate system is introduced in the figure so that the coordinates of O and C are $(0, 0)$ and $(6, 8)$ respectively. Find the equation of QR.



16C.22 HKCEE MA 2001 – I – 17

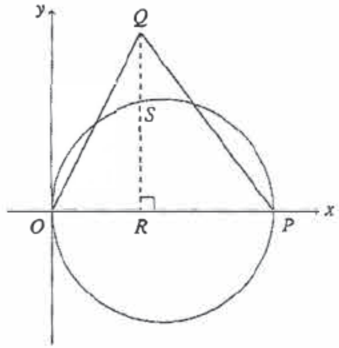


Figure (1)

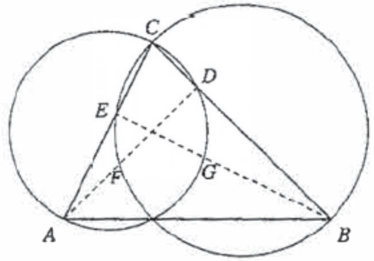


Figure (2)

(a) In Figure (1), OP is a diameter of the circle. The altitude QR of the acute angled triangle OPQ cuts the circle at S. Let the coordinates of P and S be $(p, 0)$ and (a, b) respectively.

  (i) Find the equation of the circle OPS.

  (ii) Using (i) or otherwise, show that $OS^2 = OP \cdot OQ \cos \angle POQ$.

(b) In Figure (2), ABC is an acute angled triangle. AC and BC are diameters of the circles $AGDC$ and $BCEF$ respectively.

  (i) Show that BE is an altitude of $\triangle ABC$.

  (ii) Using (a) or otherwise, compare the length of CF with that of CG. Justify your answer.

16C.23 HKCEE MA 2002 – I – 16 (Continued from 12A.21.)

In the figure, AB is a diameter of the circle $ABEG$ with centre C. The perpendicular from G to AB cuts AB at O. AE cuts OG at D. BE and OG are produced to meet at F.

Mary and John try to prove $OD \cdot OF = OG^2$ by using two different approaches.

(a) Mary tackles the problem by first proving that $\triangle AOD \sim \triangle FOB$ and $\triangle AOG \sim \triangle GOB$. Complete the following tasks for Mary.

  (i) Prove that $\triangle AOD \sim \triangle FOB$.

  (ii) Prove that $\triangle AOG \sim \triangle GOB$.

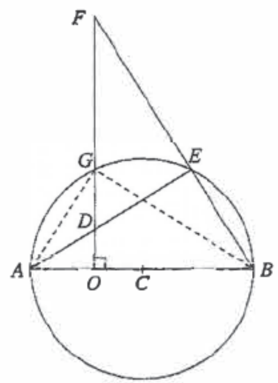
  (iii) Using (a)(i) and (a)(ii), prove that $OD \cdot OF = OG^2$.

(b) John tackles the same problem by introducing a rectangular coordinate system in the figure so that the coordinates of C, D and F are $(c, 0)$, $(0, p)$ and $(0, q)$ respectively, where c, p and q are positive numbers. He denotes the radius of the circle by r.

Complete the following tasks for John.

  (i) Express the slopes of AD and BF in terms of c, p, q and r.

  (ii) Using (b)(i), prove that $OD \cdot OF = OG^2$.



16C.24 HKCEE MA 2003 – I – 17 (Continued from **12B.16**.)

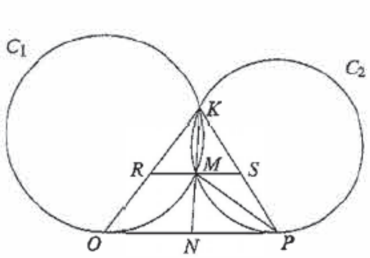


Figure (1)

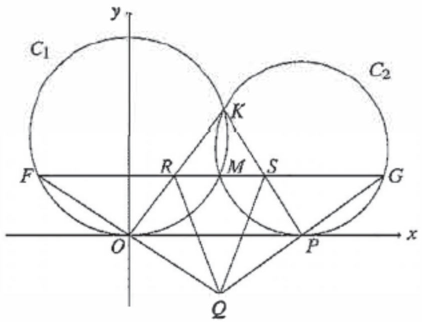


Figure (2)

(a) In Figure (1), OP is a common tangent to the circles C_1 and C_2 at the points O and P respectively. The common chord KM when produced intersects OP at N. R and S are points on KO and KP respectively such that the straight line RMS is parallel to OP.

(i) By considering triangles NPM and NKP, prove that $NP^2 = NK \cdot NM$.

(ii) Prove that $RM = MS$.

(b) A rectangular coordinate system, with O as the origin, is introduced to Figure (1) so that the coordinates of P and M are $(p, 0)$ and (a, b) respectively (see Figure (2)). The straight line RS meets C_1 and C_2 again at F and G respectively while the straight lines FO and GP meet at Q.

(i) Express FG in terms of p.

(ii) Express the coordinates of F and Q in terms of a and b.

(iii) Prove that triangle QRS is isosceles.

16C.25 HKCEE MA 2004 – I – 16 (Continued from **12B.17**.)

In the figure, BC is a tangent to the circle OAB with $BC // OA$. OA is produced to D such that $AD = OB$. BD cuts the circle at E.

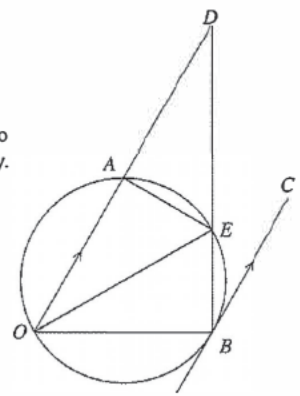
(a) Prove that $\triangle ADE \cong \triangle BOE$.

(b) Prove that $\angle BEO = 2\angle BOE$.

(c) Suppose OE is a diameter of the circle $OAEB$.

(i) Find $\angle BOE$.

(ii) A rectangular coordinate system is introduced in the figure so that the coordinates of O and B are $(0, 0)$ and $(6, 0)$ respectively. Find the equation of the circle $OAEB$.



16C.26 HKCEE MA 2005 – I – 17 (Continued from **12A.22**.)

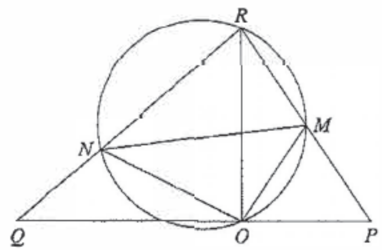


Figure (1)

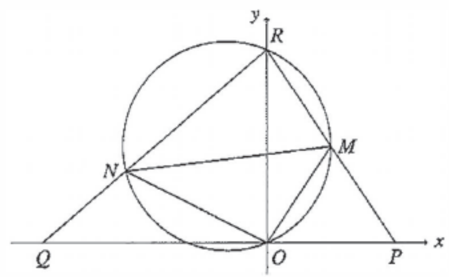


Figure (2)

(a) In Figure (1), MN is a diameter of the circle $MONR$. The chord RO is perpendicular to the straight line POQ. RNQ and RMP are straight lines.

(i) By considering triangles OQR and ORP, prove that $OR^2 = OP \cdot OQ$.

(ii) Prove that $\triangle MON \sim \triangle POR$.

(b) A rectangular coordinate system, with O as the origin, is introduced to Figure (1) so that R lies on the positive y-axis and the coordinates of P and Q are $(4, 0)$ and $(-9, 0)$ respectively (see Figure (2)).

(i) Find the coordinates of R.

(ii) If the centre of the circle $MONR$ lies in the second quadrant and $ON = \frac{3\sqrt{13}}{2}$, find the radius and the coordinates of the centre of the circle $MONR$.

16C.27 HKCEE MA 2006 I 16 (Continued from **12A.23**.)

In the figure, G and H are the circumcentre and the orthocentre of $\triangle ABC$ respectively. AH produced meets BC at O. The perpendicular from G to BC meets BC at R. BS is a diameter of the circle which passes through A, B and C.

(a) Prove that

(i) $AHCS$ is a parallelogram,

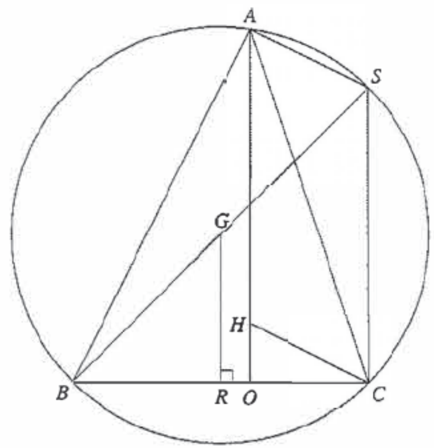
(ii) $AH = 2GR$.

(b) A rectangular coordinate system, with O as the origin, is introduced in the figure so that the coordinates of A, B and C are $(0, 12)$, $(-6, 0)$ and $(4, 0)$ respectively.

(i) Find the equation of the circle which passes through A, B and C.

(ii) Find the coordinates of H.

(iii) Are B, O, H and G concyclic? Explain your answer.



16C.28 HKCEE MA 2007 – I – 17

(Continued from **12A.24.**)

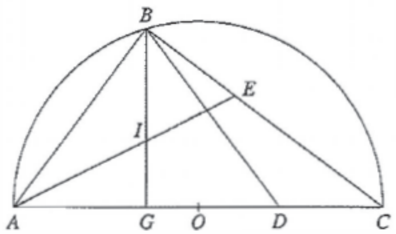


Figure (1)

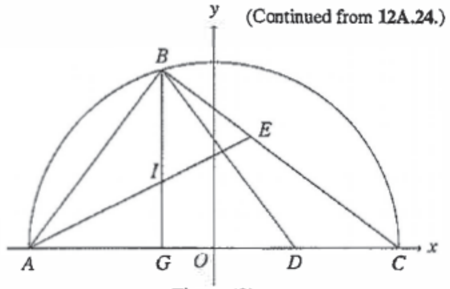


Figure (2)

(a) In Figure (1), AC is the diameter of the semi-circle ABC with centre O. D is a point lying on AC such that $AB = BD$. I is the in centre of $\triangle ABD$. AI is produced to meet BC at E. BI is produced to meet AC at G.

(i) Prove that $\triangle ABG \cong \triangle DBG$.

(ii) By considering the triangles AGI and ABE, prove that $\frac{GI}{AG} = \frac{BE}{AB}$.

(b) A rectangular coordinate system, with O as the origin, is introduced to Figure (1) so that the coordinates of C and D are $(25, 0)$ and $(11, 0)$ respectively and B lies in the second quadrant (see Figure (2)). It is found that $BE : AB = 1 : 2$.

(i) Find the coordinates of G.

(ii) Find the equation of the inscribed circle of $\triangle ABD$.

16C.29 HKCEE MA 2008 – I – 17

(Continued from **12A.25.**)

Figure (1) shows a circle passing through A, B and C. I is the in centre of $\triangle ABC$ and AI produced meets the circle at P.

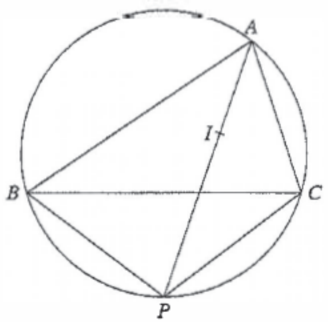


Figure (1)

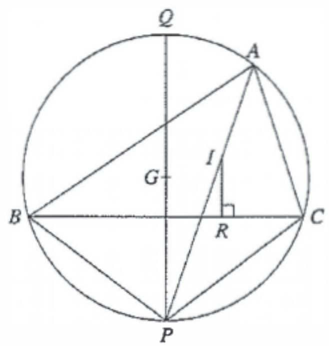


Figure (2)

(a) Prove that $BP = CP = IP$.

(b) Figure (2) is constructed by adding three points G, Q and R to Figure (1), where G is the circumcentre of $\triangle ABC$, PQ is a diameter of the circle and R is the foot of the perpendicular from I to BC. A rectangular coordinate system is then introduced in Figure (2) so that the coordinates of B, C and I are $(-80, 0)$, $(64, 0)$ and $(0, 32)$ respectively.

(i) Find the equation of the circle with centre P and radius BP.

(ii) Find the coordinates of Q.

(iii) Are B, Q, I and R concyclic? Explain your answer.

16C.30 HKCEE MA 2011 – I – 16

In the figure, $\triangle PQR$ is an isosceles triangle with $PQ = PR$. It is given that S is a point lying on QR and the orthocentre of $\triangle PQR$ lies on PS. A rectangular coordinate system is introduced in the figure so that the coordinates of P and Q are $(16, 80)$ and $(-32, -48)$ respectively. It is given that QR is parallel to the x axis.

(a) Find the equation of the perpendicular bisector of PR.

(b) Find the coordinates of the circumcentre of $\triangle PQR$.

(c) Let C be the circle which passes through P, Q and R.

(i) Find the equation of C.

(ii) Are the centre C and the in-centre of $\triangle PQR$ the same point? Explain your answer.

16C.31 HKCEE AM 1981 II 6

The circles $C_1 : x^2 + y^2 + 7y + 11 = 0$ and $C_2 : x^2 + y^2 + 6x + 4y + 8 = 0$ touch each other externally at P.

(a) Find the coordinates of P.

(b) Find the equation of the common tangent at P.

16C.32 (HKCEE AM 1981 – II – 12)

The line $L : y = mx + 2$ meets the circle $C : x^2 + y^2 = 1$ at the points $A(x_1, y_1)$ and $B(x_2, y_2)$.

(a) (i) Show that x_1 and x_2 are the roots of the quadratic equation $(m^2 + 1)x^2 + 4mx + 3 = 0$.

(ii) Hence, or otherwise, show that the length of the chord AB is $2\sqrt{\frac{m^2 - 3}{m^2 + 1}}$.

(b) Find the values of m such that

(i) L meets C at two distinct points,

(ii) L is a tangent to C,

(iii) L does not meet C.

(c) For the two tangents in (b)(ii), let the corresponding points of contact be P and Q. Find the equation of PQ.

16C.33 (HKCEE AM 1982 II 8)

M is the point $(5, 6)$, L is the line $5x + 12y = 32$ and C is the circle with M as centre and touching L.

(a) (i) Find the equation of the straight line passing through M and perpendicular to L.

(ii) Hence, or otherwise, find the equation of C.

(b) Show that C also touches the y axis.

(c) Find the equation of the tangent (other than the y-axis) to C from the origin.

(d) $P(2, 2)$ is a point on C. Q is another point on C such that PQ is a diameter. Find the equation of the circle which passes through P, Q and the origin.

16C.34 HKCEE AM 1984 – II – 6

Given the equation $x^2 + y^2 - 2kx + 4ky + 6k^2 \quad 2 = 0$.

(a) Find the range of values of k so that the equation represents a circle with radius greater than 1.

(b) *[Out of syllabus]*

16C.35 (HKCEE AM 1985 II – 5)

If the equation $x^2+y^2+kx-(2+k)y=0$ represents a circle with radius $\sqrt{5}$,

(a) find the value(s) of k;

(b) find the equation(s) of the circle(s).

16C.36 HKCEE AM 1986 – II – 10

The circles $C_1: x^2+y^2-4x+2y+1=0$ and $C_2: x^2+y^2-10x-4y+19=0$ have a common chord AB.

(a) (i) Find the equation of the line AB.

(ii) Find the equation of the circle with AB as a chord such that the area of the circle is a minimum.

(b) The circle C_1 and another circle C_3 are concentric. If AB is a tangent to C_3, find the equation of C_3.

(c) *[Out of syllabus]*

16C.37 HKCEE AM 1987 – II – 11

In the figure, A and B are the points $(8,0)$ and $(16,0)$ respectively. The equation of the circle C_1 is $x^2+y^2-16x-4y+64=0$. OH and BH are tangents to C_1.

(a) (i) Show that C_1 touches the x axis at A.

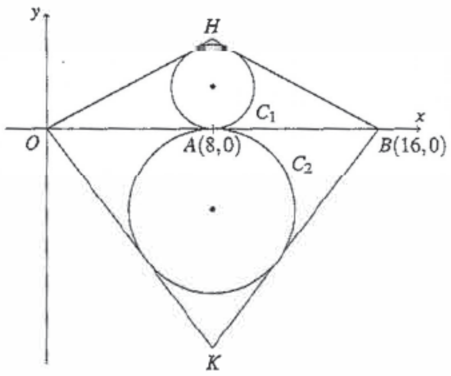
(ii) Find the equation of OH.

(iii) Find the equation of BH.

(b) In the figure, the equation of OK is $4x+3y=0$. The circle $C_2: x^2+y^2-16x+2fy+c=0$ is the inscribed circle of $\triangle OBK$ and touches the x-axis at A.

(i) Find the values of the constants c and f.

(ii) Find area of $\triangle OBH$: area of $\triangle OBK$.

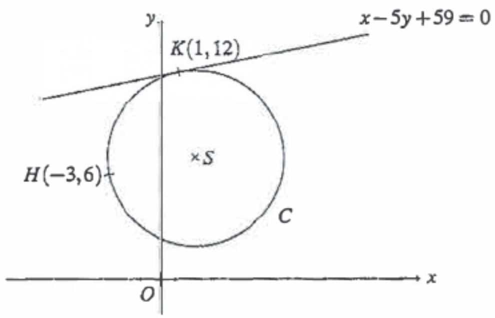


16C.38 (HKCEE AM 1988 II – 11)

In the figure, S is the centre of the circle C which passes through $H(-3,6)$ and touches the line $x \quad 5y+59=0$ at $K(1,12)$.

(a) Find the coordinates of S. Hence, or otherwise, find the equation of the circle C.

(b) The line $L: 3x-2y \quad 5=0$ cuts the circle C at A and B. Find the equation of the circle with AB as diameter.



16C.39 HKCEE AM 1993 – II – 11

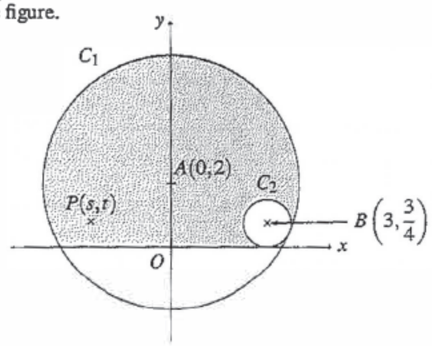
$A(0,2)$ is the centre of circle C_1 with radius 4. $B\left(3,\frac{3}{4}\right)$ is the centre of circle C_2 which touches the x axis. $P(s,t)$ is any point in the shaded region as shown in the figure.

(a) Find AB and the radius of C_2.
Hence show that C_1 and C_2 touch each other.

(b) If P is the centre of a circle which touches the x axis and C_1, show that $4t=12-s^2$.

(c) If P is the centre of a circle which touches the x-axis and C_2, show that $3t=(s \quad 3)^2$.

(d) Given that there are two circles in the shaded region, each of which touches the x-axis, C_1 and C_2. Using (b) and (c), find the equations of the two circles, giving your answers in the form $(x-h)^2+(y-k)^2=r^2$.



16C.40 HKCEE AM 1994 – II – 9

Given two points $A(5,5)$ and $B(7,1)$. Let (h,k) be the centre of a circle C which passes through A and B.

(a) Express h in terms of k.
Hence show that the equation of C is $x^2+y^2 \quad 4kx-2ky+30k-50=0$.

(b) If the tangent to C at B is parallel to the line $y=\frac{1}{2}x$, find the equation of C.

(c) *[Out of syllabus]*

16C.41 HKCEE AM 1995 – II – 10

C_1 is the circle $x^2+y^2-16x-36=0$ and C_2 is a circle centred at the point $A(-7,0)$. C_1 and C_2 touch externally as shown in the figure. $P(h,k)$ is a point in the second quadrant.

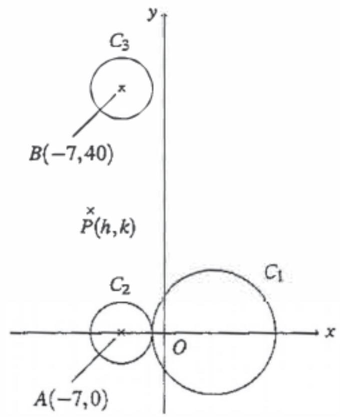
(a) Find the centre and radius of C_1.
Hence find the radius of C_2.

(b) If P is the centre of a circle which touches both C_1 and C_2 externally, show that $8h^2-k^2 \quad 8h-48=0$.

(c) C_3 is a circle centred at the point $B(-7,40)$ and of the same radius as C_2.

(i) If P is the centre of a circle which touches both C_2 and C_3 externally, write down the equation of the locus of P.

(ii) Find the equation of the circle, with centre P, which touches all the three circles C_1, C_2 and C_3 externally.



16C.42 (HKCEE AM 1996 – II – 10)

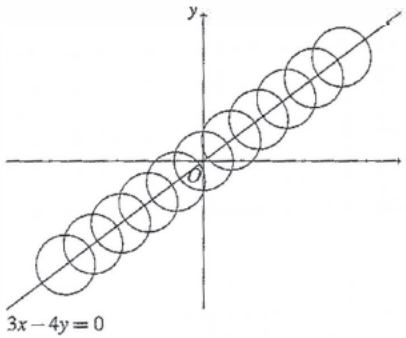
The equation $C_k: x^2+y^2-8kx-6ky+25(k^2-1)=0$, where k is real, represents a circle.

(a) (i) Find the centre of C_k in terms of k. Hence show that the centre of C_k lie on the line $3x-4y=0$ for all values of k.

(ii) Show that C_k has a radius of 5.

(b) The figure shows some C_k's for various values of k. It is given that there are two parallel lines, both of which are common tangents to all C_k's.
Write down the slope of these two common tangents.
Hence find the equations of these two common tangents.

(c) For a certain value of k, C_k cuts the x-axis at two points A and B.
Write down the distance from the centre of the circle to the x axis in terms of k.
Hence, or otherwise, find the two possible values of k such that C_k satisfies the condition $AB=8$.



16C.43 (HKCEE AM 1998 – II – 2)

Given a line $L: x-7y+3=0$ and a circle $C: (x-2)^2+(y+5)^2=a$, where a is a positive number. If L is a tangent to C, find the value of a.

16C.44 (HKCEE AM 2000 – II – 9)

A circle has the equation $(F): x^2+y^2+(4k+4)x+(3k+1)y-(8k+8)=0$, where k is real.

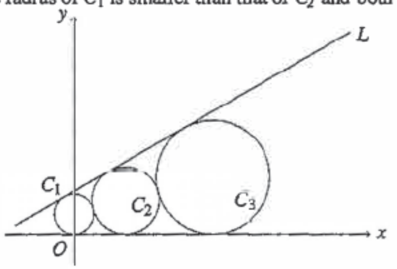
(a) Rewrite the equation (F) in the form $(x-p)^2+(y-q)^2=r^2$.

(b) C_1 and C_2 are two circles described by (F) such that the radius of C_1 is smaller than that of C_2 and both of them touch the x axis.

(i) Find the equations of C_1 and C_2.

(ii) Show that C_1 and C_2 touch each other externally.

(c) The figure shows the circles C_1 and C_2 in (b). L is a common tangent to C_1 and C_2. C_3 is a circle touching C_2, L and the x-axis. Find the equation of C_3.
(Hint: The centres of the three circles are collinear.)



16C.45 HKCEE AM 2002 15 (Continued from 12B.18.)

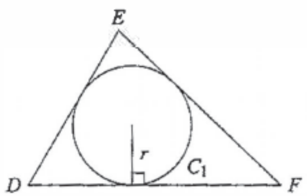


Figure (1)

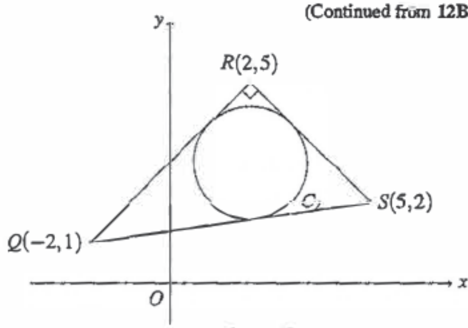


Figure (2)

(a) DEF is a triangle with perimeter p and area A. A circle C_1 of radius r is inscribed in the triangle (see Figure (1)). Show that $A=\frac{1}{2}pr$.

(b) In Figure (2), a circle C_2 is inscribed in a right-angled triangle QRS. The coordinates of Q, R and S are $(-2,1)$, $(2,5)$ and $(5,2)$ respectively.

(i) Using (a), or otherwise, find the radius of C_2.

(ii) Find the equation of C_2.

16C.46 HKCEE AM 2005 – 15

The figure shows a circle $C_1: x^2+y^2-4x-2y+4=0$ centred at point A. L is the straight line $y=kx$.

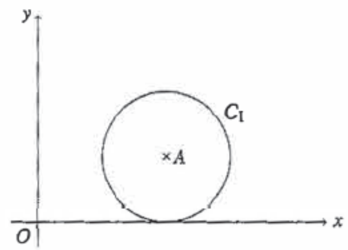
(a) Find the range of k such that C_1 and L intersect.

(b) There are two tangents from the origin O to C_1. Find the equation of the tangent L_1 other than the x-axis.

(c) Suppose that L and C_1 intersect at two distinct points P and Q. Let M be the mid-point of PQ.

(i) Show that the x coordinate of M is $\frac{k+2}{k^2+1}$.

(ii) *[Out of syllabus]*



16C.47 HKCEE AM 2006 – 14

Let J be the circle $x^2+y^2=r^2$, where $r>0$.

(a) Suppose that the straight line $L: y=mx+c$ is a tangent to J.

(i) Show that $c^2=r^2(m^2+1)$.

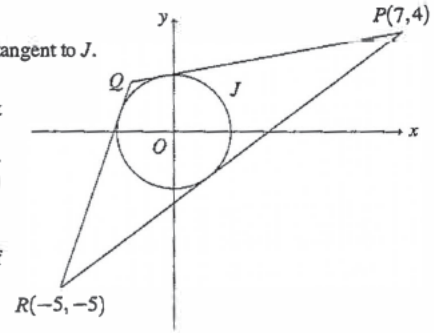
(ii) If L passes through a point (h,k), show that $(k-mh)^2=r^2(m^2+1)$.

(b) J is inscribed in a triangle PQR (see the figure). The coordinates of P and R are $(7,4)$ and $(-5,-5)$ respectively.

(i) Find the radius of J.

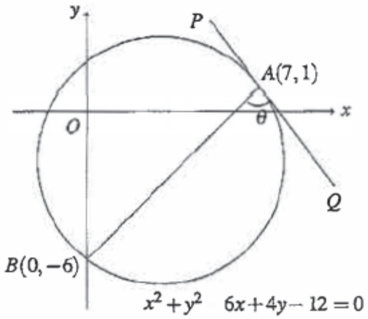
(ii) Using (a)(ii), or otherwise, find the slope of PQ.

(iii) Find the coordinates of Q.



16C.48 HKCEE AM 2010 – 7

In the figure, a tangent PQ is drawn to the circle $x^2+y^2-6x+4y-12=0$ at the point $A(7,1)$. $B(0,-6)$ is another point lying on the circle. Let θ be the acute angle between AB and PQ. Find the value of $\tan\theta$.

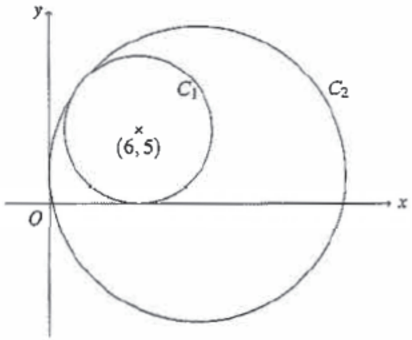


16C.49 HKCEE AM 2010 – 15

In the figure, C_1 is a circle with centre $(6,5)$ touching the x axis. C_2 is a variable circle which touches the y axis and C_1 internally.

(a) Show that the equation of locus of the centre of C_2 is $x=\frac{1}{2}y^2-5y+18$.

(b) It is known that the length of the tangent from an external point $P(0,-3)$ to C_2 is 5 and the centre of C_2 is in the first quadrant.
  (i) Find the centre of C_2.
  (ii) Find the equations of the two tangents from P to C_2.



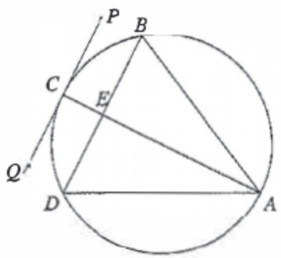
16C.50 HKDSE MA SP – I – 19

(Continued from **12B.19**.)

In the figure, the circle passes through four points A, B, C and D. PQ is the tangent to the circle at C and is parallel to BD. AC and BD intersect at E. It is given that $AB=AD$.

(a) (i) Prove that $\triangle ABE \cong \triangle ADE$.
  (ii) Are the in-centre, the orthocentre, the centroid and the circumcentre of $\triangle ABD$ collinear? Explain your answer.

(b) A rectangular coordinate system is introduced in the figure so that the coordinates of A, B and D are $(14,4)$, $(8,12)$ and $(4,4)$ respectively. Find the equation of the tangent PQ.



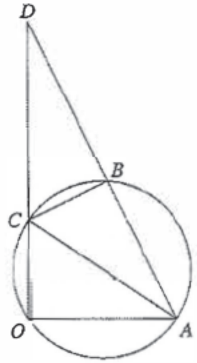
16C.51 HKDSE MA PP – I 14

(Continued from **12A.28**.)

In the figure, $OABC$ is a circle. It is given that AB produced and OC produced meet at D.

(a) Write down a pair of similar triangles in the figure.

(b) Suppose that $\angle AOD = 90°$. A rectangular coordinate system, with O as the origin, is introduced in the figure so that the coordinates of A and D are $(6,0)$ and $(0,12)$ respectively. If the ratio of the area of $\triangle BCD$ to the area of $\triangle OAD$ is $16:45$, find
  (i) the coordinates of C,
  (ii) the equation of the circle $OABC$.



16C.52 HKDSE MA 2012 – I – 17

The coordinates of the centre of the circle C are $(6,10)$. It is given that the x axis is a tangent to C.

(a) Find the equation of C.

(b) The slope and the y intercept of the straight line L is -1 and k respectively. If L cuts C at A and B, express the coordinates of the mid-point of AB in terms of k.

16C.53 HKDSE MA 2015 – I – 14

The coordinates of the points P and Q are $(4,-1)$ and $(-14,23)$ respectively.

(a) Let L be the perpendicular bisector of PQ.
  (i) Find the equation of L.
  (ii) Suppose that G is a point lying on L. Denote the x-coordinate of G by h. Let C be the circle which is centred at G and passes through P and Q.
  Prove that the equation of C is $2x^2+2y^2-4hx-(3h+59)y+13h-93=0$.

(b) The coordinates of the point R are $(26,43)$. Using (a)(ii), or otherwise, find the diameter of the circle which passes through P, Q and R.

16C.54 HKDSE MA 2016 – I – 20

(Continued from **12B.20**.)

$\triangle OPQ$ is an obtuse-angled triangle. Denote the in-centre and the circumcentre of $\triangle OPQ$ by I and J respectively. It is given that P, I and J are collinear.

(a) Prove that $OP=PQ$.

(b) A rectangular coordinate system is introduced so that the coordinates of O and Q are $(0,0)$ and $(40,30)$ respectively while the y coordinate of P is 19. Let C be the circle which passes through O, P and Q.
  (i) Find the equation of C.
  (ii) Let L_1 and L_2 be two tangents to C such that the slope of each tangent is $\frac{3}{4}$ and the y-intercept of L_1 is greater than that of L_2. L_1 cuts the x axis and the y-axis at S and T respectively while L_2 cuts the x-axis and y axis at U and V respectively. Someone claims that the area of the trapezium $STUV$ exceeds 17 000. Is the claim correct? Explain your answer.

16C.55 HKDSE MA 2018 – I – 19

The coordinates of the centre of the circle C are $(8,2)$. Denote the radius of C by r. Let L be the straight line $kx-5y-21=0$, where k is a constant. It is given that L is a tangent to C.

(a) Find the equation of C in terms of r. Hence, express r^2 in terms of k.

(b) L passes through the point $D(18,39)$.

 (i) Find r.

 (ii) It is given that L cuts the y-axis at the point E. Let F be a point such that C is the inscribed circle of $\triangle DEF$. Is $\triangle DEF$ an obtuse-angled triangle? Explain your answer.

16C.56 HKDSE MA 2019 I 19

(Continued from 7E.5.)

Let $f(x)=\dfrac{1}{1+k}(x^2+(6k-2)x+(9k+25))$, where k is a positive constant. Denote the point $(4,33)$ by F.

(a) Prove that the graph of $y=f(x)$ passes through F.

(b) The graph of $y=g(x)$ is obtained by reflecting the graph of $y=f(x)$ with respect to the y-axis and then translating the resulting graph upwards by 4 units. Let U be the vertex of the graph of $y=g(x)$. Denote the origin by O.

 (i) Using the method of completing the square, express the coordinates of U in terms of k.

 (ii) Find k such that the area of the circle passing through F, O and U is the least.

 (iii) For any positive constant k, the graph of $y=g(x)$ passes through the same point G. Let V be the vertex of the graph of $y=g(x)$ such that the area of the circle passing through F, O and V is the least. Are F, G, O and V concyclic? Explain your answer.

16C.57 HKDSE MA 2020 I 14

The coordinates of the points A and B are $(\ 10,0)$ and $(30,0)$ respectively. The circle C passes through A and B. Denote the centre of C by G. It is given that the y-coordinate of G is -15.

(a) Find the equation of C. (3 marks)

(b) The straight line L passes through B and G. Another straight line ℓ is parallel to L. Let P be a moving point in the rectangular coordinate plane such that the perpendicular distance from P to L is equal to the perpendicular distance from P to ℓ. Denote the locus of P by Γ. It is given that Γ passes through A.

 (i) Describe the geometric relationship between Γ and L.

 (ii) Find the equation of Γ.

 (iii) Suppose that Γ cuts C at another point H. Someone claims that $\angle GAH < 70^\circ$. Do you agree? Explain your answer.

(6 marks)

16D Loci in the rectangular coordinate plane

16D.1 (HKCEE MA 1981(3) I – 7)

The parabola $y^2=4ax$ passes through the points $A(1,4)$ and $B(16,\ \ 16)$. A point P divides AB internally such that $AP:PB\ \ 1:4$.

(a) Find the coordinates of P.

(b) Show that the parabola is the locus of a moving point which is equidistant from P and the line $x=-a$.

16D.2 HKCEE AM 1987 II 10

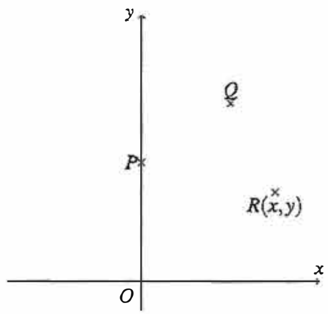
$P(x,y)$ is a variable point equidistant from the point $S(1,0)$ and the line $x+1=0$.

(a) Show that the equation of the locus of P is $y^2=4x$.

(b) *[Out of syllabus]*

16D.3 (HKCEE AM 1994 II – 4)

In the figure, $P(0,4)$ and $Q(2,6)$ are two points and $R(x,y)$ is a variable point.



(a) Suppose $R_0=(4,4)$ (not shown in the figure). Find the area of $\triangle PQR_0$.

(b) If the area of $\triangle PQR$ is 4 square units,

 (i) describe the locus of R and sketch it in the figure;

 (ii) find the equation(s) of the locus of R.

16D.4 HKCEE AM 1999 – II 10

$A(-3,0)$ and $B(-1,0)$ are two points and $P(x,y)$ is a variable point such that $PA=\sqrt{3}PB$. Let C be the locus of P.

(a) Show that the equation of C is $x^2+y^2=3$.

(b) $T(a,b)$ is a point on C. Find the equation of the tangent to C at T.

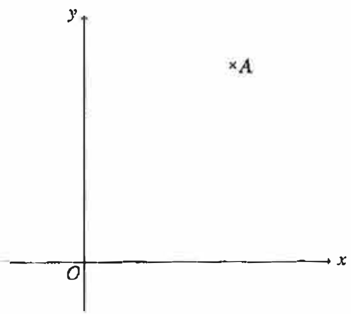
(c) The tangent from A to C touches C at a point S in the second quadrant. Find the coordinates of S.

(d) *[Out of syllabus]*

16D.5 (HKCEE AM 2004 10)

In the figure, O is the origin and A is the point $(3,4)$. P is a variable point (not shown) such that the area of $\triangle OPA$ is always equal to 2.

Describe the locus of P and sketch it in the figure.



16D.6 (HKCEE AM 2011 – 16) *[Difficult!]*

Figure (1) shows a circle $C_1 : x^2+y^2-10y+16=0$. $Z(x,y)$ is the centre of a circle which touch the x axis and C_1 externally. Let S be the locus of Z.

(a) Show that the equation of S is $y=\frac{1}{16}x^2+1$.

(b) Let C_2 and C_3 be circles touching the x-axis and C_1 externally. It is given that C_2 passes through the point $(20,16)$ and it touches C_3 externally. Suppose that both the centres of C_2 and C_3 lie in the first quadrant (see Figure (2)).

(i) Find the equation of C_2.

(ii) Without any algebraic manipulation, determine whether the following sentence is correct: "The point of contact of C_2 and C_3 lies on S."

(c) Can we draw a circle satisfying all the following conditions?

- Its centre lies on S.
- It touches the x axis.
- It touches C_1 internally.

Explain your answer.

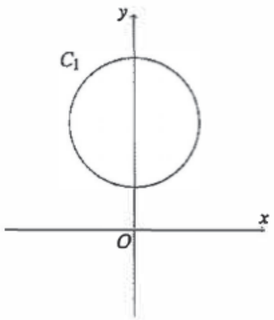


Figure (1)

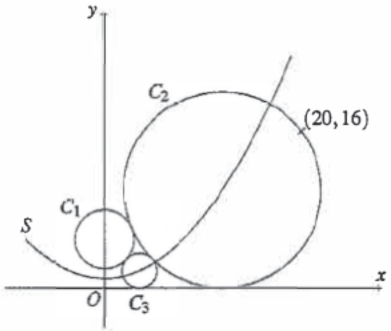


Figure (2)

16D.7 HKDSE MA SP I 13

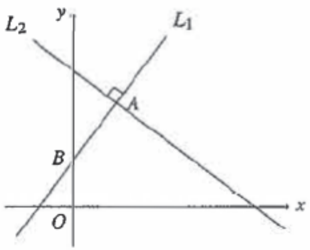
In the figure, the straight line $L_1 : 4x-3y+12=0$ and the straight line L_2 are perpendicular to each other and intersect at A. It is given that L_1 cuts the y-axis at B and L_2 passes through the point $(4,9)$.

(a) Find the equation of L_2.

(b) Q is a moving point in the coordinate plane such that $AQ=BQ$. Denote the locus of Q by Γ.

(i) Describe the geometric relationship between Γ and L_2. Explain your answer.

(ii) Find the equation of Γ.



16D.8 HKDSE MA PP – I – 8

The coordinates of the points A and B are $(-3,4)$ and $(-2,-5)$ respectively. A' is the reflection image of A with respect to the y axis. B is rotated anticlockwise about the origin O through 90° to B'.

(a) Write down the coordinates of A' and B'.

(b) Let P be a moving point in the rectangular coordinate plane such that P is equidistant from A' and B'. Find the equation of the locus of P.

16D.9 HKDSE MA 2012 – I – 14

The y-intercepts of two parallel lines L and ℓ are -1 and 3 respectively and the x intercept of L is 3. P is a moving point in the rectangular coordinate plane such that the perpendicular distance from P to L is equal to the perpendicular distance from P to ℓ. Denote the locus of P by Γ.

(a) (i) Describe the geometric relationship between Γ and L.

(ii) Find the equation of Γ.

(b) The equation of the circle C is $(x-6)^2+y^2=4$. Denote the centre of C by Q.

(i) Does Γ pass through Q? Explain your answer.

(ii) If L cuts C at A and B while Γ cuts C at H and K, find the ratio of the area of $\triangle AQH$ to the area of $\triangle BQK$.

16D.10 HKDSE MA 2013 – I – 14

The equation of the circle C is $x^2+y^2-12x-34y+225=0$. Denote the centre of C by R.

(a) Write down the coordinates of R.

(b) The equation of the straight line L is $4x+3y+50=0$. It is found that C and L do not intersect. Let P be a point lying on L such that P is nearest to R.

(i) Find the distance between P and R.

(ii) Let Q be a moving point on C. When Q is nearest to P,

(1) describe the geometric relationship between P, Q and R;

(2) find the ratio of the area of $\triangle OPQ$ to the area of $\triangle OQR$, where O is the origin.

16D.11 HKDSE MA 2014 – I – 12

The circle C passes through the point $A(6,11)$ and the centre of C is the point $G(0,3)$.

(a) Find the equation of C.

(b) P is a moving point in the rectangular coordinate plane such that $AP=GP$. Denote the locus of P by Γ.

(i) Find the equation of Γ.

(ii) Describe the geometric relationship between Γ and the line segment AG.

(iii) If Γ cuts C at Q and R, find the perimeter of the quadrilateral $AQGR$.

16D.12 HKDSE MA 2016 – I – 10

The coordinates of the points A and B are $(5,7)$ and $(13,1)$ respectively. Let P be a moving point in the rectangular coordinate plane such that P is equidistant from A and B. Denote the locus of P by Γ.

(a) Find the equation of Γ.

(b) Γ intersects the x-axis and the y axis at H and K respectively. Denote the origin by O. Let C be the circle which passes through O, H and K. Someone claims that the circumference of C exceeds 30. Is the claim correct? Explain your answer.

16D.13 HKDSE MA 2017 – I – 13

The coordinates of the points E, F and G are $(-6,5)$, $(3,11)$ and $(2,-1)$ respectively. The circle C passes through E and the centre of C is G.

(a) Find the equation of C.

(b) Prove that F lies outside C.

(c) Let H be a moving point on C. When H is farthest from F,

(i) describe the geometric relationship between F, G and H;

(ii) find the equation of the straight line which passes through F and H.

16D.14 HKDSE MA 2019 – I – 17 (Continued from **12B.21**.)

(a) Let a and p be the area and perimeter of $\triangle CDE$ respectively. Denote the radius of the inscribed circle of $\triangle CDE$ by r. Prove that $pr = 2a$.

(b) The coordinates of the points H and K are $(9, 12)$ and $(14, 0)$ respectively. Let P be a moving point in the rectangular coordinate plane such that the perpendicular distance from P to OH is equal to the perpendicular distance from P to HK, where O is the origin. Denote the locus of P by Γ.

   (i) Describe the geometric relationship between Γ and $\angle OHK$.

   (ii) Using (a), find the equation of Γ.

16E Polar coordinates

16E.1 HKCEE MA 2009 – I – 8

In a polar coordinate system, O is the pole. The polar coordinates of the points P and Q are $(k, 123°)$ and $(24, 213°)$ respectively, where k is a positive constant. It is given that $PQ = 25$.

(a) Is $\triangle OPQ$ a right-angled triangle? Explain your answer.

(b) Find the perimeter of $\triangle OPQ$.

16E.2 HKDSE MA PP – I – 6

In a polar coordinate system, the polar coordinates of the points A, B and C are $(13, 157°)$, $(14, 247°)$ and $(15, 337°)$ respectively.

(a) Let O be the pole. Are A, O and C collinear? Explain your answer.

(b) Find the area of $\triangle ABC$.

16E.3 HKDSE MA 2013 – I – 6

In a polar coordinate system, O is the pole. The polar coordinates of the points A and B are $(26, 10°)$ and $(26, 130°)$ respectively. Let L be the axis of reflectional symmetry of $\triangle OAB$.

(a) Describe the geometric relationship between L and $\angle AOB$.

(b) Find the polar coordinates of the point of intersection of L and AB.

16E.4 HKDSE MA 2016 – I – 7

In a polar coordinate system, O is the pole. The polar coordinates of the points A and B are $(12, 75°)$ and $(12, 135°)$ respectively.

(a) Find $\angle AOB$.

(b) Find the perimeter of $\triangle AOB$.

(c) Write down the number of folds of rotational symmetry of $\triangle AOB$.

16 Coordinate Geometry

16A Transformation in the rectangular coordinate plane

16A.1 HKCEE MA 2006 – I – 7

(a) $A'=(7,2),\ B'=(5,5)$

(b) $AB=\sqrt{(-2+5)^2+(7-5)^2}=\sqrt{14}$
$A'B'=\sqrt{(7-5)^2+(2-5)^2}=\sqrt{14}=AB$
$\therefore$ YES

16A.2 HKCEE MA 2009 – I – 9

(a) $A'=(-1,4),\ B'=(-5,2)$

(b) $m_{AB}=\frac{2+2}{5+1}=\frac{2}{3},\ m_{A'B'}=\frac{4-2}{-1+5}=\frac{1}{2}\neq m_{AB}$
$\therefore$ NO

16A.3 HKCEE MA 2011 – I – 8

(a) $B=(-6,-4),\ M=\left(\frac{-4-6}{2},\frac{6-4}{2}\right)=(-5,1)$

(b) $m_{OM}=\frac{1}{-5},\ m_{AB}=5$
$\because m_{OM}\cdot m_{AB}=-1$
$\therefore OM\perp AB$

16A.4 HKDSE MA SP – I – 8

(a) $A'=(5,2),\ A''=(2,5)$

(b) $m_{OA''}=\frac{5}{2},\ m_{AA'}=\frac{-3}{7}$
$\because m_{OA''}m_{AA'}=\frac{-15}{14}\neq -1$
$\therefore OA''$ is not perpendicular to AA'.

16A.5 HKDSE MA 2014 – I – 8

(a) $P'=(5,3),\ Q'=(-19,-7)$

(b) $m_{PQ}=\frac{-12}{5},\ m_{P'Q'}=\frac{10}{24}=\frac{5}{12}$
$\because m_{PQ}m_{P'Q'}=-1$
$\therefore PQ\perp P'Q'$

16A.6 HKDSE MA 2017 – I – 6

(a) $A'=(-4,-3),\ B'=(9,9)$

(b) $m_{AB}=\frac{13}{-12},\ m_{A'B'}=\frac{12}{13}$
$\because m_{AB}m_{A'B'}=-1$
$\therefore AB\perp A'B'$

16B Straight lines in the rectangular coordinate plane

16B.1 HKCEE MA 1992 – I – 5

(a) $m_{L_2}=\frac{1}{2}\Rightarrow m_{L_1}=-2$
$\therefore$ Eqn of L_1: $y-5=-2(x-10)\Rightarrow 2x+y-25=0$

(b) $\begin{cases}L_1: 2x+y-25=0\\ L_2: x-2y+5=0\end{cases}\Rightarrow (x,y)=(9,7)$

16B.2 HKCEE MA 1998 – I – 8

(a) $m_{AB}=\frac{4-1}{0+2}=\frac{3}{2}$

(b) Required eqn: $y-3=\frac{1}{\frac{3}{-2}}(x-1)\Rightarrow 2x+3y-11=0$

16B.3 HKCEE MA 1999 – I – 10

(a) $M=\left(\frac{-8+16}{2},\frac{8-4}{2}\right)=(4,2)$
$m_{AB}=\frac{12}{-24}=-\frac{1}{2}\Rightarrow m_{\ell}=2$
$\therefore$ Eqn of ℓ: $y-2=2(x-4)\Rightarrow 2x-y-6=0$

(b) Put $y=0$ into eqn of $\ell\Rightarrow x=3\Rightarrow P=(3,0)$
$BP=\sqrt{(16-3)^2+(-4-0)^2}=\sqrt{185}$

(c) $N=\left(\frac{-8+3}{2},\frac{8+0}{2}\right)=\left(-\frac{5}{2},4\right)$
$\therefore MN=\sqrt{\left(3+\frac{5}{2}\right)^2+(0-4)^2}=\sqrt{\frac{185}{4}}=\frac{\sqrt{185}}{2}$

16B.4 HKCEE MA 2000 – I – 9

(a) $m_L=\frac{4-0}{-4-6}=-\frac{2}{5}$

(b) Eqn of L: $y-0=-\frac{2}{5}(x-6)\Rightarrow 2x+5y-12=0$

(c) Put $x=0\Rightarrow y=\frac{12}{5}\Rightarrow C=\left(0,\frac{12}{5}\right)$

16B.5 HKCEE MA 2001 – I – 7

(a) $A=(-1,5),\ B=(4,3)$

(b) Eqn of AB: $\frac{y-5}{x+1}=\frac{5-3}{-1-4}=\frac{2}{-5}$
$-5(y-5)=-2(x+1)\Rightarrow 2x+5y-23=0$

16B.6 HKCEE MA 2002 – I – 8

(a) $x-2y=-8\Rightarrow \frac{x}{-8}+\frac{y}{4}=1$
$\therefore A=(-8,0),\ B=(0,4)$

(b) Mid-pt of $AB=\left(\frac{-8+0}{2},\frac{0+4}{2}\right)=(-4,2)$

16B.7 HKCEE MA 2003 – I – 12

(a) $m_{BC}=\frac{3-0}{0-2}=-\frac{3}{2}$

(b) $m_{AP}=-1\div -\frac{3}{2}=\frac{2}{3}$
$\therefore$ Eqn of AP: $y-0=\frac{2}{3}(x+1)\Rightarrow 2x-3y+2=0$

(c) (i) Put $x=0\Rightarrow y=\frac{2}{3}\Rightarrow H=\left(0,\frac{2}{3}\right)$

(ii) $m_{HB}=\frac{\frac{2}{3}-0}{0-2}=\frac{-1}{3},\ m_{AC}=\frac{3-0}{0+1}=3=\frac{-1}{m_{HB}}$
$\therefore HB\perp AC$
Hence the 3 altitudes of $\triangle ABC$ are CO, AP and HB, all passing through H.

16B.8 HKCEE MA 2004 – I – 13

(a) (i) $E=$ mid-pt of $AC=\left(\frac{2+8}{2},\frac{9+1}{2}\right)=(5,5)$

(ii) $m_{AC}=\frac{9-1}{2-8}=-\frac{4}{3}\Rightarrow m_{BD}=\frac{3}{4}$
$\therefore$ Eqn of BD: $y-5=\frac{3}{4}(x-5)\Rightarrow 3x-4y+5=0$

(b) (i) *Method 1*
$m_{AD}=\frac{-1}{7}$
$\Rightarrow BC$: $y-1=\frac{-1}{7}(x-8)\Rightarrow x+7y-15=0$

Method 2
Let BC be $x+7y+K=0$.
Put C: $(8)+7(1)+K=0\Rightarrow K=-15$
$\therefore$ Eqn of BC is $x+7y-15=0$.

(ii) $\begin{cases}BD: 3x-4y+5=0\\ BC: x+7y-15=0\end{cases}\Rightarrow B=(1,2)$
$\therefore AB=\sqrt{(2-1)^2+(9-2)^2}=\sqrt{50}$

16B.9 HKCEE MA 2005 – I – 13

(a) $A=(-2,0),\ B=(0,4)$

(b) $m_{L_1}=2\Rightarrow m_{L_2}=\frac{-1}{2}$
$\therefore$ Eqn of L_2: $y=\frac{-1}{2}x+4$

(c) $C=(8,0)$
$OC:AC=8:(8+2)=4:5$
$\therefore$ Area of $\triangle ODC$: Area of $\triangle ABC=16:25$
$\Rightarrow$ Area of $\triangle ODC$: Area of $OABD=16:(25-16)$
$=16:9$

16B.10 HKCEE MA 2006 – I – 12

(a) $M=(4,4)$

(b) $m_{AB}=\frac{1}{2}\Rightarrow m_{CM}=-2$
$\therefore$ Eqn of CM: $y-4=-2(x-4)\Rightarrow 2x+y-12=0$
Hence, put $y=0\Rightarrow C=(6,0)$

(c) (i) Eqn of BD: $\frac{y-0}{x-2}=\frac{8-0}{12-2}=\frac{4}{5}\Rightarrow 4x-5y-8=0$

(ii) $\begin{cases}CM: 2x+y-12=0\\ BD: 4x-5y-8=0\end{cases}\Rightarrow K=\left(\frac{34}{7},\frac{16}{7}\right)$

Method 1
$\frac{\text{Area of }\triangle AMC}{\text{Area of }\triangle AKC}=\frac{y\text{-coor of }M}{y\text{-coor of }K}=\frac{4}{\frac{16}{7}}=\frac{7}{4}$

Method 2
$\frac{\text{Area of }\triangle AMC}{\text{Area of }\triangle AKC}=\frac{MC}{KC}=\frac{\sqrt{(4-6)^2+(4-0)^2}}{\sqrt{(6-\frac{34}{7})^2+(0-\frac{16}{7})^2}}$
$=\frac{\sqrt{20}}{\sqrt{\frac{320}{49}}}=\frac{7}{4}$

Method 3
Let $MK:KC=r:s\Rightarrow \frac{16}{7}=\frac{s(4)+r(0)}{r+s}$
$16r+16s=28s$
$r:s=12:16=3:4$
$\therefore \frac{\text{Area of }\triangle AMC}{\text{Area of }\triangle AKC}=\frac{MC}{KC}=\frac{7}{4}$

16B.11 HKCEE MA 2007 – I – 13

(a) Eqn of AB: $y-3=\frac{-4}{3}(x-10)\Rightarrow 4x+3y-49=0$

(b) Put $x=4\Rightarrow y=11\Rightarrow h=11$

(c) (i) (Since $\triangle ABC$ is isosceles, A should lie 'above' the mid-point fo BC.)
$\frac{k+10}{2}=4\Rightarrow k=-2$

(ii) Area of $\triangle ABC=\frac{(10+2)(11-3)}{2}=48$
$AC=\sqrt{(4+2)^2+(11-3)^2}=10$
$\therefore BD=\frac{2\times \text{Area of }\triangle ABC}{AC}=\frac{48}{5}$

16B.12 HKCEE MA 2008 – I – 12

(a) $B=(-3,4),\ C=(4,-3)$

(b) $m_{OB}=\frac{4}{-3},\ m_{OC}=\frac{-3}{4}\neq m_{OB}$
$\therefore$ NO

(c) $m_{CD}=\frac{-1}{m_{BC}}=1$
$\therefore$ Eqn of CD: $y+3=1(x-4)\Rightarrow x-y-7=0$
$\because$ D is translated horizontally from A,
$\therefore$ y-coordinate of $D=y$-coordinat eof $A=3$
Put into eqn of $CD\Rightarrow x=10\Rightarrow D=(10,3)$

16B.13 HKCEE MA 2010 – I – 12

(a) Eqn of AB: $\frac{y-24}{x-6}=\frac{18-24}{-2-6}=\frac{3}{4}\Rightarrow 3x-4y+78=0$

(b) Let $C=(x,0)$.
$m_{AC}=\frac{-1}{m_{AB}}=\frac{-4}{3}$
$\frac{24-0}{6-x}=\frac{-4}{3}\Rightarrow x=24\Rightarrow C=(24,0)$

(c) $AB=\sqrt{(24-18)^2+(6+2)^2}=10$
$AC=\sqrt{(24-6)^2+(0-24)^2}=30$
$\therefore$ Area of $\triangle ABC=\frac{10\times 30}{2}=150$

(d) $\frac{BD}{DC}=\frac{\text{Area of }\triangle ABD}{\text{Area of }\triangle ADC}\Rightarrow \frac{r}{1}=\frac{90}{150-90}\Rightarrow r=1.5$

16B.14 HKCEE AM 1982 – II – 2

Method 1
Eqn of AB: $\frac{y-1}{x+1}=\frac{-1-1}{3+1}=\frac{-1}{2}\Rightarrow x+2y-1=0$

Let P be the pt of division. $\begin{cases}x+2y-1=0\\ x-y-1=0\end{cases}\Rightarrow P=(1,0)$

Let $AP:PB=r:1\Rightarrow 0=\frac{-1+(1)r}{r+1}=\frac{r-1}{r+1}\Rightarrow r=1$
$\therefore$ The required ratio is 1 : 1.

Method 2
Let the point of division be P, and $AP:PB=r:1$.
$P=\left(\frac{3+(-1)r}{r+1},\frac{-1+(1)r}{r+1}\right)=\left(\frac{3-r}{r+1},\frac{r-1}{r+1}\right)$
If P lies on $x-y-1=0$,
$\left(\frac{3-r}{r+1}\right)-\left(\frac{r-1}{r+1}\right)-1=0\Rightarrow r=1$
$\therefore$ The required ratio is 1 : 1.

16B.15 <u>HKCEE AM 1982 – II – 10</u>

(a) $\begin{cases}3x\;\;2y\;\;8=0\\x-y-2=0\end{cases}\Rightarrow P=(4,2)$

Eqn of L_1: $y\;\;2=\frac{1}{2}(x\;\;4)\Rightarrow x+2y-8=0$

Eqn of L_2: $y\;\;2=2(x-4)\Rightarrow 2x-y-6=0$

16B.16 <u>(HKCEE AM 1985 – II – 10)</u>

(a) <u>*Method 1* – Use collinearity of points</u>

Let $R=(r,h)$ and $S=(s,h)$

$m_{RC}=m_{AC}\Rightarrow\frac{h}{r-1}=\frac{2-0}{0-1}\Rightarrow r=1-\frac{h}{2}$

$m_{SB}=m_{AB}\Rightarrow\frac{h}{s+3}=\frac{2-0}{0+3}\Rightarrow s=\frac{3}{2}h-3$

$\therefore S=\left(\frac{3}{2}h-3,h\right),\ R=\left(1\;\;\frac{h}{2},h\right)$

<u>*Method 2* – Use eqns of straight lines</u>

Eqn of AB: $\frac{y-0}{x+3}=\frac{2\;\;0}{0+3}\Rightarrow 2x-3y+6=0$

Put $y=h\Rightarrow x=\frac{3}{2}h-3\Rightarrow S=\left(\frac{3}{2}h-3,h\right)$

Eqn of AC: $\frac{y-0}{x-1}=\frac{2\;\;0}{0-1}\Rightarrow 2x+y-2=0$

Put $y=h\Rightarrow x=1-\frac{h}{2}\Rightarrow R=\left(1-\frac{h}{2},h\right)$

<u>*Method 3* – Use similar triangles</u>

$\Delta BSP\sim\Delta BAO\Rightarrow\frac{h}{2}=\frac{BP}{3}\Rightarrow BP=\frac{3}{2}h$

$\therefore$ x-coordinate of $S=-3+\frac{3}{2}h\Rightarrow S=\left(\frac{3}{2}h-3,h\right)$

$\Delta AOC\sim\Delta RQC\Rightarrow\frac{2}{h}=\frac{1}{QC}\Rightarrow QC=\frac{h}{2}$

$\therefore$ x-coordinate of $R=1-\frac{h}{2}\Rightarrow R=\left(1-\frac{h}{2},h\right)$

(b) $RS=\left(1-\frac{h}{2}\right)-\left(\frac{3}{2}h-3\right)=4-2h$

When $PQRS$ is a square,

$PS=RS\Rightarrow h=4-2h\Rightarrow h=\frac{4}{3}\Rightarrow A_1=h^2=\frac{16}{9}$

Area of $PQRS=h(4-2h)=\;\;2(h^2-2h)$

$=-2(h-1)^2+2\Rightarrow A_2=2$

$A_3=\frac{2\times4}{2}=4$

$\therefore A_1:A_2:A_3=\frac{16}{9}:2:4=8:9:18$

(c) $M=$ mid-pt of $PR=\left(\frac{h}{2}-1,\frac{h}{2}\right)$

i.e. $x=\frac{h}{2}-1,\ y=\frac{h}{2}$

Put into $x\;\;y+1=0$:

$\text{LHS}=\left(\frac{h}{2}-1\right)\;\;\left(\frac{h}{2}\right)+1=0=\text{RHS}$

$\therefore$ M lies on $x-y+1=0$

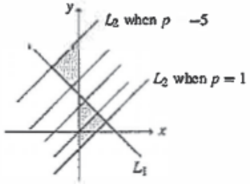
16B.17 <u>(HKCEE AM 1984 – II – 4)</u>

(a) $\begin{cases}L_1:x+y=4\\L_2:x-y=2p\end{cases}\Rightarrow(x,y)=(2+p,2\;\;p)$

(b) y-intercept of $L_1=4$, y-intercept of $L_2=-2p$

$\therefore$ Area of $\Delta=\frac{[4-(-2p)](2+p)}{2}$

$9=(2+p)^2\Rightarrow p=-5$ or 1



16B.18 <u>HKCEE AM 1988 – II – 2</u>

(a) $P\;\;\left(\frac{7k+1}{k+1},\frac{4k+2}{k+1}\right)$

(b) When P lies on $7x-3y-28=0$,

$7\left(\frac{7k+1}{k+1}\right)-3\left(\frac{4k+2}{k+1}\right)\;\;28=0$

$7(7k+1)-3(4k+2)\;\;28(k+1)=0$

$9k\;\;27=0\Rightarrow k=3$

$\therefore$ The ratio is 3 : 1.

16B.19 <u>HKCEE AM 1990 – II – 7</u>

<u>*Method 1* – Use algebra to find D</u>

Eqn of AB: $\frac{x}{3}+\frac{y}{5}=1\Rightarrow 5x+3y-15=0$

Area of $\Delta OAB=\frac{5\times3}{2}=\frac{15}{2}\Rightarrow$ Area of $\Delta BCD=\frac{15}{4}$

Let $D=(h,k)$. Then

$\begin{cases}5h+3k\;\;15=0\\\frac{15}{4}=\frac{(5-1)h}{2}=2h\end{cases}\Rightarrow D=\left(\frac{15}{8},\frac{15}{8}\right)$

<u>*Method 2* – Use ratios of areas to find D</u>

Area of $\Delta OAB=\frac{15}{2}$, $\Delta OAC=\frac{3}{2}$, $\Delta BCD=\frac{15}{4}$

$\Rightarrow$ Area of $\Delta ACD=\frac{15}{2}-\frac{15}{4}-\frac{3}{2}=\frac{9}{4}$

$\Rightarrow\frac{BD}{DA}=\frac{\text{Area of }\Delta BCD}{\text{Area of }\Delta ACD}=\frac{\frac{15}{4}}{\frac{9}{4}}=\frac{5}{3}$

$\therefore D=\left(\frac{3(0)+5(3)}{5+3},\frac{3(5)+5(0)}{5+3}\right)=\left(\frac{15}{8},\frac{15}{8}\right)$

<u>*Hence,*</u>

Eqn of CD: $\frac{y\;\;1}{x-0}=\frac{\frac{15}{8}-1}{\frac{15}{8}-0}-\frac{7}{15}\Rightarrow 7x-15y+15=0$

16B.20 <u>(HKCEE AM 1996 – II – 8)</u>

$\begin{cases}L_1:2x-y-4=0\\L_2:x-2y+4=0\end{cases}\Rightarrow(x,y)=(4,4)$

$\therefore$ Eqn of required line: $\frac{y\;\;0}{x-0}=\frac{4\;\;0}{4-0}\Rightarrow y=x$

16B.21 <u>(HKCEE AM 1998 – II – 5)</u>

(a) $\begin{cases}L_1:2x+y-3=0\\L_2:x-3y+1=0\end{cases}\Rightarrow P=\left(\frac{8}{7},\frac{5}{7}\right)$

(b) Eqn of L: $\frac{y\;\;0}{x\;\;0}=\frac{\frac{5}{7}-0}{\frac{8}{7}-0}\Rightarrow y=\frac{5}{8}x$

16B.22 <u>HKCEE AM 2005 – 6</u>

(a) $\tan\theta=m_{L_1}=\;\;2$

(b) $\angle OQP=\theta\Rightarrow\angle QOP=180°\;\;2\theta$

$\therefore$ Eqn of L_2: $y=x\tan\angle QOP=x\tan(180°\;\;2\theta)$

$=\;\;x\tan2\theta$

$=-x\cdot\frac{2\tan\theta}{1-\tan^2\theta}$

$=-x\cdot\frac{2(\;\;2)}{1-(-2)^2}$

$\Rightarrow y=\frac{4}{5}x$

16B.23 <u>(HKCEE AM 2009 – 3)</u>

$\begin{cases}L_1:x-2y+3=0\\L_2:2x-y-1=0\end{cases}\Rightarrow P=\left(\frac{5}{3},\frac{7}{3}\right)$

<u>*Method 1*</u>

Let the eqn of L be $\frac{x}{a}+\frac{y}{a}=1$, where $a>0$.

$\because$ P lies on L

$\therefore\left(\frac{5}{3}\right)+\left(\frac{7}{3}\right)\;\;a\Rightarrow a=4$

$\therefore$ Required line: $\frac{x}{4}+\frac{y}{4}=1\Rightarrow x+y-4=0$

<u>*Method 2*</u>

Let L be $y-\frac{7}{3}=m\left(x-\frac{5}{3}\right)\Rightarrow 3mx\;\;3y+7\;\;5m=0$

$\Rightarrow$ x-intercept $=\frac{5m-7}{3m}$, y-intercept $=\frac{7\;\;5m}{3}$

$\Rightarrow\frac{5m\;\;7}{3m}=\frac{7\;\;5m}{3}\Rightarrow 5m-7=-m(5m-7)$

$m=\frac{7}{5}$ or -1

However, when $m=\frac{7}{5}$, L becomes $7x-5y=0$, which has zero x- and y-intercepts. Rejected.

$\therefore$ Eqn of L is: $3(\;\;1)x\;\;3y+7\;\;5(-1)=0\Rightarrow x+y-4=0$

16B.24 <u>HKCEE AM 2010 – 6</u>

$\begin{cases}L_1:x-3y+7=0\\L_2:3x-y-11=0\end{cases}\Rightarrow(x,y)=(5,4)$

$\therefore$ Eqn of required line: $\frac{y-1}{x-2}=\frac{4-1}{5-2}=1\Rightarrow x-y-1=0$

16C Circles in the rectangular coordinate plane

16C.1 <u>HKCEE MA 1980(1/3 I) – B – 15</u>

(a) Put $y=0\Rightarrow x^2-10x+16=0\Rightarrow x=2$ or 8

$\therefore A=(2,0),\ B=(8,0)$

Put $x=0\Rightarrow y^2+8y+16=0\Rightarrow y=-4$

$\therefore T=(0,\;\;4)$

(b) (i) $m_{TB}=\frac{0+4}{8-0}=\frac{1}{2}$

$\therefore$ Eqn of AC: $y\;\;0=\frac{1}{2}(x\;\;2)\Rightarrow x-2y-2=0$

(ii) $\begin{cases}x^2+y^2-10x+8y+16=0\\x-2y\;\;2=0\end{cases}$

$(2y+2)^2+y^2-10(2y+2)+8y+16=0$

$5y^2-8y=0$

$y=0$ or $\frac{8}{5}$

Put $y=\frac{8}{5}\Rightarrow x=\frac{26}{5}\Rightarrow C=\left(\frac{26}{5},\frac{8}{5}\right)$

16C.2 <u>HKCEE MA 1981(1/3) – I – 13</u>

(a) $x^2+y^2=15^2\Rightarrow x^2+y^2-225=0$

(b) $OP=\frac{OT}{\sin\angle OPT}=\frac{OT}{\sin\theta}=\frac{15}{\frac{3}{\sqrt{3^2+4^2}}}=25$

(c) $P=(25,0)$

$\therefore$ Eqn of TP: $y\;\;0=\frac{3}{4}(x\;\;25)\Rightarrow 3x-4y-75=0$

(d) By geometry, $OCPT$ is a rectangle.

i.e. Eqn of OC: $y=\frac{3}{4}x$

(e) Let $C=(h,k)$. Then $k=\frac{3}{4}h$

$15=CP=\sqrt{(h-25)^2+(\frac{3}{4}h)^2}$

$225=\frac{25}{16}h^2-50h+625$

$h^2-32h+256=0\Rightarrow h=16\Rightarrow C=(16,12)$

Hence, eqn of circle is $(x-16)^2+(y-12)^2=15^2$

$x^2+y^2-32x-24y+175=0$

16C.3 <u>HKCEE MA 1982(I) – I – 13</u>

(a) $C:x^2+y^2-14y+40=0\Rightarrow x^2+(y-7)^2=3^2$

$\therefore$ Centre $=(0,7)$, Radius $=3$

(b) $m_L=\frac{4}{3}\Rightarrow m_{L'}=\frac{3}{4}$

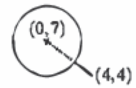
$\therefore$ Eqn of L': $y=\frac{-3}{4}x+7$

(c) $\begin{cases}L:4x-3y-4=0\\L':y=\frac{-3}{4}x+7\end{cases}\Rightarrow(x,y)=(4,4)$

(d) <u>Distance between centre of C and $(4,4)$</u>

$=\sqrt{(0-4)^2+(7-4)^2}=5$

$\Rightarrow$ Shortest dist $=5-$ radius $=2$



16C.4 HKCEE MA 1983(A/B) – I – 9

(a) Let $B=(b,0)$.
$1=m_{AB}=\frac{2\ \ 0}{8-6}\ \Rightarrow\ b=6\ \Rightarrow\ B=(6,0)$

(b) Let $C=(c,0)$. Since $\triangle ABC$ is isosceles, A lies 'above' the mid-point of BC.
$\therefore\ \frac{c+6}{2}=8\ \Rightarrow\ c=10\ \Rightarrow\ C=(10,0)$

(c) Eqn of AC: $\frac{y-0}{x-10}=\frac{2\ \ 0}{8-10}\ \Rightarrow\ y=-x+10$
$\therefore\ D=(0,10)$

(d) $BD=\sqrt{6^2+10^2}=\sqrt{136}$
Mid-pt of $BD=\left(\frac{6+0}{2},\frac{0+10}{2}\right)=(3,5)$

Eqn of circle OBD is $(x-3)^2+(y-5)^2=\left(\frac{\sqrt{136}}{2}\right)^2$
$\Rightarrow\ x^2+y^2\ \ 6x\ \ 10y=0$

Put A int othe equation:
LHS $=(8)^2+(2)^2\ \ 6(8)\ \ 10(2)=0=$ RHS
$\therefore$ A lies on the circle.

16C.5 HKCEE MA 1984(A/B) – I – 9

(a) $\begin{cases}x^2+y^2=4\\ y=k\ \ x\end{cases}\ \Rightarrow\ x^2+(k\ \ x)^2=4$
$2x^2-2kx+k^2\ \ 4=0\ldots(*)$
$\Delta=4k^2\ \ 8(k^2-4)=0\ \Rightarrow\ k=\pm\sqrt{8}$

(b) (i) If $A(2,0)$ is one fo the intersections of C and L, 2 is a root of the equation $(*)$
$2(2)^2\ \ 2k(2)+(2)^2\ \ 4=0\ \Rightarrow\ k=2$
Then $(*)$ becomes $2x^2\ \ 4x=0\ \Rightarrow\ x=2$ or 0
$\therefore\ B=(0,k\ \ 0)=(0,2)$

(ii) $AB=\sqrt{(2\ \ 0)^2+(0-2)^2}=\sqrt{8}$
Mid-pt of $AB=\left(\frac{2+0}{2},\frac{0+2}{2}\right)=(1,1)$

. Eqn of circle is $(x-1)^2+(y-1)^2=\left(\frac{\sqrt{8}}{2}\right)^2$
$\Rightarrow\ x^2+y^2\ \ 2x\ \ 2y=0$

16C.6 HKCEE MA 1985(A/B) I – 9

(a) $AB=\sqrt{(2-7)^2+(0\ \ 5)^2}=\sqrt{50}$
Mid-pt of $AB=\left(\frac{2+7}{2},\frac{0+5}{2}\right)=\left(\frac{9}{2},\frac{5}{2}\right)$

$\therefore$ Eqn of circle is $\left(x\ \ \frac{9}{2}\right)^2+\left(y\ \ \frac{5}{2}\right)^2=\left(\frac{\sqrt{50}}{2}\right)^2$
$\Rightarrow\ x^2+y^2-9x\ \ 5y+14=0$

(b) ~~$P=\left(\frac{4(2)+1(7)}{1+4},\frac{4(0)+1(5)}{1+4}\right)=(3,1)$~~

(c) (i) $m_{AB}=\frac{0-5}{2\ \ 7}=1\ \Rightarrow\ m_{HPK}=\ \ 1$
$\therefore$ Eqn of HPK: $y\ \ 1=-1(x\ \ 3)\ \Rightarrow\ x+y-4=0$

(ii) $\begin{cases}x^2+y^2-9x\ \ 5y+14=0\\ x+y-4=0\end{cases}$
$\Rightarrow\ x^2+(4\ \ x)^2\ \ 9x\ \ 5(4\ \ x)+14=0$
$2x^2-12x+10=0$
$x=1$ or 5
$\Rightarrow\ y=3$ or $\ \ 1$
$\therefore\ H=(1,3),\ K=(5,\ \ 1)$

16C.7 HKCEE MA 1986(A/B) – I – 8

(a) $\begin{cases}x^2+y^2-6x-8y=0\\ y-x\ \ 6=0\end{cases}$
$\Rightarrow\ x^2+(x+6)^2-6x-8(x+6)=0$
$2x^2\ \ 2x-12=0$
$x=3$ or $\ \ 2$
$y=9$ or 4

$\therefore\ B=(3,9),\ C=(\ \ 2,4)$

(b) Put $y=0\ \Rightarrow\ x=0$ or $6\ \Rightarrow\ A=(6,0)$
Put $x=0\ \Rightarrow\ y=0$ or $8\ \Rightarrow\ D=(0,8)$

(c) $\angle ADO=\tan^{-1}\frac{AO}{DO}=\tan^{-1}\frac{6}{8}=37°$ (nearest degree)
$\therefore\ \angle ABO=\angle ACO=\angle ADO=37°$

(d) Area of $\triangle ACO=\frac{6\times 4}{2}=12$

16C.8 HKCEE MA 1987(A/B) – I – 8

(a) Eqn of ℓ: $y\ \ 0=1(x+2)\ \Rightarrow\ x\ \ y+2=0$

(b) x-coordinate of $C=x$-coordinate of mid-pt of $OB=2$
Put $x=2$ into $\ell\ \Rightarrow\ y=4\ \Rightarrow\ C=(2,4)$

(c) Let the centre of the circle be $(2,k)$.
$k^2+4=(4\ \ k)^2$
$k^2+4=16-8k+k^2\ \Rightarrow\ k=\frac{3}{2}$

$\therefore$ Eqn of circle: $(x\ \ 2)^2+\left(y-\frac{3}{2}\right)^2=\left(4\ \ \frac{3}{2}\right)^2$
$\Rightarrow\ x^2+y^2\ \ 4x-3y=0$

(d) $\begin{cases}x^2+y^2-4x-3y=0\\ x-y+2=0\end{cases}$
$\Rightarrow\ x^2+(x+2)^2-4x-3(x+2)=0$
$2x^2\ \ 3x-2=0\ \Rightarrow\ x=2$ or $\frac{1}{2}$

$\therefore\ D=\left(\frac{1}{2},\frac{1}{2}+2\right)=\left(\frac{1}{2},\frac{5}{2}\right)$

16C.9 HKCEE MA 1988 – I – 7

(a) $(2,\ \ 5)$

(b) Radius of $C=x$-coordinate of centre $=2$
$\therefore\ \sqrt{2^2+5^2-k}=2\ \Rightarrow\ k=5$

16C.10 HKCEE MA 1989 – I – 8

(a) $E=(1,2)$

(b) $\begin{cases}x^2+y^2-2x\ \ 4y\ \ 20=0\\ x+7y-40=0\end{cases}$
$\Rightarrow\ (40\ \ 7y)^2+y^2-2(40\ \ 7y)\ \ 4y\ \ 20=0$
$50y^2\ \ 55y+1500=0$
$y=5$ or 6
$x=5$ or -2

$\therefore\ P=(\ \ 2,6),\ Q=(5,5)$

(c) $PQ=\sqrt{(\ \ 2\ \ 5)^2+(6\ \ 5)^2}=\sqrt{50}$
Mid-pt of $PQ=\left(\frac{-2+5}{2},\frac{6+5}{2}\right)=\left(\frac{3}{2},\frac{11}{2}\right)$

$\therefore$ Eqn of $\mathscr{C}_2$: $\left(x\ \ \frac{3}{2}\right)^2+\left(y\ \ \frac{11}{2}\right)^2=\left(\frac{\sqrt{50}}{2}\right)^2$
$\Rightarrow\ x^2+y^2-3x-11y+20=0$

(d) Put $E(1,2)$ into $\mathscr{C}_2$:
LHS $=(1)^2+(2)^2-3(1)-11(2)+20=0=$ RHS
$\therefore$ E lies on $\mathscr{C}_2\ \Rightarrow\ \angle EPQ=90°$

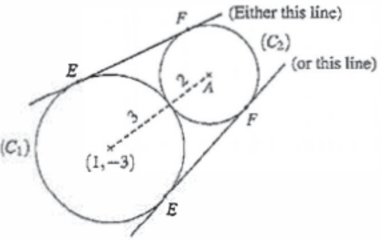
16C.11 HKCEE MA 1990 – I – 8

(a) $(C_1):(x\ \ 1)^2+(y+3)^2=3^2$
$\therefore$ Centre $=(1,\ \ 3)$, Radius $=3$

(b) Required distance $=\sqrt{(1-5)^2+(\ \ 3-0)^2}=5>3$
$\therefore$ Outside

(c) (i) $s=5-3=2$
(ii) Eqn of C_2: $(x\ \ 5)^2+(y-0)^2=2^2$
$\Rightarrow\ x^2+y^2-10x+21=0$

(d)



16C.12 HKCEE MA 1991 – 1 – 9

(a) $S:(x\ \ 2)^2+(y-1)^2=1^2$
$\therefore\ C=(2,1),\ A=(2,0)$

(b) $\begin{cases}y=mx\\ x^2+y^2\ \ 4x-2y+4=0\end{cases}$
$x^2+(mx)^2\ \ 4x\ \ 2(mx)+4=0$
$(1+m^2)x^2-2(2+m)x+4=0$
$\Delta=4(2+m)^2-16(1+m^2)=0$
$(2+m)^2\ \ 4(1+m^2)=0$
$3m^2-4m=0\ \Rightarrow\ m=0$ (rej.) or $\frac{4}{3}$

(c) (i) $\because\ \angle OBC=\angle OAC=90°$ (tangent properties)
$\therefore\ \angle OBC+\angle OAC=180°$
$\therefore$ O, A, C and B are concyclic. (opp $\angle$s supp.)

(ii) $OC=\sqrt{2^2+1^2}=\sqrt{5}$
Mid-pt of $OC=\left(\frac{2+0}{2},\frac{1+0}{2}\right)=\left(1,\frac{1}{2}\right)$

$\therefore$ Eqn of circle: $(x-1)^2+\left(y-\frac{1}{2}\right)^2=\left(\frac{\sqrt{5}}{2}\right)^2$
$\Rightarrow\ x^2+y^2-2x-y=0$

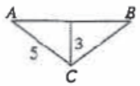
16C.13 HKCEE MA 1992 – I – 13

(a) $x^2+y^2\ \ 18x\ \ 14y+105=0\ \Rightarrow\ (x-9)^2+(y-7)^2=5^2$
$\therefore\ C=(9,7)$, Radius $=5$

(b) $x^2+(mx)^2-18x\ \ 14(mx)+105=0$
$(1+m^2)x^2\ \ 2(9+7m)x+105=0$
$\therefore\ x_1x_2=$ product of roots $=\frac{105}{1+m^2}$

(c) $OA=\sqrt{x_1^2+y_1^2}=\sqrt{x_1^2+(mx_1)^2}=x_1\sqrt{1+m^2}$
Similarly, $OB=x_2\sqrt{1+m^2}$
$\therefore\ OA\cdot OB=(1+m^2)x_1x_2=(1+m^2)\cdot\frac{105}{1+m^2}=105$

(d) $AB=2\sqrt{5^2-3^2}=8$
$OA\cdot(OA+AB)=105$
$OA^2+8OA-105=0$
$\Rightarrow\ OA=\ \ 15$ (rej.) or 7



16C.14 HKCEE MA 1993 – I – 8

(a) $L_1:\frac{y\ \ 7}{x\ \ 0}=\frac{2\ \ 7}{10-0}=\frac{1}{2}\ \Rightarrow\ x+2y-14=0$

(b) $m_{L_2}=\frac{1}{\frac{-1}{2}}=2$
$\therefore$ Eqn of L_2: $y-0=2(x-4)\ \Rightarrow\ 2x\ \ y-8\ \ 0$
$\begin{cases}x+2y-14=0\\ 2x-y-8=0\end{cases}\ \Rightarrow\ D=(x,y)=(6,4)$

(c) $P=\left(\frac{1(0)+k(10)}{k+1},\frac{1(7)+k(2)}{k+1}\right)=\left(\frac{10k}{k+1},\frac{7+2k}{k+1}\right)$
If P lies on the circle,
$\left[\left(\frac{10k}{k+1}\right)-4\right]^2+\left(\frac{7+2k}{k+1}\right)^2=30$
$(6k\ \ 4)^2+(7+2k)^2=30(k+1)^2$
$10k^2\ \ 80k+35=0$
$k=\frac{16\pm\sqrt{200}}{4}=4\pm\frac{5\sqrt{2}}{2}$

$\because\ \frac{AD}{DB}=\frac{6-0}{10-6}=\frac{3}{2}$
$\therefore\ k<\frac{3}{2}$ if P lies bet weenA and D.
i.e. $\frac{AP}{PB}=k=4-\frac{5\sqrt{2}}{2}$

16C.15 HKCEE MA 1994 – I – 12

(a) $A=(10,0)$, Radius of $C_2=7$

(b) $\frac{RO}{RA}=\frac{OQ}{AP}\ \Rightarrow\ \frac{RO}{RO+10}=\frac{1}{7}\ \Rightarrow\ RO=\frac{5}{3}$
$\therefore$ x-coordinate of $R=\ \ \frac{5}{3}$

(c) $m_{QP}=\tan\angle QRO=\frac{OQ}{QR}=\frac{1}{\sqrt{(\frac{5}{3})^2-1^2}}=\frac{3}{4}$

(d) Eqn of QP: $y\ \ 0=\frac{3}{4}\left(x+\frac{5}{3}\right)\ \Rightarrow\ 3x-4y+5=0$

(e) By symmetry, the other tangent is:
$y-0=\frac{-3}{4}\left(x+\frac{5}{3}\right)\ \Rightarrow\ 3x+4y+5=0$

16C.16 HKCEE MA 1995 – I – 10

(a) Eqn of AB: $\frac{y\ \ 7}{x\ \ 9}=\frac{9-7}{1-9}=\frac{1}{4}\ \Rightarrow\ x+4y-37=0$

(b) Mid-pt of $AB=\left(\frac{1+9}{2},\frac{9+7}{2}\right)=(5,8)$
Slope of $\perp$ bisector of $AB=4$
$\therefore$ Eqn of $\perp$ bisector is: $y\ \ 8=4(x\ \ 5)\ \Rightarrow\ y=4x\ \ 12$
$\begin{cases}4x-3y+12=0\\ y=4x-12\end{cases}\ \Rightarrow\ G=(6,12)$

(c) Radius $=\sqrt{(6\ \ 1)^2+(12\ \ 9)^2}=\sqrt{34}$
$\therefore$ Eqn of $\mathscr{C}$: $(x-6)^2+(y\ \ 12)^2=34$
$x^2+y^2\ \ 12x\ \ 24y+146=0$

(d) (i) Let the mid-pt of DE be (m,n). Then G is the mid-pt of $(5,8)$ and (m,n).
$\therefore\ \left(\frac{5+m}{2},\frac{8+n}{2}\right)=(6,12)\Rightarrow G=(m,n)=(7,16)$

(ii) $m_{DE}=m_{AB}=\frac{1}{4}$
$\therefore$ Eqn of DE: $y-16=\frac{1}{4}(x-7)$
$\Rightarrow\ x+4y\ \ 57=0$

16C.17 <u>HKCEE MA 1996 – I – 11</u>

(a) (i) $\mathscr{C}_1 : (x-0)^2+(y-2)^2=2^2 \Rightarrow x^2+y^2 \quad 4y=0$

(ii) $B=(0,4) \Rightarrow$ Eqn of L: $y=2x+4$

(b) $\begin{cases} L: y=2x+4 \\ \mathscr{C}_2 : x^2+(y-2)^2=25 \end{cases}$

$x^2+(2x+2)^2=25$

$5x^2+8x-21=0 \Rightarrow x=-3 \text{ or } \frac{7}{5} \Rightarrow y=-2 \text{ or } \frac{34}{5}$

$\therefore Q=\left(\frac{7}{5},\frac{34}{5}\right),\ R=(-3,-2)$

(c) (i) Req. pt = mid-pt of $QR=\left(\frac{-4}{5},\frac{12}{5}\right)$

(ii) Req. pt = Intersection of AQ and $\mathscr{C}_1$

= the pt 'P' with $AP:PQ=2:(5-2)$

$=\left(\frac{3(0)+2(\frac{7}{5})}{2+3},\frac{3(2)+2(\frac{34}{5})}{2+3}\right)=\left(\frac{14}{25},\frac{98}{25}\right)$

16C.18 <u>HKCEE MA 1997 – I – 16</u>

(a) (i) $\angle EAB=90°$ (tangent ⊥ radius)

$\therefore \angle FEA+\angle EAB=90°+90°=180°$

$\therefore AB//EF$ (int. ∠s supp.)

(ii) $\angle FDE=\angle BDC$ (vert. opp. ∠s)

$=\angle DBC$ (base ∠s, isos. △)

$=\angle FED$ (alt. ∠s, $AB//EF$)

$\therefore FD=FE$ (sides opp. equal ∠s)

(iii) If the circle touches AE at E, then its centre lies on EF.

If ED is a chord, the centre lies on the ⊥ bisector of ED.

∴ The intersection of these two lines, F, is the centre of the circle described.

(b) $C=\left(\frac{6-2}{2},\frac{3-1}{2}\right)=(2,1)$

$\because FD=FE$,

∴ Let $F=\left(-\frac{4}{2}\cdot 2,k\right)=(-3,k)$

F, D, C collinear $\Rightarrow m_{FD} \quad m_{CD}$

$\frac{k-3}{-3+2}=\frac{3-1}{-2-2} \Rightarrow k=\frac{7}{2}$

$\therefore F=\left(-3,\frac{7}{2}\right)$

16C.19 <u>HKCEE MA 1998 – I – 15</u>

(a) Centre of $C_2=(11,-8)$, Radius of $C_2=7$

Dist btwn the 2 centres $=\sqrt{(11-5)^2+(-8-0)^2}=10$

∴ Radius of $C_1=10-7=3$

∴ Eqn of C_1: $(x-5)^2+(y-0)^2=3^2$

$\Rightarrow x^2+y^2-10x+16=0$

(b) Let the tangent be $y=mx$.

$\begin{cases} y=mx \\ x^2+y^2-10x+16=0 \end{cases} \Rightarrow (1+m^2)x^2-10x+16=0$

$\Delta=100-64(1+m^2)=0 \Rightarrow m=\pm\frac{1}{2}$

∴ The tangents are $y=\pm\frac{1}{2}x$

(c) $\begin{cases} y=\frac{-1}{2}x \\ (x-11)^2+(y+8)^2=49 \end{cases} \Rightarrow \frac{5}{4}x^2-30x+136=0$

Sum of rts $=\frac{30}{\frac{5}{4}}=24 \Rightarrow$ x-coor of mid-pt of $AB=12$

$\Rightarrow$ y-coor $=\frac{-1}{2}(12)=-6 \Rightarrow$ The mid-pt $=(12,-6)$

16C.20 <u>HKCEE MA 1999 – I – 16</u>

(a) (i) $\angle BFE=\angle BDE$ (∠s in the same segment)

$=\angle BAC$ (corr. ∠s, $AC//DE$)

∴ A, F, B and C are concyclic.

(converse of ∠s in the same segment)

(ii) ∵ $\angle ABC=90°$ (given)

∴ AC is a diameter of circle $AFBC$.

(converse of ∠ in sem-circle)

$\Rightarrow M$ is the centre of circle $AFBC \Rightarrow MB=MF$

(b) (i) $m_{PQ}=\frac{17-0}{0+17}=1$

$m_{RS}=\frac{7-0}{-2+9}=1=m_{PQ}$

$\therefore PQ//RS$

(ii) Eqn of QS: $\frac{y-17}{x-0}=\frac{17-7}{0+2} \Rightarrow y=5x+17$

$\begin{cases} y=5x+17 \\ x^2+y^2+10x-6y+9=0 \end{cases}$

$x^2+(5x+17)^2+10x \quad 6(5x+17)+9=0$

$26x^2+150x+196=0$

$x=-2 \text{ or } -\frac{49}{13}$

$\therefore T=\left(-\frac{49}{13},5\left(-\frac{49}{13}\right)+17\right)=\left(-\frac{49}{13},-\frac{24}{13}\right)$

(iii) *<u>Method 1</u>*

Let the mid-pt of PQ be $N=\left(\frac{-17}{2},\frac{17}{2}\right)$

$NO \quad \sqrt{\left(\frac{-17}{2}\right)^2+\left(\frac{17}{2}\right)^2}=\sqrt{\frac{289}{2}}$

$NT=\sqrt{\left(\frac{-49}{13}+\frac{17}{2}\right)^2+\left(\frac{-24}{13}-\frac{17}{2}\right)^2}=\sqrt{\frac{3365}{26}}$

Hence, $NT \neq NO$.

If P, Q, O and T are concyclic, the result of (a)(ii) should apply, i.e. $NO=NT$. Thus they are not concyclic.

<u>Method 2</u>

$\because m_{PT} \quad m_{QT}=\frac{0+\frac{24}{13}}{-17+\frac{49}{13}}\cdot\frac{17+\frac{24}{13}}{0+\frac{49}{13}}=\frac{-30}{43}\neq -1$

$\therefore \angle PTQ \neq 90°$

Thus, $\angle PTQ+\angle POQ \neq 90°+90°=180°$, and P, Q, O and T are not concyclic.

16C.21 <u>HKCEE MA 2000 – I – 16</u>

(a) In $\triangle OCP$, $\angle CPO=90°$ (tangent ⊥ radius)

$\angle PCO=180°-30°-90°$ (∠ sum of △)

$\therefore \angle PQO=60°\div 2=30°$ (∠ at centre twice ∠ at ⊙^{ce})

(b) (i) $\angle SOC=\angle POC=30°$ (tangent properties)

$\angle PQR=180°-\angle POS$ (opp. ∠s, cyclic quad.)

$=120°$

$\Rightarrow \angle RQO=120°-30°=90°$

∴ RQ is tangent to the circle at Q.

(converse of tangent ⊥ radius)

(ii) $OC=\sqrt{6^2+8^2}=10$

$CQ=CP=OC\sin 30°=5$

$\because OC:CQ=10:5=2:1$

$\therefore Q=(9,12)$

$m_{OC}=\frac{4}{3} \Rightarrow m_{QR}=\frac{-3}{4}$

∴ Eqn of QR: $y \quad 12=\frac{-3}{4}(x-9)$

$\Rightarrow 3x+4y-21=0$

16C.22 <u>HKCEE MA 2001 – I – 17</u>

(a) (i) Centre $=\left(\frac{p}{2},0\right)$, Radius $=\frac{p}{2}$

∴ Eqn of OPS: $\left(x-\frac{p}{2}\right)^2+y^2=\left(\frac{p}{2}\right)^2$

$\Rightarrow x^2+y^2-px=0$

(ii) *<u>'Hence'</u>*

$S(a,b)$ lies on the circle

$\Rightarrow a^2+b^2-pa=0 \Rightarrow a^2+b^2=pa$

$\therefore OS^2=(a-0)^2+(b-0)^2=a^2+b^2$

$=pa$

$=OP\cdot OQ\cos\angle POQ$

<u>'Otherwise'</u>

$\angle OSP=90°$ (∠ in semi-circle)

In $\triangle OPS$ and $\triangle OSR$,

$\angle POS=\angle SOR$ (common)

$\angle ORS=\angle OSP=90°$ (proved)

$\therefore \triangle OPS \sim \triangle OSR$ (AA)

$\Rightarrow \frac{OS}{OR}=\frac{OP}{OS}$ (corr. sides, ~ △s)

$OS^2=OP\cdot OR$

$=OP\cdot OQ\cos\angle POQ$

(b) (i) In circle BCE, $\angle CEB=90°$ (∠ in semi-circle)

i.e. BE is an altitude of $\triangle ABC$.

(ii) By (a), $CG^2=AC\cdot BC\cos\angle ACB$

Similarly, AD is an altitude of $\triangle ABC$ by considering circle ACD.

$\Rightarrow CF^2=BC\cdot AC\cos\angle ACB=CG^2$

$\therefore CF=CG$

16C.23 <u>HKCEE MA 2002 – I – 16</u>

(b) (i) $A=(c \quad r,0)$, $B=(c+r,0)$

$m_{AD}=\frac{p \quad 0}{0-(c-r)}=\frac{p}{r-c}$

$m_{BF}=\frac{q-0}{0-(c+r)}=\frac{q}{r+c}$

(ii) $AD \perp BF \Rightarrow \frac{p}{r \quad c}\cdot\frac{-q}{r+c}=-1$

$pq=r^2-c^2$

i.e. $OD\cdot OF=CG^2-OC^2$

$=OG^2$

16C.24 <u>HKCEE MA 2003 – I – 17</u>

(a) (i) In $\triangle NPM$ and $\triangle NKP$,

$\angle PNM=\angle KNP$ (common)

$\angle NPM=\angle NKP$ (∠ in alt. segment)

$\angle PMN=\angle KPN$ (∠ sum of △)

$\therefore \triangle NPM \sim \triangle NKP$ (AAA)

$\Rightarrow \frac{NP}{NM}=\frac{NK}{NP}$ (corr. sides, ~ △s)

$NP^2=NK\cdot NM$

(ii) $\because RS//OP$ (given)

$\therefore \triangle KRM \sim \triangle KON$ and $\triangle KSM \sim \triangle KPN$

$\Rightarrow \frac{RM}{ON}=\frac{KM}{KN}$ and $\frac{SM}{PN}=\frac{KM}{KN}$

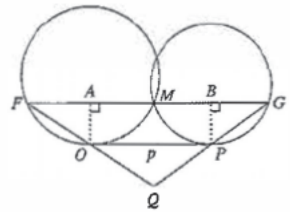
$\Rightarrow \frac{RM}{ON}=\frac{SM}{PN}$

Similar to (a), we have $NO^2=NK\cdot NM$

$\therefore NP=NO$

Hence, $RM=MS$.

(b) (i)



With the notation above, note that OA (extended) and PB (extended) are diameters of C_1 and C_2 respectively.

$\therefore FA=AM$ and $MB=BG$

(⊥ from centre to chord bisects chord)

Hence, $FG=2AM+2MB=2AB=2p$

(ii) $\because M=(a,b)$ and $FA=AM$,

$\therefore F=(\quad a,b)$

Since $\triangle QOP \sim \triangle QFG$ and $FG=2OP$, we have

$FQ=2OQ \Rightarrow O$ is the mid-pt of FQ

$\Rightarrow Q=(a, \quad b)$

(iii) Note that QM is vertical. Thus $QM \perp RS$.

In $\triangle QMR$ and $\triangle QMS$,

$QM=QM$ (common)

$RM=SM$ (proved)

$\angle QMR=\angle QMS=90°$ (proved)

$\therefore \triangle QMR \cong \triangle QMS$ (SAS)

$\Rightarrow QR=QS$ (corr. sides, ≅ △s)

i.e. $\triangle QRS$ is isosceles.

16C 25 <u>HKCEE MA 2004 – I – 16</u>

(a) In $\triangle ADE$ and $\triangle BOE$,

$\angle ADE=\angle EBC$ (alt. ∠s, $OD//BC$)

$=\angle BOE$ (∠ in alt. segment)

$\angle DAE=\angle OBE$ (ext. ∠, cyclic quad.)

$AD=BO$ (given)

$\therefore \triangle ADE \cong \triangle BOE$ (ASA)

(b) $DE=OE$ (corr. sides, ≅ △s)

$\angle BOE=\angle ADE$ (proved)

$=\angle AOE$ (base ∠s, isos. △)

i.e. $\angle AOB=2\angle BOE$

$\therefore \angle BEO=\angle AED$ (corr. ∠s, ≅ △s)

$=\angle AOB$ (ext. ∠, cyclic quad.)

$=2\angle BOE$ (proved)

(c) Suppose OE is a diameter of the circle $OAEB$.

(i) $\angle OBE=90°$ (∠ in semi-circle)

In $\triangle OBE$, $\angle BOE=180°-90°-(2\angle BOE)$

(∠ sum of △)

$3\angle BOE=90° \Rightarrow \angle BOE=30°$

(ii) $OB=6 \Rightarrow BE=OB\tan\angle BOE \Rightarrow E=(6,2\sqrt{3})$

$OE=\frac{OB}{\cos 30°}=4\sqrt{3}$

Mid-pt of $OE=(3,\sqrt{3})$

∴ Eqn of circle: $(x-3)^2+(y-\sqrt{3})^2=\left(\frac{4\sqrt{3}}{2}\right)^2$

$\Rightarrow x^2+y^2-6x \quad 2\sqrt{3}y=0$

16C.26 HKCEE MA 2005 – I – 17

(a) (i) $\because$ MN is a diameter (given)

$\therefore$ $\angle NOM = \angle QRP = 90°$ ($\angle$ in semi-circle)

In $\triangle OQR$ and $\triangle ORP$,

$\angle ROQ = \angle POR = 90°$ (given)

$\angle QRO = \angle QRP - \angle PRO = 90° - \angle PRO$

$\angle POR = 180° - \angle ROP - \angle PRO$ ($\angle$ sum of $\triangle$)

$= 90° - \angle PRO$

$\Rightarrow \angle QPO = \angle PRO$

$\angle RQO = \angle PRO$ ($\angle$ sum of $\triangle$)

$\therefore \triangle OQR \sim \triangle ORP$ (AAA)

$\Rightarrow \dfrac{OR}{OQ} = \dfrac{OP}{OR}$ (corr. sides, $\sim\triangle$s)

$OR^2 = OP \cdot OQ$

(ii) In $\triangle MON$ and $\triangle POR$,

$\angle NMO = \angle QRO$ ($\angle$s in the same segment)

$= \angle RPO$ (proved)

$\angle MON = \angle POR$ (proved)

$\angle MNO = \angle RQO$ ($\angle$ sum of $\triangle$)

$\therefore \triangle MON \sim \triangle RQO$ (AAA)

(b) (i) $OR = \sqrt{OP \cdot OQ} = \sqrt{4 \cdot 9} = 6 \Rightarrow R = (0, 6)$

(ii) In $\triangle POR$, $PR = \sqrt{4^2 + 6^2} = \sqrt{52}$

$\dfrac{MN}{ON} = \dfrac{PR}{OR} = \dfrac{\sqrt{52}}{6} \Rightarrow MN = \dfrac{\sqrt{13}}{3} \cdot \dfrac{3\sqrt{13}}{2} = \dfrac{13}{2}$

$\therefore$ Radius $= \dfrac{13}{2} \div 2 = \dfrac{13}{4}$

Let the centre be $(h, 6 \div 2) = (h, 3)$

(since it lies on the $\perp$ bisector of OR).

$\Rightarrow \sqrt{(h-0)^2 + (3-0)^2} = \dfrac{13}{4} \Rightarrow h = -\dfrac{5}{2}$ $(h < 0)$

$\therefore$ The centre is $\left(-\dfrac{5}{2}, 3\right)$

16C.27 HKCEE MA 2006 – I – 16

(a) (i) $\because$ G is the circumcentre (given)

$\therefore$ $SC \perp BC$ and $SA \perp AB$ ($\angle$ in semi-circle)

$\because$ H is the orthocentre (given)

$\therefore$ $AH \perp BC$ and $CH \perp AB$

Thus, $SC // AH$ and $SA // CH \Rightarrow AHCS$ is a //gram

(ii) *Method 1*

$\because$ $\angle GRB = \angle SCB = 90°$ (proved)

$\therefore$ $GR // SC$ (corr. $\angle$s equal)

$\because$ $BG = GS =$ radius

$\therefore$ $BR = RC$ (intercept thm)

$\Rightarrow SC = 2GR$ (mid-pt thm)

Hence, $AH = SC = 2GR$ (property of //gram)

Method 2

$\because$ $BG = GS =$ radius

and $BR = RC$ ($\perp$ from centre to chord bisects chord)

$\Rightarrow SC = 2GR$ (mid-pt thm)

Hence, $AH = SC = 2GR$ (property of //gram)

(b) (i) Let the circle be $x^2 + y^2 + Dx + Ey + F = 0$

$$\begin{cases} 0^2 + 12^2 + 0D + 12E + F = 0 \\ (-6)^2 + 0^2 - 6D + 0E + F = 0 \\ 4^2 + 0^2 + 4D + 0E + F = 0 \end{cases} \Rightarrow \begin{cases} D = 2 \\ E = -10 \\ F = -24 \end{cases}$$

$\therefore$ The circle is $x^2 + y^2 + 2x - 10y - 24 = 0$.

(ii) $G = (-1, 5) \Rightarrow GR = 5$

$\therefore H = (0, 12 - 2 \times 5) = (0, 2)$ (by (a)(ii))

(iii) $m_{BG} \cdot m_{GH} = \dfrac{5-0}{-1+6} \cdot \dfrac{5-2}{-1-0} = -3 \neq -1$

$\therefore \angle BGH \neq 90° \Rightarrow \angle BOH + \angle BGH \neq 180°$

Hence, B, O, H and G are not concyclic.

16C.28 HKCEE MA 2007 – I – 17

(a) (i) $\because$ I is the incentre of $\triangle ABD$ (given)

$\therefore$ $\angle ABG = \angle DBG$ and $\angle BAE = \angle CAE$

In $\triangle ABG$ and $\triangle DBG$,

$\angle ABG = \angle DBG$ (proved)

$AB = DB$ (given)

$BG = BG$ (common)

$\therefore \triangle ABG \cong \triangle DBG$ (SAS)

(ii) $\because$ $\triangle ABD$ is isosceles and $\angle ABG = \angle DBG$

$\therefore$ $\angle BGA = 90°$ (property of isos. $\triangle$)

In $\triangle AGI$ and $\triangle ABE$,

$\angle AGI = 90° = \angle ABE$ ($\angle$ in semi-circle)

$\angle IAG = \angle EAB$ (proved)

$\angle AIG = \angle AEB$ ($\angle$ sum of $\triangle$)

$\therefore \triangle AGI \sim \triangle ABE$ (AAA)

$\Rightarrow \dfrac{GI}{AG} = \dfrac{BE}{AB}$ (corr. sides, $\sim\triangle$s)

(b) (i) $\because$ $AG = DG$

$\therefore$ $AG =$ (Diameter $- CD$) $\div 2$

$= (25 \times 2 - (25 - 11)) \div 2 = 18$

$\therefore G = (-25 + 18, 0) = (-7, 0)$

(ii) By (a)(ii), $GI = \dfrac{1}{2} \times AG = 9 \Rightarrow I = (-7, 9)$

Radius of inscribed circle $= GI = 9$

$\therefore$ Eqn of circle is $(x+7)^2 + (y-9)^2 = 9^2$

$\Rightarrow x^2 + y^2 + 14x - 18y + 49 = 0$

16C.29 HKCEE MA 2008 – I – 17

(a) *Method 1*

$\because$ I is the incentre of $\triangle ABC$ (given)

$\therefore$ $\angle BAP = \angle CAP$

$\therefore$ $BP = CP$ (equal $\angle$s, equal chords)

Method 2

$\because$ I is the incentre of $\triangle ABC$ (given)

$\therefore$ $\angle BAP = \angle CAP$

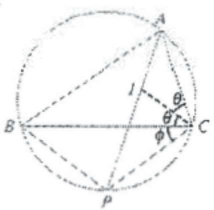
$\angle BCP = \angle BAP$ ($\angle$s in the same segment)

$= \angle CAP$ (proved)

$= \angle CBP$ ($\angle$s in the same segment)

$\Rightarrow BP = CP$ (sides opp. equal $\angle$s)

Both methods



Join CI. Let $\angle ACI = \angle BCI = \theta$ and $\angle BCP = \phi$.

$\angle PAC = \phi$ (equal chords, equal $\angle$s)

$\Rightarrow \angle PIC = \angle PAC + \angle ACI = \theta + \phi$ (ext. $\angle$ of $\triangle$)

$= \angle PCI$

$\therefore$ $IP = CP$ (sides opp. equal $\angle$s)

i.e. $BP = CP = IP$

(b) (i) Let $P = \left(\dfrac{-80+64}{2}, k\right) = (-8, k)$

$\because$ $BP = IP$

$\therefore$ $(-8+380)^2 + (k-0)^2 = (-8-0)^2 + (k-32)^2$

$5184 + k^2 = 64 + k^2 - 64k + 1024$

$k = -64 \Rightarrow P = (-8, -64)$

$\therefore$ $P = (-8, -64)$

Radius of circle $BIC = \sqrt{5184 + (-64)^2} = \sqrt{9280}$

$\therefore$ Eqn of circle: $(x+8)^2 + (y+64)^2 = 9280$

$\Rightarrow x^2 + y^2 + 16y + 128y - 5120 = 0$

(ii) *Method 1*

$\because$ $GB = GP$

$\therefore$ $(-8+80)^2 + (g-0)^2 = (g+64)^2$

$72^2 + g^2 = g^2 + 128g + 64^2$

$g = 8.5$

$\therefore$ $Q = (-8, -64 + 2GP)$

$= (-8, -64 + 2(8.5 + 64)) = (-8, 81)$

Method 2

Let the equation of circle be $x^2 + y^2 + Dx + Ey + F = 0$

$$\begin{cases} (-80)^2 + 0^2 - 80D + 0E + F = 0 \\ 64^2 + 0^2 + 64D + 0E + F = 0 \\ (-8)^2 + (-64)^2 - 8D - 64E + F = 0 \end{cases} \Rightarrow \begin{cases} D = 16 \\ E = -17 \\ F = -5120 \end{cases}$$

$\therefore$ Eqn of circle is $x^2 + y^2 + 16x - 17y - 5120 = 0$

Put $x = -8 \Rightarrow y^2 - 17y - 5184 = 0$

$\Rightarrow y = 81$ or $-64 \Rightarrow Q = (-8, 81)$

(iii) *Method 1*

$m_{BQ} \cdot m_{IQ} = \dfrac{81-0}{-8+80} \cdot \dfrac{81-32}{-8-0} = -\dfrac{441}{64} \neq -1$

$\Rightarrow \angle BQI \neq 90° \Rightarrow \angle BQI + \angle BRI \neq 180°$

$\therefore$ They are not concyclic.

Method 2

Mid-pt of $BI = \left(\dfrac{-80+0}{2}, \dfrac{0+32}{2}\right) = (-40, 16)$

$BI = \sqrt{80^2 + 32^2} = \sqrt{7424}$

$\therefore$ Eqn of circle BRI:

$(x+40)^2 + (y-16)^2 = (\sqrt{7424} \div 2)^2$

$x^2 + y^2 + 80x - 32y = 0$

Put $Q(-8, 81)$ into the equation:

LHS $= (-8)^2 + (81)^2 + 80(-8) - 32(81)$

$= 3393 \neq$ RHS

Thus, Q does not lie on the circle through B, R and I.

The 4 points are not concyclic.

16C.30 HKCEE MA 2011 – I – 16

(a) $S = (16, -48)$

$R = (-32 + 2 \times (16 + 32), -48) = (64, -48)$

Method 1

Mid-pt of $PR = \left(\dfrac{16+64}{2}, \dfrac{80-48}{2}\right) = (40, 16)$

$m_{PR} = \dfrac{-48-80}{64-16} = \dfrac{-8}{3}$

$\therefore$ Eqn of $\perp$ bisector: $y - 16 = \dfrac{-1}{\frac{-8}{3}}(x - 40)$

$\Rightarrow 3x - 8y + 8 = 0$

Method 2

$\sqrt{(x-16)^2 + (y-80)^2} = \sqrt{(x-64)^2 + (y+48)^2}$

$x^2 + y^2 - 32x - 160y + 6656 = x^2 + y^2 - 128x + 96y + 6400$

$96x - 256y + 256 = 0 \Rightarrow 3x - 8y + 8 = 0$

(b) Since $PQ = PR$ and $PS \perp QR$, PS is the $\perp$ bisector of QR. (property of isos. $\triangle$)

Thus the circumcentre of $\triangle PQR$ is the intersection of the line in (a) and PS.

Put $x = 16$ into the eqn in (a) $\Rightarrow y = 7 \Rightarrow (16, 7)$

(c) (i) Radius $= 80 - 7 = 73$

$\therefore$ Eqn of C: $(x-16)^2 + (y-7)^2 = 73^2$

$\Rightarrow x^2 + y^2 - 32x - 14y - 5024 = 0$

(ii) If the centre of C is the in-centre of $\triangle PQR$, its distances to each of PR, QR and PQ would also be the same (the radii of the inscribed circle).

From (a), the foot of $\perp$ from centre to $PR = (40, 16)$

$\Rightarrow$ Dist from centre to $PR = \sqrt{(16-40)^2 + (7-16)^2} = \sqrt{657}$

Dist from centre to $QR = 7 - (-48) = 56 \neq \sqrt{657}$

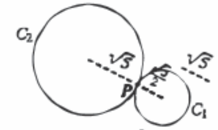
Therefore, the centre of C cannot be the in-centre of $\triangle PQR$. The claim is disagreed.

16C.31 HKCEE AM 1981 – II – 6

(a) C_1: Centre $= \left(0, -\dfrac{7}{2}\right)$, Radius $= \sqrt{\left(\dfrac{7}{2}\right)^2 - 11} = \dfrac{\sqrt{5}}{2}$

C_2: Centre $= (-3, -2)$, Radius $= \sqrt{3^2 + 2^2 - 8} = \sqrt{5}$

$\therefore P = \left(\dfrac{2(0) + 1(-3)}{1+2}, \dfrac{2(-\frac{7}{2}) + 1(-2)}{1+2}\right) = (-1, -3)$



(b) Slope of line joining centres $= \dfrac{-\frac{7}{2} + 2}{0 + 3} = \dfrac{-1}{2}$

$\therefore$ Eqn of tg: $y + 3 = \dfrac{-1}{\frac{-1}{2}}(x + 1) \Rightarrow 2x - y - 1 = 0$

16C.32 (HKCEE AM 1981 – II – 12)

(a) (i) $\begin{cases} L: y = mx + 2 \\ C: x^2 + y^2 = 1 \end{cases} \Rightarrow x^2 + (mx+2)^2 = 1$

$\Rightarrow (1+m^2)x^2 + 4mx + 3 = 0$

$\therefore$ x_1 and x_2 are the roots of this equation.

(ii) $x_1 + x_2 = \dfrac{-4m}{1+m^2}$, $x_1 x_2 = \dfrac{3}{1+m^2}$

$\Rightarrow AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(x_1 - x_2)^2 + (mx_1 + 2 - mx_2 - 2)^2}$

$= \sqrt{(x_1 - x_2)^2 + m^2(x_1 - x_2)^2}$

$= \sqrt{(1+m^2)[(x_1 + x_2)^2 - 4x_1x_2]}$

$= \sqrt{(1+m^2)\left[\dfrac{16m^2}{(1+m^2)^2} - \dfrac{12}{1+m^2}\right]}$

$= \sqrt{\dfrac{16m^2 - 12(1+m^2)}{1+m^2}} = 2\sqrt{\dfrac{m^2 - 3}{m^2 + 1}}$

(b) (i) 2 distinct pts $\Rightarrow 2\sqrt{\dfrac{m^2 - 3}{m^2 + 1}} > 0 \Rightarrow m^2 - 3 > 0$

$\Rightarrow m < -\sqrt{3}$ or $m > \sqrt{3}$

(ii) Tg to $C \Rightarrow 2\sqrt{\dfrac{m^2 - 3}{m^2 + 1}} = 0 \Rightarrow m = \pm\sqrt{3}$

(iii) No intsn $\Rightarrow \dfrac{m^2 - 3}{m^2 + 1} < 0 \Rightarrow -\sqrt{3} < m < \sqrt{3}$

(c) For $m = \pm\sqrt{3}$, the eqn in (a)(i) becomes

$10x^2 \pm 4\sqrt{3}x + 3 = 0 \Rightarrow x = \dfrac{\mp 4\sqrt{3} \pm \sqrt{0}}{20} = \mp\dfrac{\sqrt{3}}{5}$

$\Rightarrow y = \pm\sqrt{3}\left(\mp\dfrac{\sqrt{3}}{5}\right) + 2 = \dfrac{8}{5}$

$\therefore$ Eqn of PQ is $y = \dfrac{8}{5}$ (since it is horizontal)

16C.33 (HKCEE AM 1982 – II – 8)

(a) (i) $m_L = \dfrac{-5}{12}$

$\therefore$ Req eqn: $y-6=\dfrac{-1}{\frac{-5}{12}}(x-5) \Rightarrow y=\dfrac{12}{5}x-6$

(ii) *'Hence'*

$\begin{cases}5x+12y=32\\ y=\frac{12}{5}x-6\end{cases} \Rightarrow (x,y)=\left(\dfrac{40}{13},\dfrac{18}{13}\right)$

Radius of circle $=\sqrt{\left(5-\dfrac{40}{13}\right)^2+\left(6-\dfrac{18}{13}\right)^2}=5$

. Eqn of C: $(x-5)^2+(y-6)^2=5^2$

$\Rightarrow x^2+y^2-10x-12y+36=0$

'Otherwise'

Let C be $(x-5)^2+(y-6)^2=r^2$.

$\begin{cases}5x+12y=32\\ (x-5)^2+(y-6)^2=r^2\end{cases}$

$\Rightarrow (x-5)^2+\left(\dfrac{32-5x}{12}-6\right)^2=r^2$

$\dfrac{169}{144}x^2-\dfrac{65}{9}x+\dfrac{325}{9}-r^2=0$

$\Delta=\left(\dfrac{65}{9}\right)^2-4\cdot\dfrac{169}{144}\left(\dfrac{325}{9}-r^2\right)=0 \Rightarrow r^2=25$

$\therefore$ Eqn of C: $(x-5)^2+(y-6)^2=5^2$

$\Rightarrow x^2+y^2-10x-12y+36=0$

(b) *Method 1*

x-coordinate of centre $=5=$ radius

$\therefore$ C touches the y-axis.

Method 2

Put $x=0 \Rightarrow y^2-12y+36=0 \Rightarrow y=6$ (repeated)

$\therefore$ y-axis is tangent to C.

(c) Let the tangent be $y=mx$.

$\begin{cases}y=mx\\ x^2+y^2-10x-12y+36=0\end{cases}$

$\Rightarrow (1+m^2)x^2-2(5+6m)x+36=0$

$\Delta=4(5+6m)^2-4\cdot 36(1+m^2)=0 \Rightarrow m=\dfrac{5}{12}$

$\therefore$ The required tangent is $y=\dfrac{5}{12}x$.

(d) Let $Q=(m,n)$ Since M is the mid-pt of PQ,

$\left(\dfrac{2+m}{2},\dfrac{2+n}{2}\right)=(5,6) \Rightarrow (m,n)=(8,10)$

Let $x^2+y^2+Dx+Ey+F=0$ be the circle through P, Q and O.

$\begin{cases}0^2+0^2+0D+0E+F=0\\ 2^2+2^2+2D+2E+F=0\\ 8^2+10^2+8D+10E+F=0\end{cases} \Rightarrow \begin{cases}D=62\\ E=-66\\ F=0\end{cases}$

$\therefore$ The circle is $x^2+y^2+62x-66y=0$.

16C.34 HKCEE AM 1984 – II – 6

(a) $x^2+y^2-2kx+4ky+6k^2-2=0$

Radius $=\sqrt{(-k)^2+(2k)^2-(6k^2-2)}>1$

$k^2+4k^2-6k^2+2>1^2$

$k^2<1$

$-1<k<1$

16C.35 (HKCEE AM 1985 – II – 5)

(a) Radius $=\sqrt{\left(\dfrac{k}{2}\right)^2+\left(\dfrac{2+k}{2}\right)^2}=\sqrt{5}$

$\dfrac{k^2}{4}+1+k+\dfrac{k^2}{4}=5$

$k^2+2k-8=0 \Rightarrow k=-4$ or 2

(b) $k=-4 \Rightarrow x^2+y^2-4x+2y=0$

$k=2 \Rightarrow x^2+y^2+2x-4y=0$

16C.36 HKCEE AM 1986 – II – 10

(a) (i) $\begin{cases}C_1: x^2+y^2-4x+2y+1=0\\ C_2: x^2+y^2-10x-4y+19=0\end{cases}$

$\Rightarrow 6x+6y\quad 18=0 \Rightarrow y=3-x$

$\Rightarrow x^2+(3-x)^2-4x+2(3-x)+1=0$

$2x^2-12x+16=0$

$x=2$ or 4

$y=1$ or -1

Hence, A and B are $(2,1)$ and $(4,-1)$.

$\therefore$ Eqn of AB: $\dfrac{y-1}{x-2}=\dfrac{-1-1}{4-2}=\dfrac{-1}{2}$

$\Rightarrow x+2y\quad 4=0$

(ii) The required circle has AB as a diameter.

Mid-pt of $AB=\left(\dfrac{2+4}{2},\dfrac{1-1}{2}\right)=(3,0)$

$AB=\sqrt{(4-2)^2+(-1-1)^2}=\sqrt{8}$

$\therefore$ Req. circle is: $(x-3)^2+(y-0)^2=\left(\dfrac{\sqrt{8}}{2}\right)^2$

$\Rightarrow x^2+y^2-6x+7=0$

(b) Centre of $C_3=$ Centre of $C_1=(2,-1)$

Radius of $C_3=$ Dist. from $(2,-1)$ to AB

$=\sqrt{(\text{Radius of } C_1)^2-(\frac{1}{2}AB)^2}$

$=\sqrt{(2)^2+(1)^2-1-2}=\sqrt{2}$

$\therefore$ Eqn of C_3: $(x-2)^2+(y+1)^2=2$

$\Rightarrow x^2+y^2-4x+2y+3=0$

16C.37 HKCEE AM 1987 – II – 11

(a) (i) *Method 1*

$C_1: (x-8)^2+(y\quad 2)^2=2^2$

$\Rightarrow$ Radius $=2=y$-coordinate of centre

$\therefore$ C_1 touches the x-axis, and the point of contact is (x-coordinate of centre, 0) $=(8,0)=A$.

Method 2

Put $y=0 \Rightarrow x^2-16x+64=0 \Rightarrow x=8$ (repeated)

$\therefore$ $A(8,0)$ is the only pt of contact of C_1 and x-axis.

(ii) Let OH be $y=mx$.

$\begin{cases}y=mx\\ x^2+y^2-16x-4y+64=0\end{cases}$

$\Rightarrow x^2+(mx)^2-16x-4(mx)+64=0$

$(1+m^2)x^2-4(4+m)x+64=0$

$\Delta=16(4+m)^2-4\cdot 64(1+m^2)=0$

$m^2+8m+16-16-16m^2=0$

$15m^2-8m=0$

$m=0$ or $\dfrac{8}{15}$

.. Eqn of OH is $y=\dfrac{8}{15}x$.

(iii) By symmetry, $m_{BH}=\dfrac{-8}{15}$

$\therefore$ Eqn of BH: $y-0=\dfrac{-8}{15}(x-16)$

$\Rightarrow y=\dfrac{-8}{15}+\dfrac{128}{15}$

(b) (i) Sub $A \Rightarrow 8^2+0^2-16(8)+0+c=0 \Rightarrow c=64$

Method 1

$\begin{cases}4x+3y=0\\ x^2+y^2-16x+2fy+64=0\end{cases}$

$x^2+\left(\dfrac{-4}{3}x\right)^2-16x+2f\left(\dfrac{-4}{3}x\right)+64=0$

$\dfrac{25}{9}x^2-8\left(2+\dfrac{f}{3}\right)x+64=0$

$\Delta=64\left(2+\dfrac{f}{3}\right)^2-4\ \dfrac{25}{9}\cdot 64=0$

$\left(2+\dfrac{f}{3}\right)^2=\dfrac{100}{9}$

$2+\dfrac{f}{3}=\pm\dfrac{10}{3}$

$f=4$ or -16

Since the centre is in Quad IV, $f>0$. $\therefore$ $f=4$

Method 2

Suppose the point of contact of OK and C_2 is P. Then $OP=OA=8$. Let $P=\left(p,\dfrac{-4}{3}p\right)$.

$\sqrt{(p)^2+\left(\dfrac{-4}{3}p\right)^2}=8$

$\dfrac{25}{9}p^2=64 \Rightarrow p=\pm\dfrac{24}{5}$

As P is in Quad IV, $p=\dfrac{24}{5} \Rightarrow P=\left(\dfrac{24}{5},\dfrac{-32}{5}\right)$

Put into C_2:

$\left(\frac{24}{5}\right)^2+\left(\frac{-32}{5}\right)^2-16\left(\frac{24}{5}\right)+2f\left(-\frac{32}{5}\right)+64=0$

$\frac{256}{5}-\frac{64}{5}f=0$

$f=4$

(ii) Put $x=8$ into OH and OK respectively.

$OH \Rightarrow y=\dfrac{8}{15}(8)=\dfrac{64}{15} \Rightarrow H=\left(8,\dfrac{64}{15}\right)$

$OK \Rightarrow y=\dfrac{-4}{3}(8)=\dfrac{-32}{3} \Rightarrow K=\left(8,\dfrac{-32}{3}\right)$

$\therefore$ $\dfrac{\text{Area of }\Delta OBH}{\text{Area of }\Delta OBK}\quad\dfrac{y\text{-coor of }H}{-(y\text{-coor of }K)}=\dfrac{\frac{64}{15}}{\frac{32}{3}}=\dfrac{2}{5}$

16C.38 (HKCEE AM 1988 – II – 11)

(a) *Method 1*

Let $S=(h,k)$.

$\because$ $KS\perp(x-5y+59=0)$

$\therefore$ $\dfrac{k-12}{h-1}=m_{KS}=\dfrac{-1}{\frac{1}{5}}=-5 \Rightarrow k=-5h+17$

$\because$ $SK=SH$

$\therefore$ $(h-1)^2+(k-12)^2=(h+3)^2+(k-6)^2$

$-2h-24k+145=6h-12k+45 \Rightarrow 2h+3k=25$

Solving, $h=2$, $k=7 \Rightarrow S=(2,7)$

Method 2

Eqn of KS: $y-12=\dfrac{-1}{\frac{1}{5}}(x-1) \Rightarrow y=-5x+17$

Eqn of $\perp$ bisector of HK:

$(x-1)^2+(y-12)^2=(x+3)^2+(y-6)^2$

$\Rightarrow 2x+3y=25$

Solving, $(x,y)=(2,7) \Rightarrow S=(2,7)$

(Note how different concepts gave similar calculations.)

Hence,

Radius of $C=\sqrt{(1-2)^2+(12-7)^2}=\sqrt{26}$

$\Rightarrow$ Eqn of C: $(x-2)^2+(y-7)^2=26$

$\Rightarrow x^2+y^2-4x-14y+27=0$

(b) $\begin{cases}L: 3x-2y-5=0\\ C: x^2+y^2-4x-14y+27=0\end{cases}$

$\Rightarrow x^2+\left(\dfrac{3x-5}{2}\right)^2-4x-14\left(\dfrac{3x-5}{2}\right)+27=0$

$\dfrac{13}{4}x^2-\dfrac{65}{2}x+\dfrac{273}{4}=0$

$x=3$ or 7

$\Rightarrow y=2$ or 8

$\therefore$ A and B are $(3,2)$ and $(7,8)$.

$\Rightarrow$ Centre of circle $=\left(\dfrac{7+3}{2},\dfrac{8+2}{2}\right)=(5,5)$

Radius $\dfrac{1}{2}\sqrt{(7-3)^2+(8-2)^2}=\dfrac{1}{2}\sqrt{52}=\sqrt{13}$

$\therefore$ Eqn of circle: $(x-5)^2+(y-5)^2=13$

$\Rightarrow x^2+y^2-10x-10y+37=0$

16C.39 HKCEE AM 1993 – II – 11

(a) $AB=\sqrt{(3-0)^2+\left(\dfrac{3}{4}-2\right)^2}=\dfrac{13}{4}$

Radius of $C_2=y$-coordinate of $B=\dfrac{3}{4}$

$\because$ Radius of C_1- Radius of $C_2=4-\dfrac{3}{4}=\dfrac{13}{4}=AB$

$\therefore$ C_1 and C_2 touch internally.

(b) $AP=4-$ Radius of circle

$s^2+(t-2)^2=(4-t)^2$

$s^2+t^2-4t+4=16-8t+t^2 \Rightarrow 4t=12-s^2$

(c) $BP=\dfrac{13}{4}+$ Radius of circle

$(s-3)^2+\left(t-\dfrac{3}{4}\right)^2=\left(\dfrac{3}{4}+t\right)^2$

$(s-3)^2=\left(t+\dfrac{3}{4}\right)^2-\left(t-\dfrac{3}{4}\right)^2=3t$

(d) $\begin{cases}4t=12-s^2\\ 3t=(s-3)^2\end{cases}$

$\Rightarrow 3(12-s^2)=4(s-3)^2$

$36-3s^2=4s^2-24s+36$

$7s^2-24s=0$

$s=0$ or $\dfrac{24}{7} \Rightarrow t=3$ or $\dfrac{3}{49}$

$\therefore$ The required circles are $(x-0)^2+(y-3)^2=3^2$ and $\left(x-\dfrac{24}{7}\right)^2+\left(y-\dfrac{3}{49}\right)^2=\left(\dfrac{3}{49}\right)^2$.

16C.40 HKCEE AM 1994 – II – 9

(a) $(h-5)^2+(k-5)^2=(h-7)^2+(k-1)^2$

$-10h+25-10k+25=-14h+49-2k+1$

$4h=8k \Rightarrow h=2k$

Hence, the equation of C is

$(x-h)^2+(y-k)^2=(h-5)^2+(k-4)^2$

$x^2+y^2-2hx-2ky=-10h+25-10k+25$

$x^2+y^2-2(2k)x-2ky+10(2k)+10k-50=0$

$x^2+y^2-4kx-2ky+30k-50=0$

(b) Denote the centre of C by G.

$m_{BG}=\dfrac{-1}{\frac{1}{2}}=-2$

$\dfrac{k-1}{h-7}=-2 \Rightarrow k-1=-2(2k\quad 7) \Rightarrow k=3$

$\therefore$ Eqn of C is $x^2+y^2-4(3)x-2(3)y+30(3)-50=0$

$\Rightarrow x^2+y^2-12x\quad 6y+40=0$

16C.41 <u>HKCEE AM 1995 – II 10</u>

(a) $C_1:(x-8)^2+(y-0)^2=10^2$
$\therefore$ Centre $=(8,0)$, Radius $=10$
$\therefore$ Radius of $C_2=$ (Dist. btwn centres of C_1 and C_2) -10
$=15-10=5$

(b) $\sqrt{(h-8)^2+(k-0)^2}-10=\sqrt{(h+7)^2+(k-0)^2}-5$
$h^2+14h+49+k^2=\left(\sqrt{h^2-16h+64+k^2}-5\right)^2$
$30h-40=10\sqrt{h^2-16h+64+k^2}$
$(3h-4)^2=h^2-16h+64+k^2$
$9h^2-24h+16=h^2-16h+64+k^2$
$8h^2-k^2-8h-48=0$

(c) (i) $y=\dfrac{40+0}{2}=20$
(The centre lies on the ⊥ bisector of the segment joining the two centres. This is true because the radii of C_2 and C_3 are the same.)

(ii) From (c)(i), $k=20$
Put into the result of (b):
$8h^2-(20)^2-8h-48=0$
$h^2-h-56=0 \Rightarrow h=8$ (rej.) or -7
Centre $=(-7,20)$, Radius $=20-5-15$
$\therefore$ Eqn of req. circle: $(x+7)^2+(y-20)^2=15^2$
$\Rightarrow x^2+y^2+14x-40y+224=0$

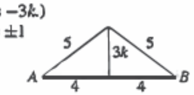
16C.42 <u>(HKCEE AM 1996 – II 10)</u>

(a) (i) Centre $=(4k,3k)$
Put into the line: LHS $=3(4k)-4(3k)=0=$ RHS
$\therefore$ The centre lies on $3x-4y=0$.

(ii) Radius $=\sqrt{(4k)^2+(3k)^2-25(k^2-1)}=\sqrt{25}=5$

(b) Slope $=\dfrac{3}{4}$
Pick a value of k for C_k, e.g. $C_0: x^2+y^2-25=0$.
Let the equation of tangent be $y=\dfrac{3}{4}x+b$.
$\begin{cases} y=\frac{3}{4}x+b \\ x^2+y^2-25=0 \end{cases} \Rightarrow x^2+\left(\dfrac{3}{4}+b\right)^2-25=0$
$\dfrac{25}{16}x^2+\dfrac{3}{2}bx+b^2-25=0$
$\Delta=\left(\dfrac{3}{2}b\right)^2-4\cdot\dfrac{25}{16}(b^2-25)=0 \Rightarrow b=\pm\dfrac{25}{4}$
$\therefore$ The tangents are $y=\dfrac{3}{4}x\pm\dfrac{25}{4}$.

(c) Distance $=y$-coordinate of centre $=3k$
(If k is negative, the distance is $-3k$.)
$\therefore 5^2=(3k)^2+(4)^2 \Rightarrow k=\pm1$



16C.43 <u>(HKCEE AM 1998 – II 2)</u>

$\begin{cases} L: x-7y+3=0 \\ C:(x-2)^2+(y+5)^2=a \end{cases}$
$\Rightarrow (7y-3-2)^2+(y+5)^2=a \Rightarrow 50y^2-60y+50-a=0$
$\therefore \Delta=3600-4\cdot50(50-a)=0 \Rightarrow 18-(50-a)=0$
$\Rightarrow a=32$

16C.44 <u>(HKCEE AM 2000 – II 9)</u>

(a) $(x+2k+2)^2+\left(y+\dfrac{3k+1}{2}\right)^2=(8k+8)+(2k+2)^2+\left(\dfrac{3k+1}{2}\right)^2$
$(x+2k+2)^2+\left(y+\dfrac{3k+1}{2}\right)^2=\dfrac{25}{4}k^2+\dfrac{35}{2}k+\dfrac{49}{4}$
$(x+2k+2)^2+\left(y+\dfrac{3k+1}{2}\right)^2=\left(\dfrac{5k+7}{2}\right)^2$

(b) (i) $\because$ Touches x-axis
$\therefore \dfrac{3k+1}{2}=\pm\left(\dfrac{5k+7}{2}\right) \Rightarrow k=-3$ or -1
$\therefore$ The circles are $x^2+(y-1)^2=1$ (C_1) and $(x-4)^2+(y-4)^2=16$ (C_2)

(ii) Dist. between centres $=\sqrt{(4-0)^2+(4-1)^2}$
$=5=1+4$
$\therefore$ Touch externally

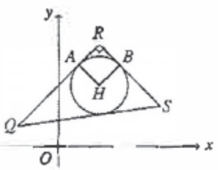
(c) Let the centre of C_3 be (a,b).
$\because$ Collinear with centres of C_1 and C_2
$\therefore \dfrac{b-1}{a-0}=\dfrac{4-1}{4-0}=\dfrac{3}{4} \Rightarrow b=\dfrac{3}{4}a+1$
$\because$ Touches x-axis
$\therefore$ Radius $=b$
$\because$ Touches C_2 externally
$\therefore \sqrt{(a-4)^2+(b-4)^2}=4+b$
$a^2-8a+16+b^2-8b+16=(4+b)^2$
$a^2-8a+16-8b=+8b$
$a^2-8a+16=16b$
$=16\left(\dfrac{3}{4}a+1\right)$
$a^2-20a=0$
$\Rightarrow a=0$ or $20 \Rightarrow b=1$ or 16
$\because (0,1)$ is the centre of C_1
$\therefore C_3$ is $(x-20)^2+(y-16)^2=16^2$

16C.45 <u>HKCEE AM 2002 – 15</u>

(a) Suppose the centre is G. Then
$A=$ Area of $\triangle GDE+$ Area of $\triangle GEF+$ Area of $\triangle GFD$
$=\dfrac{1}{2}DE\cdot r+\dfrac{1}{2}EF\cdot r+\dfrac{1}{2}FD\cdot r$
$=\dfrac{1}{2}(DE+EF+FD)r=\dfrac{1}{2}pr$

(b) (i) Perimeter of $\triangle QRS$
$=\sqrt{4^2+4^2}+\sqrt{3^2+3^2}+\sqrt{7^2+1^2}$
$=4\sqrt{2}+3\sqrt{2}+5\sqrt{2}=12\sqrt{2}$
$\therefore$ Radius of $C_2=\dfrac{\frac{1}{2}\cdot4\sqrt{2}\cdot3\sqrt{2}}{\frac{1}{2}\cdot12\sqrt{2}}=\sqrt{2}$

(ii) Denote the points where C_2 touches QR and RS by A and B respectively. Also let H be the centre of C_2. Then $RAHB$ is a square.



i.e. $RA=AH=HB=BR=\sqrt{2}$
$RH=\sqrt{(\sqrt{2})^2+(\sqrt{2})^2}=2$
$\because m_{RA}=\dfrac{5-1}{2+2}=1$ and $m_{RS}=\dfrac{5-2}{2-5}=-1$
$\therefore RH$ is vertical.
Thus, $H=(2,5-2)=(2,3)$.
$\therefore$ Eqn of C_2 is $(x-2)^2+(y-3)^2=2$

16C.46 <u>HKCEE AM 2005 – 15</u>

(a) $\begin{cases} L: y=kx \\ C: x^2+y^2-4x-2y+4=0 \end{cases}$
$\Rightarrow x^2+(kx)^2-4x-2(kx)+4=0$
$(1+k^2)x^2-2(2+k)x+4=0 \ldots (*)$
$\Delta=4(2+k)^2-4(4)(1+k^2)>0$
$k^2+4k+4-4-4k^2>0$
$3k^2-4k<0 \Rightarrow 0<k<\dfrac{4}{3}$

(b) From (a), equation of the tangent is $y=\dfrac{4}{3}x$.

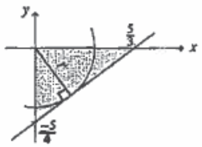
(c) (i) The x-coordinates of P and Q are the roots of $(*)$.
$\Rightarrow$ Sum of roots $=\dfrac{2(2+k)}{1+k^2}$
$\therefore$ x-coordinate of $M=\dfrac{\text{Sum of roots}}{2}=\dfrac{2+k}{1+k^2}$

16C.47 <u>HKCEE AM 2006 – 14</u>

(a) (i) $\begin{cases} L: y=mx+c \\ J: x^2+y^2=r^2 \end{cases}$
$\Rightarrow x^2+(mx+c)^2=r^2$
$(1+m^2)x^2+2mcx+c^2-r^2=0$
$\Delta=4m^2c^2-4(1+m^2)(c^2-r^2)=0$
$m^2c^2-c^2-m^2c^2+r^2+r^2m^2=0$
$c^2=r^2(m^2+1)$

(ii) Put (h,k) into L: $k=mh+c$
$\therefore (k-mh)^2=c^2=r^2(m^2+1)$

(b) (i) PR: $\dfrac{y-4}{x-7}=\dfrac{-5-4}{-5-7}=\dfrac{3}{4} \Rightarrow 3x-4y-5=0$
$\Rightarrow$ x-intercept $=\dfrac{5}{3}$
y-intercept $=\dfrac{-5}{4}$



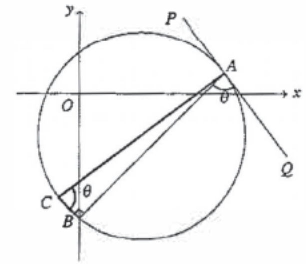
In the shaded triangle,
$\dfrac{1}{2}r\sqrt{\left(\dfrac{5}{3}\right)^2+\left(\dfrac{5}{4}\right)^2}=\dfrac{1}{2}\cdot\dfrac{5}{3}\cdot\dfrac{5}{4}=$ Area
$\Rightarrow r=\dfrac{25}{12}\div\dfrac{25}{12}=1$

(ii) Use (a)(ii) with $(h,k)=(7,4)$ and $r=1$.
$(4-7m)^2=m^2+1$
$48m^2-56m+15=0 \Rightarrow m=\dfrac{3}{4}$ or $\dfrac{5}{12}$
$\therefore m_{PQ}=\dfrac{5}{12}$

(iii) Use (a)(ii) with $(h,k)=R=(-5,-5)$ and $r=1$.
$(-5+5m)^2=m^2+1$
$24m^2-50m+24=0 \Rightarrow m=\dfrac{3}{4}$ or $\dfrac{4}{3}$
$\therefore m_{QR}=\dfrac{4}{3}$
Let $Q=(a,b)$. Then
$\begin{cases} \dfrac{b-4}{a-7}=\dfrac{5}{12} \Rightarrow 5a-12b=-13 \\ \dfrac{b+5}{a+5}=\dfrac{4}{3} \Rightarrow 4a-3b=-5 \end{cases}$
$\Rightarrow Q=(a,b)=\left(\dfrac{-7}{11},\dfrac{9}{11}\right)$

16C.48 <u>HKCEE AM 2010 – 7</u>

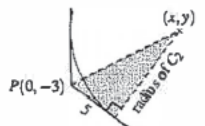
Centre $=(3,-2)$, Radius $=5$
Let $C(m,n)$ be the diametrically opposite pt of A on the circle.
Then $\left(\dfrac{m+7}{2},\dfrac{n+1}{2}\right)=(3,-2) \Rightarrow C=(m,n)=(-1,-5)$
$\because \angle ACB=\theta$ (∠ in alt. segment)
and $\angle ABC=90°$ (∠ in semi-circle)
$\therefore \tan\theta=\dfrac{AB}{BC}=\dfrac{\sqrt{(7-0)^2+(1+6)^2}}{\sqrt{(0+1)^2+(-6+5)^2}}=7$



16C.49 <u>HKCEE AM 2010 – 15</u>

(a) Let the centre of C_2 be (x,y).
Dist. between centres $=$ Radius of C_2- Radius of C_1
$(x-6)^2+(y-5)^2=(x-5)^2$
$-12x+36+y^2-10y=-10x$
$y^2-10y+36=2x \Rightarrow x=\dfrac{1}{2}y^2-5y+18$

(b) (i) By Pyth. thm, $(x-0)^2+(y+3)^2=5^2+x^2$
$(y+3)^2=5^2$
$y=2$ or -8 (rej)
$\Rightarrow x=\frac{1}{2}(2)^2-5(2)+18$
$=10$
$\therefore$ Centre of $C_2=(10,2)$



(ii) Eqn of C_2: $(x-10)^2+(y-2)^2=10^2$
Let the eqns of tangents be $y=mx-3$.
$\begin{cases} y=mx-3 \\ (x-10)^2+(y-2)^2=100 \end{cases}$
$\Rightarrow (x-10)^2+(mx-5)^2=100$
$(1+m^2)x^2-10(m+2)x+25=0$
$\Delta=100(m+2)^2-100(1+m^2)=0$
$m^2+4m+m-1-m^2=0 \Rightarrow m=\dfrac{-3}{4}$
$\therefore$ Eqns of tgs are $y=\dfrac{-3}{4}x-3$ and $x=0$ (y-axis).

16C.50 HKDSE MA SP – I – 19

(a) (i) Join B and C.

$\angle DAE = \angle DBC$ ($\angle$s in the same segment)
$= \angle PCB$ (alt. $\angle$s, $PQ//BD$)
$= \angle BAE$ ($\angle$ in alt. segment)

In $\triangle ABE$ and $\triangle ADE$,
$AB = AD$ (given)
$\angle BAE = \angle DAE$ (proved)
$AE = AE$ (common)
$\therefore \triangle ABE \cong \triangle ADE$ (SAS)

(ii) $\because \angle BAE = \angle DAE$ (corr. $\angle$s, $\cong \triangle$s)
$\therefore AE$ is an $\angle$ bisector of $\triangle ABD$.
Hence, $AE \perp BD$ (property of isos. $\triangle$)
$\Rightarrow AE$ is an altitude of $\triangle ABD$.
$BE = DE$ (property of isos. $\triangle$)
$\Rightarrow AE$ is a median of $\triangle ABD$.
$\Rightarrow AE$ is a $\perp$ bisector of $\triangle ABD$.
Thus, the in centre, orthocentre, centroid and circumcentre of $\triangle ABE$ all lie on AE. They are collinear.

(b) $m_{PQ} = m_{BD} = \dfrac{12-4}{8-4} = 2$
From (a)(ii), AC is a diameter of the circle.

Method 1
Let the circle be $x^2+y^2+Dx+Ey+F=0$.

$$\begin{cases}14^2+4^2+14D+4E+F=0\\8^2+12^2+8D+12E+F=0\\4^2+4^2+4D+4E+F=0\end{cases} \Rightarrow \begin{cases}D=-18\\E=-13\\F=92\end{cases}$$

$\therefore$ The circle is $x^2+y^2-18x-13y+92=0$.
$\Rightarrow$ Centre $=(9,6.5)$

Method 2
Eqn of $\perp$ bisector of BD (i.e. AC):

$$\sqrt{(x-8)^2+(y-12)^2} = \sqrt{(x-4)^2+(y-4)^2}$$
$$-16x+64-24y+144 = -8x+16-8y+16$$
$$x+2y-22=0$$

Eqn of $\perp$ bisector of AD: $x = \dfrac{14+4}{2} = 9$
($\because$ AD is parallel to the x-axis.)

Solving $\begin{cases}x+2y-22=0\\x=9\end{cases} \Rightarrow$ Circumcentre $=(9,6.5)$

Method 3
Let the centre be $\left(\dfrac{14+4}{2},k\right)=(9,k)$.

Radius $= \sqrt{(9-8)^2+(k-12)^2} = \sqrt{(9-4)^2+(k-4)^2}$
$$k^2-24k+145 = k^2-8k+41$$
$$k=6.5$$

$\therefore$ Centre $=(9,6.5)$

Hence,
Let $C=(m,n)$. Then
$\left(\dfrac{m+14}{2},\dfrac{n+4}{2}\right)=(9,6.5) \Rightarrow C=(m,n)=(4,9)$
$\therefore$ Eqn of PQ: $y-9=2(x-4) \Rightarrow 2x-y+1=0$

16C.51 HKDSE MA PP – I – 14

(a) $\triangle BCD \sim \triangle OAD$

(b) (i) $AD = \sqrt{6^2+12^2} = \sqrt{180}$

$\dfrac{CD}{AD} = \sqrt{\dfrac{16}{45}} \Rightarrow CD = \sqrt{\dfrac{16}{45}\times 180} = 8$

$\therefore C=(0,12-8)=(0,4)$

(ii) AC is a diameter of the circle.

Mid-pt of $AC = \left(\dfrac{6+0}{2},\dfrac{0+4}{2}\right)=(3,2)$

$AC = \sqrt{6^2+4^2} = \sqrt{52}$

$\therefore$ Eqn of circle $OABC$: $(x-3)^2+(y-2)^2 = \left(\dfrac{\sqrt{52}}{2}\right)^2$
$\Rightarrow x^2+y^2-6x-4y=0$

16C.52 HKDSE MA 2012 – I – 17

(a) Radius $= y$-coordinate of centre $=10$
$\therefore$ Eqn of C: $(x-6)^2+(y-10)^2=100$

(b) Eqn of L: $y=-x+k$

$$\begin{cases}y=-x+k\\(x-6)^2+(y-10)^2=100\end{cases}$$
$\Rightarrow (x-6)^2+(-x+k-10)^2=100$
$2x^2+(8-2k)x+(k^2-20k+36)=0$

Sum of roots $= \dfrac{8-2k}{2} = k-4$

$\Rightarrow$ x-coordinate of mid-pt of $AB = \dfrac{k-4}{2}$

y-coordinate of mid-pt of $AB = -\left(\dfrac{k-4}{2}\right)+k$

$\therefore$ Mid-point of $AB = \left(\dfrac{k-4}{2},\dfrac{k+4}{2}\right)$

16C.53 HKDSE MA 2015 – I – 14

(a) (i) *Method 1*

Mid-pt of $PQ = \left(\dfrac{4-14}{2},\dfrac{-1+23}{2}\right)=(-5,11)$

$m_{PQ} = \dfrac{-1-23}{4+14} = \dfrac{-4}{3}$

$\therefore$ Eqn of L: $y-11 = \dfrac{-1}{\frac{-4}{3}}(x+5) \Rightarrow y=\dfrac{3}{4}x+\dfrac{59}{4}$

Method 2

$$\sqrt{(x-4)^2+(y+1)^2} = \sqrt{(x+14)^2+(y-23)^2}$$
$$-8x+16+2y+1 = 28x+196-46y+529$$
$$3x-4y+59=0$$

(ii) Centre $=\left(h,\dfrac{3h+59}{4}\right)$

Radius $= \sqrt{(4-h)^2+\left(-1-\dfrac{3h+59}{4}\right)^2}$

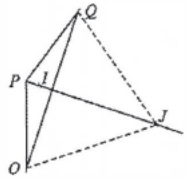
Eqn of C:
$$(x-h)^2+\left(y-\tfrac{3h+59}{4}\right)^2 = (4-h)^2+\left(-1-\tfrac{3h+59}{4}\right)^2$$
$$x^2-2hx+y^2-\tfrac{3h+59}{2}y = 16-8h+1+\tfrac{3h+59}{2}$$
$$x^2+y^2-2hx-\tfrac{3h+59}{2}y+\tfrac{13h-93}{2}=0$$
$$2x^2+2y^2-4hx-(3h+59)y+13h-93=0$$

(b) If C passes through R,
$$2(26)^2+2(43)^2-4h(26)-(3h+59)(43)+13h-93=0$$
$$2420-220h=0$$
$$h=11$$

$\therefore$ Diameter $= 2\sqrt{(4-11)^2+\left(-1-\tfrac{3(11)+59}{4}\right)^2} = 50$

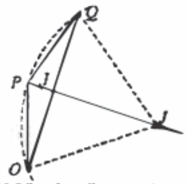
16C.54 HKDSE MA 2016 – I – 20

(a) *Method 1*



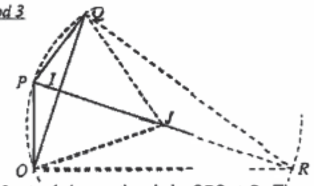
Let $\angle OPJ = \angle QPJ = \theta$. (in-centre)
$OJ=PJ=QJ$ (radii)
In $\triangle POJ$, $\angle POJ = \angle OPJ = \theta$ (base $\angle$s, isos. $\triangle$)
In $\triangle PQJ$, $\angle PQJ = \angle QPJ = \theta$ (base $\angle$s, isos. $\triangle$)
In $\triangle POJ$ and $\triangle PQJ$,
$\angle OPJ = \angle QPJ = \theta$ (in-centre)
$\angle POJ = \angle PQJ = \theta$ (proved)
$PJ = PJ$ (common)
$\therefore \triangle POJ \cong \triangle PQJ$ (AAS)
$\therefore PO = PQ$ (corr. sides, $\cong \triangle$s)

Method 2



Let $\angle OPJ = \angle QPJ = \theta$. (in-centre)
$OJ=PJ=QJ$ (radii)
In $\triangle POJ$, $\angle POJ = \angle OPJ = \theta$ (base $\angle$s, isos. $\triangle$)
$\Rightarrow \angle PJO = 180° - 2\theta$ ($\angle$ sum of $\triangle$)
$\Rightarrow \angle PQO = (180°-2\theta)\div 2 = 90°-\theta$
($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
In $\triangle PQJ$, $\angle PQJ = \angle QPJ = \theta$ (base $\angle$s, isos. $\triangle$)
$\Rightarrow \angle PJQ = 180° - 2\theta$ ($\angle$ sum of $\triangle$)
$\Rightarrow \angle POQ = (180°-2\theta)\div 2 = 90°-\theta$
($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
$\because \angle PQO = \angle POQ = 90°-\theta$ (proved)
$\therefore PO = PQ$ (sides opp. equal $\angle$s)

Method 3



Let PJ extended meet the circle OPQ at R. Then PR is a diameter of the circle.
$\therefore \angle POR = \angle PQR = 90°$ ($\angle$ in semi-circle)
Let $\angle OPR = \angle QPR = \theta$. (in-centre)
In $\triangle OPR$, $PO = PR\cos\theta$
In $\triangle QPR$, $PQ = PR\cos\theta$
$\therefore PO = PQ$

(b) (i) Let $P=(x,19)$. By (a),
$$OP = PQ$$
$$\sqrt{x^2+19^2} = \sqrt{(x-40)^2+(19-30)^2}$$
$$x^2+361 = x^2-80x+1600+121$$
$$x=17 \Rightarrow P=(17,19)$$

Method 1
Let C be $x^2+y^2+Dx+Ey+F=0$.

$$\begin{cases}0^2+0^2+0+0+F=0\\17^2+19^2+17D+19E+F=0\\40^2+30^2+40D+30E+F=0\end{cases} \Rightarrow \begin{cases}D=-112\\E=66\\F=0\end{cases}$$

$\therefore$ Eqn of C is $x^2+y^2-112x+66y=0$.

Method 2
The centre J lies on the $\perp$ bisector of OQ.

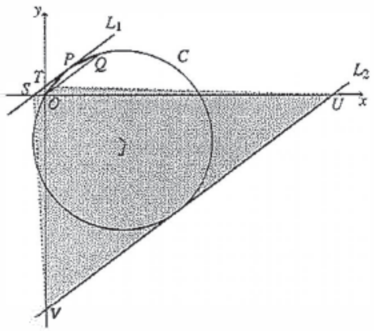
$$\begin{cases}\text{Mid-pt of } OQ = \left(\dfrac{40}{2},\dfrac{30}{2}\right)=(20,15)\\ m_{OQ} = \dfrac{30}{40} = \dfrac{3}{4} \Rightarrow m_{\perp \text{ bisector}} = \dfrac{-4}{3}\end{cases}$$

$\therefore$ Eqn of $\perp$ bisector: $y-15 = \dfrac{-4}{3}(x-20)$
$\Rightarrow y = \dfrac{125-4x}{3}$

Let $J=(h,k)$. Then
$$\begin{cases}k=\dfrac{125-4h}{3}\\(h-17)^2+(k-19)^2=(h-0)^2+(k-0)^2\end{cases}$$
$$h^2-34h+289+k^2-38k+361 = h^2+k^2$$
$$-34h-38\left(\frac{125-4h}{3}\right)+650=0$$
$$\frac{50}{3}h-\frac{2800}{3}=0$$
$$h=56 \Rightarrow k=-33$$

$\therefore$ Eqn of C is
$(x-56)^2+(y+33)^2=(0-56)^2+(0+33)^2$
$\Rightarrow x^2+y^2-112x+66y=0$

(ii)



Approach One – Find L_1 and L_2

Method 1
Let L_1 and L_2 be $y=\dfrac{3}{4}x+c$.

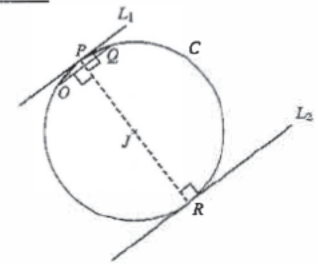
$$\begin{cases}y=\dfrac{3}{4}x+c\\x^2+y^2-112x+66y=0\end{cases}$$
$$x^2+\left(\frac{3}{4}x+c\right)^2-112x+66\left(\frac{3}{4}x+c\right)=0$$
$$\frac{25}{16}x^2+\left(\frac{3c-125}{2}\right)x+(c^2+66c)=0$$

$$\Delta=\frac{(3c-125)^2}{4}-4\cdot\frac{25}{16}\cdot(c^2+66c)=0$$
$$-16c^2-2400c+15625=0$$
$$c=-\frac{625}{4}\text{ or }\frac{25}{4}$$

$$\therefore L_1\text{ is } y=\frac{3}{4}x+\frac{25}{4}\Rightarrow\begin{cases}S=\left(\frac{-25}{3},0\right)\\T=\left(0,\frac{25}{4}\right)\end{cases}$$

$$L_2\text{ is } y=\frac{3}{4}x-\frac{625}{4}\Rightarrow\begin{cases}U=\left(\frac{625}{3},0\right)\\V=\left(0,\frac{-625}{4}\right)\end{cases}$$

Method 2



$\because$ $OP=PQ$ and $\angle OPJ=\angle QPJ$ (proved)
$\therefore$ $OQ\perp PJ$ (property of isos. $\triangle$)
$\Rightarrow$ $L_1\perp PJ$ ($OQ//L_1$)
$\Rightarrow$ L_1 is tangent to C at P.
(converse of $\angle$s in the same segment)

$\therefore$ Eqn of L_1: $y\ \ 19=\frac{3}{4}(x-17)\Rightarrow y=\frac{3}{4}x+\frac{25}{4}$

$\Rightarrow S=\left(\frac{-25}{3},0\right),\ T=\left(0,\frac{25}{4}\right)$

Let the diameter of C through P meet C again at $R(r,s)$. Then $\left(\frac{17+r}{2},\frac{19+s}{2}\right)=J=(56,-33)$

$\Rightarrow R=(95,-85)$

$\because$ L_2 is tangent to C at R

$\therefore$ Eqn of L_2: $y+85=\frac{3}{4}(x\ \ 95)\Rightarrow y=\frac{3}{4}x-\frac{625}{4}$

$\Rightarrow U=\left(\frac{625}{3},0\right),\ V=\left(0,\frac{-625}{4}\right)$

Therefore, (Method 1)

Area of trapezium $STUV$

$=$ Area of $\triangle STU+$ Area of $\triangle SVU$

$$=\frac{(\frac{625}{3}+\frac{25}{3})(\frac{25}{4})}{2}+\frac{(\frac{625}{3}+\frac{25}{3})(\frac{625}{4})}{2}$$

$$=\frac{105625}{6}=17604.2>17000\Rightarrow\text{YES}$$

Therefore, (Method 2)

$$ST\ \ \sqrt{\left(0+\frac{25}{3}\right)^2+\left(\frac{25}{4}-0\right)^2}=\frac{125}{12}$$

$$UV=\sqrt{\left(\frac{625}{3}\ \ 0\right)^2+\left(\ \frac{625}{4}-0\right)^2}=\frac{3125}{12}$$

Height of $STUV=$ Diameter of $C=130$

$$\therefore\text{ Area of }STUV=\frac{(\frac{125}{12}+\frac{3125}{12})(130)}{2}$$
$$=\frac{105625}{6}>17000\Rightarrow\text{YES}$$

Approach Two – Find S,T,U,V without L_1 and L_2

Method 3

Let the foots of perpendiculars from P and Q to the x-axis be M and N respectively. Note that $OQ//L_1//L_2$.

$\because$ $\triangle SPM\sim\triangle OQN$

$\therefore$ $\frac{PM}{SM}=\frac{QN}{ON}=\frac{3}{4}\Rightarrow SM=\frac{4}{3}(19)=\frac{76}{3}$

$\Rightarrow S=\left(17\ \ \frac{76}{3},0\right)=\left(\frac{\ 25}{3},0\right)$

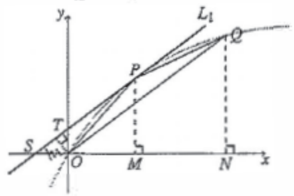
In $\triangle OST$, $OT=\frac{3}{4}OS=\frac{25}{4}\Rightarrow T=\left(0,\frac{25}{4}\right)$

Area of $\triangle OST=\frac{1}{2}\times\frac{25}{3}\times\frac{25}{4}\ \ \frac{625}{4}$

$ST-\sqrt{\left(\frac{25}{3}\right)^2+\left(\frac{25}{4}\right)^2}-\frac{125}{12}$

$\Rightarrow$ Height of $\triangle OST$ from O to ST ('h_1')

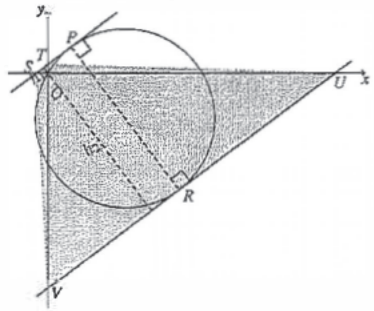
$$=\frac{2\times\frac{625}{24}}{\frac{125}{12}}=5$$



Referring to Method 2, PR is the height of trapezium $STUV$ as $PR\perp L_1$.

$\therefore$ Height of $\triangle OUV$ from O to UV ('h_2')

$=$ Diameter of C $\ h_1=2\sqrt{56^2+33^2}\ \ 5=125$



$\because$ $\triangle OST\sim\triangle OUV$

$\therefore$ $\frac{OV}{OT}=\frac{OU}{OS}=\frac{h_2}{h_1}=25$

Area of $\triangle OUV=\left(\frac{h_2}{h_1}\right)^2$(Area of $\triangle OST$)
$=625$(Area of $\triangle OST$)

Area of $\triangle OTU=\left(\frac{OU}{OS}\right)$(Area of $\triangle OST$)
$=25$(Area of $\triangle OST$)

Area of $\triangle OSV=\left(\frac{OV}{OT}\right)$(Area of $\triangle OST$)
$=25$(Area of $\triangle OST$)

$\therefore$ Area of $STUV=(1+625+25+25)$(A of $\triangle OST$)
$$=\frac{105625}{6}>17000\Rightarrow\text{YES}$$

Approach Three – A hybrid of Methods 1 and 3

Method 4

Let L_1 and L_2 be $y=\frac{3}{4}x+c$

$$\begin{cases}y=\frac{3}{4}x+c\\x^2+y^2\ \ 112x+66y=0\end{cases}$$

$$x^2+\left(\frac{3}{4}x+c\right)^2-112x+66\left(\frac{3}{4}x+c\right)=0$$
$$\frac{25}{16}x^2+\left(\frac{3c-125}{2}\right)x+(c^2+66c)=0$$
$$\Delta=\frac{(3c\ \ 125)^2}{4}-4\cdot\frac{25}{16}\cdot(c^2+66c)=0$$
$$-16c^2\ \ 2400c+15625=0$$
$$c=\ \ \frac{625}{4}\text{ or }\frac{25}{4}$$

$\therefore$ $OT=\frac{25}{4},\ OV=\frac{625}{4}$

$\Rightarrow\frac{OV}{OT}=25\Rightarrow\frac{OU}{OS}=25$ ($\because\triangle OST\sim\triangle OUV$)

Thus,

Area of $\triangle OUV=(25)^2$(Area of $\triangle OST$)

Area of $\triangle OTU=\left(\frac{OU}{OS}\right)$(Area of $\triangle OST$)
$=25$(Area of $\triangle OST$)

Area of $\triangle OSV=\left(\frac{OV}{OT}\right)$(Area of $\triangle OST$)
$=25$(Area of $\triangle OST$)

Besides, for $\triangle OST$, $\frac{OT}{OS}=\text{slope}=\frac{3}{4}\Rightarrow OS=\frac{25}{3}$

$\Rightarrow$ Area $=\frac{1}{2}\times\frac{25}{3}\times\frac{25}{4}=\frac{625}{24}$

$\therefore$ Area of $STUV=(1+625+25+25)$(A of $\triangle OST$)
$$=\frac{105625}{6}>17000\Rightarrow\text{YES}$$

16C.55 HKDSE MA 2018 – I – 19

(a) Eqn of C: $(x\ \ 8)^2+(y-2)^2=r^2$

$$\begin{cases}L:kx-5y-21=0\\C:(x\ \ 8)^2+(y\ \ 2)=r^2\end{cases}$$

$$\Rightarrow(x\ \ 8)^2+\left(\frac{kx\ \ 21}{5}-2\right)^2=r^2$$
$$(x-8)^2+\left(\frac{k}{5}x\ \ \frac{31}{5}\right)^2-r^2=0$$
$$\left(1+\frac{k^2}{25}\right)x^2\ \ 2\left(\frac{31}{25}k+8\right)x+\frac{2561}{25}-r^2=0$$
$$\Delta=0=4\left(\frac{31}{25}k+8\right)^2-4\left(1+\frac{k^2}{25}\right)\left(\frac{2561}{25}-r^2\right)$$
$$\left(\frac{31}{25}k+8\right)^2=\left(1+\frac{k^2}{25}\right)\left(\frac{2561}{25}-r^2\right)$$
$$\frac{2561}{25}\ \ r^2=\frac{(31k+200)^2}{25(25+k^2)}$$
$$r^2=\frac{2561(25+k^2)\ \ (31k+200)^2}{25(25+k^2)}$$
$$=\frac{961-496k+64k^2}{25+k^2}$$

(b) (i) Put D into L:

$k(18)\ \ 5(39)-21=0\Rightarrow k=12$

$$r=\sqrt{\frac{961-496(12)+64(12)^2}{25+(12)^2}}=5$$

(ii) $E=\left(0,\frac{-21}{5}\right)$

Denote the centre of C be G, which is the in-centre of $\triangle DEF$.

$DG=\sqrt{(18-8)^2+(39\ \ 2)^2}-\sqrt{1469}$

$\Rightarrow\angle GDE=\sin^{-1}\frac{r}{DG}=7.49586^\circ$

$\Rightarrow\angle FDE=2\angle GDE=14.99172^\circ$

$EG=\sqrt{(0\ \ 8)^2+(\frac{-21}{5}-2)^2}=\sqrt{\frac{2561}{25}}$

$\Rightarrow\angle GED=\sin^{\ 1}\frac{r}{EG}=29.60445^\circ$

$\Rightarrow\angle FED=2\angle GED=59.20890^\circ$

$\therefore$ $\angle DFE=180^\circ-14.99172^\circ\ \ 59.20890^\circ$
$=105.6^\circ>90^\circ$

$\therefore$ YES

16C.56 HKDSE MA 2019 – I – 19

(a) $f(4)=\frac{1}{1+k}((4)^2+(6k-2)(4)+(9k+25))$

$=\frac{1}{1+k}(33+33k)=33$

Hence, the graph passes through F.

(b) (i) $g(x)=f(-x)+4$

$$=\frac{1}{1+k}\Big((\ \ x)^2+(6k\ \ 2)(\ \ x)+(9k+25)\Big)+4$$
$$=\frac{1}{1+k}\Big(x^2\ \ (6k\ \ 2)x+(3k\ \ 1)^2$$
$$(3k\ \ 1)^2+(9k+25)\Big)+4$$
$$=\frac{1}{1+k}\Big((x\ \ 3k+1)^2-9k^2+3k+24\Big)+4$$
$$\frac{1}{1+k}\Big((x\ \ 3k+1)^2-3(1+k)(3k-8)\Big)+4$$
$$\frac{1}{1+k}(x\ \ 3k+1)^2\ \ 3(3k\ \ 8)+4$$
$$=\frac{1}{1+k}(x\ \ 3k+1)^2+28-9k$$

$\therefore$ $U=(3k\ \ 1,28\ \ 9k)$

(ii) As F varies, the circle is the smallest when OU is the diameter.

Method 1

$FO\perp FU\Rightarrow m_{FO}\ m_{FU}=-1$

$$\frac{28-9k\ \ (28\ \ 9k)\ \ 33}{3k\ \ 1\ \ \ (3k-1)-4}=-1$$
$$(28-9k)^2\ \ 33(28\ \ 9k)=\ \ (3k-1)^2+4(3k\ \ 1)$$
$$90k^2\ \ 225k-135=0$$
$$k=3\text{ or }\frac{1}{2}\text{ (rej.)}$$

Method 2

Mid-pt of $OU=\left(2,\frac{33}{2}\right)$

$$\sqrt{(3k-1\ \ 2)^2+(28-9k-\tfrac{33}{2})^2}=\sqrt{2^2+(\tfrac{33}{2})^2}$$
$$(3k\ \ 1)^2\ \ 4(3k-1)+(28\ \ 9k)^2$$
$$33(28\ \ 9k)=0$$
$$90k^2-225k-135=0$$
$$k=3\text{ or }\frac{-1}{2}\text{ (rej.)}$$

(iii) The fixed point G is the image of F after the above transformations. i.e. $G=(\;4,37)$.
Also, $V=(3(3)-1,28-9(3))=(8,1)$

Method 1

$\because\ m_{GF}\cdot m_{GO}=\frac{37\;33}{-4\;4}\cdot\frac{37-0}{-4-0}=\frac{37}{8}\neq -1$

$\therefore$ G is not on the circle with FO as diameter (which is the circle through F, O and V). $\Rightarrow$ NO

Method 2

The circle through $F(4,33)$, $O(0,0)$ and $V(8,1)$ is
$(x\;\;2)^2+(y\;\;\frac{33}{2})^2=2^2+(\frac{33}{2})^2$
$\Rightarrow x^2+y^2-4x\;\;33y=0$
Put $G(\;4,37)$: LHS $=180\neq$ RHS $\Rightarrow$ NO

Method 2'

Let the circle through $F(4,33)$, $O(0,0)$ and $V(8,1)$ be $x^2+y^2+dx+ey+f=0$.

$$\begin{cases}4^2+33^2+4d+33e+f=0\\0^2+0^2+0d+0e+f=0\\8^2+1^2+8d+e+f=0\end{cases}\Rightarrow\begin{cases}d=\;\;4\\e=-33\\f=0\end{cases}$$

Thus, the eqn of circle FOV is $x^2+y^2-4x-33y=0$.
Put $G(-4,37)$: LHS $=180\neq$ RHS $\Rightarrow$ NO

16C.57 HKDSE MA 2020 – I – 14

14a Let M be the mid-point of AB.
Then, $GM\perp AB$ (line joining centre to mid-pt. of chord ⊥ chord).
Since AB is horizontal, GM is vertical.
The x-coordinate of $G=\frac{-30+10}{2}$
$=-10$
The radius of $C=AG$
$=\sqrt{(-10-10)^2+[0-(-15)]^2}$
$=25$
Therefore, the equation of C is $(x-10)^2+[y-(-15)]^2\;\;25^2$, i.e.
$x^2+y^2\;\;20x+30y-300=0$.
bi Γ and L are parallel.
ii Since Γ and L are parallel, we know that the slope of Γ is equal to the slope of L, i.e. $\frac{-15-0}{10-30}\;\;\frac{3}{4}$.
Let $P=(x,y)$.
$y-0=\frac{3}{4}[x-(-10)]$
$3x-4y+30=0$
Therefore, the equation of Γ is $3x\;\;4y+30=0$.
iii Let θ be the inclination of AG and ϕ be the inclination of AH.
Note that $0°\leq\theta<180°$ and $0°\leq\phi<180°$.
$\tan\theta$ The slope of AG
$\tan\theta=\frac{15\;\;0}{10-(-10)}$
$\tan\theta=-\frac{3}{4}$
$\theta=180°-36.86989765°$
$\theta=143.1301024°$
$\tan\phi$ The slope of AH
$\tan\phi=\frac{3}{4}$
$\phi=36.86989765°$

$\angle BAG+\theta=180°$ (adj. ∠s on st. line)
$\angle BAG+143.1301024°=180°$
$\angle BAG=36.86989765°$
$\angle GAH=\angle BAG+\angle BAH$
$=\angle BAG+\phi$
$=36.86989765°+36.86989765°$
$=73.73979529°$
$>70°$
Therefore, the claim is disagreed with.

16D Loci in the rectangular coordinate plane

16D.1 (HKCEE MA 1981(3) – I – 7)

(a) $P=\left(\frac{4(1)+1(16)}{1+4},\frac{4(4)+1(-16)}{1+4}\right)=(4,0)$

(b) Put A into the parabola: $(4)^2=4a(1)\Rightarrow a=4$
Hence, the parabola is $y^2=16x$.
Eqn of locus: $(x+a)^2=(x\;\;4)^2+(y-0)^2$
$x^2+8x+16=x^2\;\;8x+16+y^2$
$y^2=16x$
which is the given parabola.

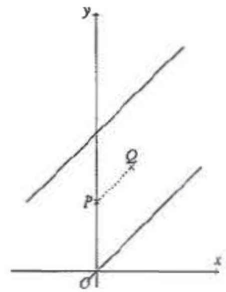
16D.2 HKCEE AM 1987 – II – 10

(a) $(x+1)^2=(x-1)^2+(y\;\;0)^2$
$x^2+2x+1=x^2\;\;2x+1+y^2$
$y^2=4x$

16D.3 (HKCEE AM 1994 II – 4)

(a) (Note that PR_0 is parallel to the x-axis. Thus:)
Area $=\frac{(4\;\;0)(6-4)}{2}=4$

(b) (i) A pair of lines parallel and equidistant to PQ



(ii) $m_{PQ}=\frac{6-4}{2-0}=-1$
Since R_0 is a point on the locus (from (a)), the line parallel to PQ and through $R_0(4,4)$ is:
$y\;\;4=1(x\;\;4)\Rightarrow y=x$
Thus, the equations are $y=x$ and $y=x+4+4=x+8$.

16D.4 HKCEE AM 1999 – II – 10

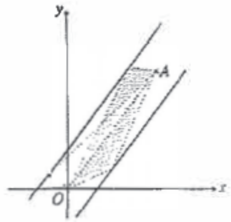
(a) $(x+3)^2+(y\;\;0)^2=3[(x+1)^2+(y\;\;0)^2]$
$x^2+6x+9+y^2=3x^2+6x+3+3y^2$
$2x^2+2y^2=6\Rightarrow x^2+y^2=3$

(b) Slope of segment joining centre and $T=\frac{b}{a}$
$\Rightarrow$ Slope of tg $=\;\;\frac{a}{b}$
$\therefore$ Eqn of tg: $y\;\;b=-\frac{a}{b}(x-a)$
$by\;\;b^2=-ax+a^2$
$ax+by\;\;(a^2+b^2)=0$
$ax+by\;\;3=0$ ($\because$ (a,b) lies on C)

(c) If the tangent in (b) passes through A,
$a(-3)+b(0)-3=0\Rightarrow a=-1$
$\Rightarrow b=\pm\sqrt{3-a^2}=\pm\sqrt{2}$
Since S is in Quad II, $S=(a,b)=(-1,\sqrt{2})$

16D.5 (HKCEE AM 2004 – 10)

A pair of straight lines parallel and equisdistant to OA
$\because\ OA=\sqrt{3^2+4^2}=5$
$\therefore$ Dist. from the lines to $OA=\frac{2\times 2}{5}=0.8$



16D.6 (HKCEE AM 2011 – 16)

(a) Centre of $C_1=(0,5)$, Radius of $C_1=\sqrt{5^2\;\;16}=3$
Radius of the unknown circle $=y$
$\because$ It touches C_1 externally
$\therefore\ \sqrt{(x-0)^2+(y-5)^2}=y+3$
$x^2+y^2-10y+25=y^2+6y+9$
$x^2+16=16y\Rightarrow y=\frac{1}{16}x^2+1$

(b) (i) Let (h,k) be the centre of C_2.
Then $k=\frac{1}{16}h^2+1$.
Radius $=k=\sqrt{(h\;\;20)^2+(k\;\;16)^2}$
$k^2=h^2+k^2\;\;40h\;\;32k+656$
$0=h^2-40h\;\;32\left(\frac{1}{16}h^2+1\right)+656$
$0=\;\;h^2\;\;40h+624$
$h=12$ or $\;\;52$ (rej.)
$\therefore\ (h,k)=\left(12,\frac{1}{16}(12)^2+1\right)=(12,10)$
$\Rightarrow$ Eqn of C_2: $(x\;\;12)^2+(y-10)^2=10^2$
$\Rightarrow x^2+y^2\;\;24x\;\;20y+144=0$

(ii) The point of contact is collinear with the 2 centres, which are both points on S. However, for a parabola opening upwards, the line segment joining 2 points on the parabola (we call it a 'secant' line) must lie above the parabola.
$\therefore$ The sentence is not correct.

(c) A circle that satisfies the first two conditions will touch C_1 *externally*. Hence, it cannot satisfy the last condition.
$\therefore$ NO

16D.7 HKDSE MA SP – I – 13

(a) $m_{L_1}=\frac{4}{3}\Rightarrow m_{L_2}=\frac{3}{4}$
$\therefore$ Eqn of L_2: $y\;\;9=\frac{-3}{4}(x-4)\Rightarrow 3x+4y\;\;48=0$

(b) (i) Γ is the perpendicular bisector of AB.
$\therefore\ \Gamma//L_2$

(ii) *Method 1*

$$\begin{cases}L_1:4x\;\;3y+12=0\\L_2:3x+4y\;\;48=0\end{cases}\Rightarrow A=(3.84,9.12)$$

$B=(0,4)$
$\therefore$ Eqn of Γ is:
$(x-3.84)^2+(y\;\;9.12)^2=(x-0)^2+(y\;\;4)^2$
$-7.68x\;\;18.24y+97.92=-8y+16$
$3x+4y\;\;32=0$

Method 2

y-int of $L_1=4$, y-int of $L_2=12$
$\Rightarrow$ y-intercept of $\Gamma=\frac{4+12}{2}=8$
$\therefore$ Eqn of Γ is $y=\frac{-3}{4}x+8$

16D.8 HKDSE MA PP – I – 8

(a) $A'(3,4)$, $B'(5,-2)$

(b) Eqn: $(x\;\;3)^2+(y-4)^2=(x-5)^2+(y+2)^2$
$6x\;\;8y+25=\;\;10x\Rightarrow 4x-8y+25=0$

16D.9 HKDSE MA 2012 – I – 14

(a) (i) $\Gamma//L$

(ii) y-intercept of $\Gamma=\frac{(\;1)+(\;3)}{2}=\;\;2$
$m_L=\frac{0+1}{3\;\;0}=\frac{1}{3}$
$\therefore$ Eqn of Γ: $y=\frac{1}{3}x\;\;2$

(b) (i) Put Q into the eqn of Γ:
RHS $=\frac{1}{3}(6)-2=0$ LHS
$\therefore$ Γ passes through Q.

(ii) $QH=QK=$ radius
(In fact, HQK is a diameter of the circle.)
Besides, since A and B lie on L, their perpendicular distances to Γ is the distance between L and Γ.
i.e., The height of $\triangle AQH$ with QH as base and the height of $\triangle BQK$ with QK as base are the same.
$\therefore$ Area of $\triangle AQH$: Area of $\triangle BQK=1:1$

16D.10 HKDSE MA 2013 – I – 14

(a) $R\;\;(6,17)$

(b) (i) *Method 1*

$m_L=\frac{-4}{3}$
$\Rightarrow$ Eqn of PR: $y\;\;17=\frac{-1}{\frac{-4}{3}}(x\;\;6)\Rightarrow y=\frac{3}{4}x+\frac{25}{2}$

$$\begin{cases}PR:y=\frac{3}{4}x+\frac{25}{2}\\L:4x+3y+50=0\end{cases}\Rightarrow P=(\;\;14,2)$$

Method 2

Let $P=(a,b)$.
$\because\ PR\perp L$
$\therefore\ m_{PR}=-1\div\frac{-4}{3}=\frac{3}{4}\Rightarrow\frac{b-17}{a-6}=\frac{3}{4}$

$$\begin{cases}4a+3b+50=0\\\frac{b-17}{a\;\;6}=\frac{3}{4}\end{cases}\Rightarrow(a,b)=(-14,2)$$

Hence

$PR=\sqrt{(\;\;14\;\;6)^2+(2\;\;17)^2}=25$

(ii) (1) P, Q and R are collinear.
(2) $QR=$ radius of circle $=\sqrt{6^2+17^2-225}=10$
$\therefore\ \frac{\text{Area of }\triangle OPQ}{\text{Area of }\triangle OQR}=\frac{PQ}{QR}=\frac{25\;\;10}{10}=\frac{3}{2}$

16D.11 HKDSE MA 2014 – I – 12

(a) Radius of $C = \sqrt{(6-0)^2 + (11-3)^2} = 10$

$\therefore$ Eqn of C: $(x-0)^2+(y-3)^2 = 10^2$

$\Rightarrow x^2+y^2-6y-91=0$

(b) (i) Eqn of Γ:

$(x-6)^2+(y-11)^2 = (x-0)^2+(y-3)^2$

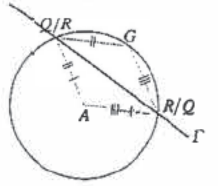
$-12x-22y+157 = -6y+9$

$3x+4y-37=0$

(ii) Γ is the perpendicular bisector of AG.

(iii) The quadrilateral is a rhombus.

$\therefore$ Perimeter $= 4 \times$ Radius $= 40$



16D.12 HKDSE MA 2016 I – 10

(a) Eqn of Γ:

$(x\ \ 5)^2+(y-7)^2 = (x-13)^2+(y-1)^2$

$-10x-14y+74 = -26x-2y+170$

$4x-3y-24=0$

(b) $H=(6,0)$, $K=(0,-8)$

Since $\angle HOK = 90°$, HK is a diameter of C.

Diameter $= \sqrt{6^2+8^2} = 10$

$\therefore$ Circumference of $C = 10\pi = 31.4 > 30$

$\therefore$ YES

16D.13 HKDSE MA 2017 – I – 13

(a) Radius $= \sqrt{(-6-2)^2+(5+1)^2} = 10$

$\therefore$ Eqn of C: $(x-2)^2+(y+1)^2 = 10^2$

$\Rightarrow x^2+y^2-4x+2y-95=0$

(b) *Method 1* – From the standard form

$FG = \sqrt{(-3-2)^2+(11+1)^2} = 13 >$ Radius

$\therefore$ Outside

Method 2 – From the general form

Put F: LHS $= (\ \ 3)^2+(11)^2-4(-3)+2(11)-95$

$= 69 > 0$

$\therefore$ Outside

(c) (i) F, G and H are collinear.

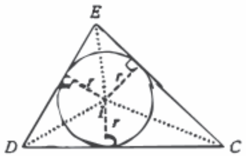
(ii) Req. eqn: $\dfrac{y+1}{x-2} = \dfrac{11+1}{3-2} \Rightarrow 12x+5y-19=0$

16D.14 HKDSE MA 2019 – I – 17

(a) Let I be the in-centre of $\triangle CDE$. Then the perpendiculars from I to CD, DE and EC are all r.

$a\quad \dfrac{r\cdot CD}{2}+\dfrac{r\cdot DE}{2}+\dfrac{r\cdot EC}{2}$

$= \dfrac{r(CD+DE+EC)}{2} = \dfrac{r(p)}{2} \Rightarrow pr = 2a$



(b) (i) Γ is the angle bisector of $\angle OHK$.

(ii) $OK = 14$

$OH = \sqrt{9^2+12^2} = 15$

$HK = \sqrt{(9-14)^2+(12-0)^2} = 13$

Perimeter of $\triangle OHK = 42$

Area of $\triangle OHK = \dfrac{14 \times 12}{2} = 84$

From (a), radius of inscribed circle $= \dfrac{42 \times 84}{2} = 4$

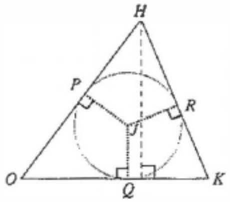
Let the in-centre be $J(h, 4)$.

Method 1

By tangent properties,

$OQ = OP = h \Rightarrow \begin{cases} HR = HP = 15-h \\ KR = KQ = 14-h \end{cases}$

$\therefore HK = 13 = (15-h)+(14-h) \Rightarrow h = 8$



Method 2

Let the inscribed circle touch OH at P.

In $\triangle OJP$, $OP^2 = OJ^2 - PJ^2$

$= (\sqrt{h^2+4^2})^2 - 4^2 = h^2$

In $\triangle HJP$, $PH^2 = HJ^2 - PJ^2$

$= (\sqrt{(h-9)^2+(4-12)^2})^2 - 4^2$

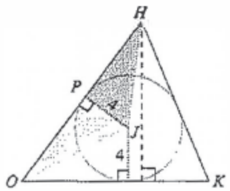
$= h^2 - 18h + 129$

$\therefore \quad OP + PH = OH$

$h + \sqrt{h^2-18h+129} = 15$

$h^2 - 18h + 129 = 225 - 30h + h^2$

$h = 8$



Hence,

$J = (8, 4)$

Eqn of HJ (i.e. Γ) is $\dfrac{y-4}{x\ \ 8} = \dfrac{12\ \ 4}{9-8}$

$\Rightarrow y = 8x - 60$

16E Polar coordinates

16E.1 HKCEE MA 2009 – I – 8

(a) $\angle POQ = 213° - 123° = 90°$

$\therefore \triangle OPQ$ is right-angled.

(b) $k^2+24^2 = 25^2 \Rightarrow k = 7$

$\therefore$ Perimeter $= 7+24+25 = 56$

16E.2 HKDSE MA PP – I – 6

(a) $\angle AOC = 337°\ \ 157° = 180°$

$\therefore$ A, O and C are collinear.

(b) $\angle AOB = 247° - 157° = 90°$

$\therefore$ OB is the height of $\triangle ABC$ with AC as base.

$\therefore$ Area $= \dfrac{(13+15)\times 14}{2} = 196$

16E.3 HKDSE MA 2013 – I – 6

(a) L bisects $\angle AOB$.

(b) Suppose L intersects AB at P.

$\angle AOP = \dfrac{130° - 10°}{2} = 60°$, $OP = OA\cos 60° = 13$

$\therefore$ The intersection $= P = (13, 10°+60°) = (13, 70°)$

16E.4 HKDSE MA 2016 – I – 7

(a) $\angle AOB = 135° - 75° = 60°$

(b) $OA = OB = 12$ and $\angle AOB = 60°$

$\Rightarrow \triangle AOB$ is equilateral.

$\therefore$ Perimeter $= 12 \times 3 = 36$

(c) 3