12 Geometry of Circles

12A Angles and chords in circles

12A.1 HKCEE MA 1980(1/1*/3) – I 10

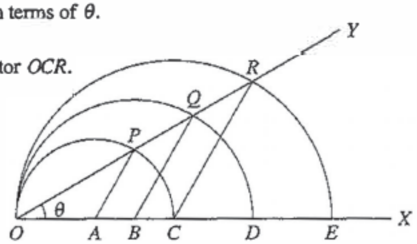
(Continued from 15A.1.)

A, *B* and *C* are three points on the line *OX* such that $OA = 2$, $OB = 3$ and $OC = 4$. With *A*, *B*, *C* as centres and *OA*, *OB*, *OC* as radii, three semi-circles are drawn as shown in the figure. A line *OY* cuts the three semi circles at *P*, *Q*, *R* respectively.

(a) If $\angle YOX = \theta$, express $\angle PAX$, $\angle QBX$ and $\angle RCX$ in terms of θ.

(b) Find the following ratios:
area of sector *OAP* : area of sector *OBQ* : area of sector *OCR*.

(c) If $RD \perp OX$, calculate the angle θ.



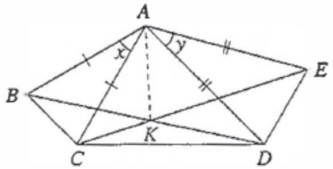
12A.2 HKCEE MA 1980(1*) – I 14

In the figure, $AB = AC$, $AD = AE$, $x = y$. Straight lines *BD* and *CE* intersect at *K*.

(a) Prove that $\triangle ABD$ and $\triangle ACE$ are congruent.

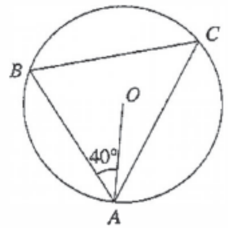
(b) Prove that *ABCK* is a cyclic quadrilateral.

(c) Besides the quadrilateral *ABCK*, there is another cyclic quadrilateral in the figure. Write it down (proof is not required).



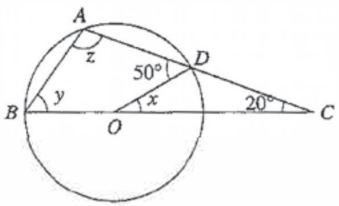
12A.3 HKCEE MA 1981(2) I 7

In the figure, *O* is the centre of circle *ABC*. $\angle OAB = 40°$. Calculate $\angle BCA$.



12A.4 HKCEE MA 1982(2) – I 6

In the figure, *O* is the centre of the circle *BAD*. *BOC* and *ADC* are straight lines. If $\angle ADO = 50°$ and $\angle ACB = 20°$, find *x*, *y* and *z*.



12A.5 HKCEE MA 1982(2) I 13

(Continued from 11B.1.)

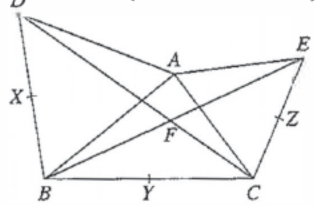
In the figure, $\triangle ADB$ and $\triangle ACE$ are equilateral triangles. *DC* and *BE* intersect at *F*.

(a) Prove that $DC = BE$. [Hint: Consider $\triangle ADC$ and $\triangle ABE$.]

(b) (i) Prove that *A*, *D*, *B* and *F* are concyclic.

(ii) Find $\angle BFD$.

(c) Let the mid points of *DB*, *BC* and *CE* be *X*, *Y* and *Z* respectively. Find the angles of $\triangle XYZ$.



12A.6 HKCEE MA 1989 – I – 4

AB is a diameter of a circle and *M* is a point on the circumference. *C* is a point on *BM* produced such that $BM = MC$.

(a) Draw a diagram to represent the above information.

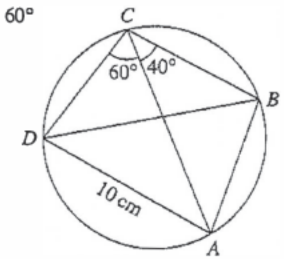
(b) Show that *AM* bisects $\angle BAC$.

12A.7 HKCEE MA 1989 I – 6

(To continue as 14A.4.)

In the figure, *ABCD* is a cyclic quadrilateral with $AD = 10$ cm, $\angle ACD = 60°$ and $\angle ACB = 40°$.

(a) Find $\angle ABD$ and $\angle BAD$.



12A.8 HKCEE MA 1990 I – 9

In the figure, *AB* is a diameter of the circle *ADB* and *ABC* is an isosceles triangle with $AB = AC$.

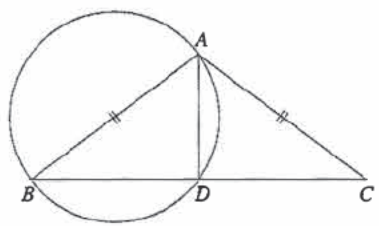
(a) Prove that $\triangle ABD$ and $\triangle ACD$ are congruent.

(b) The tangent to the circle at *D* cuts *AC* at the point *E*. Prove that $\triangle ABD$ and $\triangle ADE$ are similar.

(c) In (b), let $AB = 5$ and $BD = 4$.

(i) Find *DE*.

(ii) *CA* is produced to meet the circle at the point *F*. Find *AF*.



12A.9 <u>HKCEE MA 1992 I – 11</u>

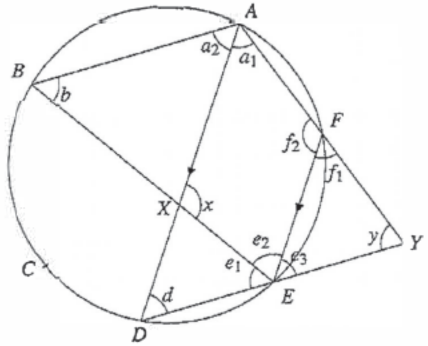
In the figure, A, B, C, D, E and F are points on a circle such that $AD//FE$ and $\widehat{BCD} = \widehat{AFE}$. AD intersects BE at X. AF and DE are produced to meet at Y.

(a) Prove that $\triangle EFY$ is isosceles.

(b) Prove that $BA//DE$.

(c) Prove that A, X, E, Y are concyclic.

(d) If $b = 47°$, find f_1, y and x.



12A.10 <u>HKCEE MA 1993 – I – 11</u>

The figure shows a semicircle with diameter AD and centre O. The chords AC and BD meet at P. Q is the foot of the perpendicular from P to AD.

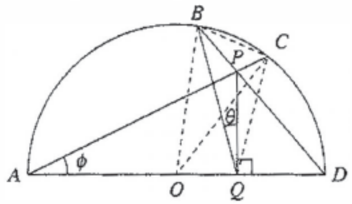
(a) Show that A, Q, P, B are concyclic.

(b) Let $\angle BQP = \theta$. Find, in terms of θ,

(i) $\angle BQC$,

(ii) $\angle BOC$.

(c) Let $\angle CAD = \phi$. Find $\angle CBQ$ in terms of ϕ.



12A.11 <u>HKCEE MA 1994 I – 13</u>

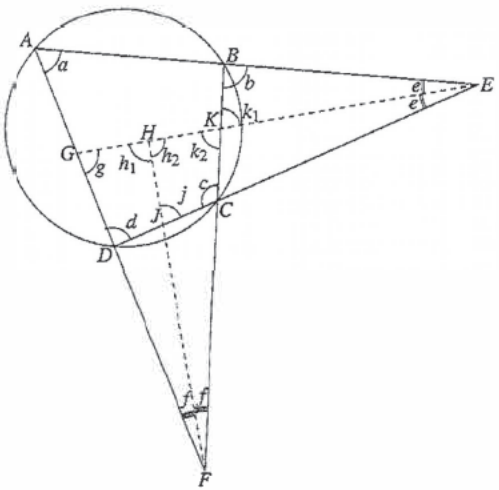
In the figure, A, B, C, D are points on a circle and ABE, $GHKE$, $DJCE$, $AGDF$, HJF, $BKCF$ are straight lines. FH bisects $\angle AFB$ and GE bisects $\angle AED$.

(a) Prove that $\angle FGH = \angle FKH$.

(b) Prove that $FH \perp GK$.

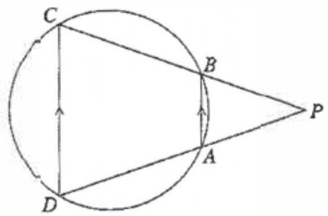
(c) (i) If $\angle AED = \angle AFB$, prove that D, J, H, G are concyclic.

(ii) If $\angle AED = 28°$ and $\angle AFB = 46°$, find $\angle BCD$.



12A.12 <u>HKCEE MA 1996 – I – 6</u>

In the figure, A, B, C, D are points on a circle. CB and DA are produced to meet at P. If $AB//DC$, prove that $AP = BP$.



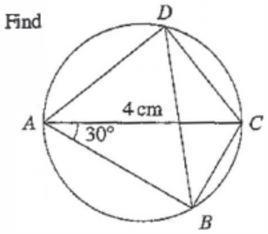
12A.13 <u>HKCEE MA 1997 – I – 9</u>

In the figure, AC is a diameter of the circle. $AC = 4$ cm and $\angle BAC = 30°$. Find

(a) $\angle BDC$ and $\angle ADB$,

(b) $\widehat{AB} : \widehat{BC}$,

(c) $AB : BC$.

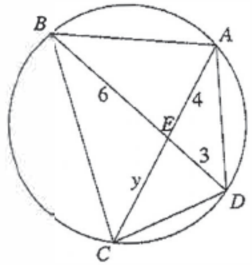


12A.14 <u>HKCEE MA 1998 – I – 6</u>

In the figure, A, B, C, D are points on a circle. AC and BD meet at E.

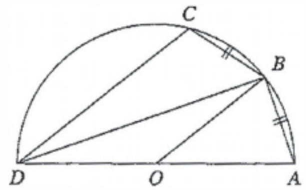
(a) Which triangle is similar to $\triangle ECD$?

(b) Find y.



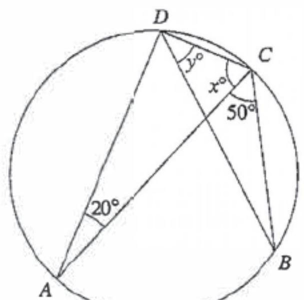
12A.15 <u>HKCEE MA 1998 – I – 14</u>

In the figure, O is the centre of the semicircle $ABCD$ and $AB = BC$. Show that $BO//CD$.



12A.16 <u>HKCEE MA 1999 – I – 5</u>

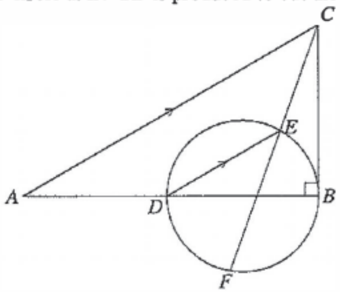
In the figure, A, B, C, D are points on a circle and AC is a diameter. Find x and y.



12A.17 <u>HKCEE MA 1999 – I – 16</u> (To continue as **16C.20**.)

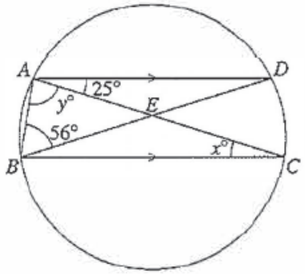
(a) In the figure, ABC is a triangle right angled at B. D is a point on AB. A circle is drawn with DB as a diameter. The line through D and parallel to AC cuts the circle at E. CE is produced to cut the circle at F.

- (i) Prove that A, F, B and C are concyclic.
- (ii) If M is the mid point of AC, explain why $MB = MF$.



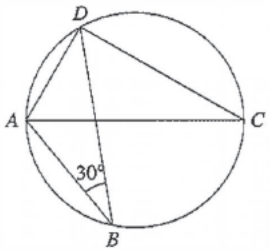
12A.18 <u>HKCEE MA 2000 – I – 7</u>

In the figure, AD and BC are two parallel chords of the circle. AC and BD intersect at E. Find x and y.



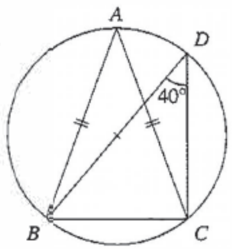
12A.19 <u>HKCEE MA 2001 – I – 5</u>

In the figure, AC is a diameter of the circle. Find $\angle DAC$.



12A.20 <u>HKCEE MA 2002 – I – 9</u>

In the figure, BD is a diameter of the circle $ABCD$. $AB = AC$ and $\angle BDC = 40°$. Find $\angle ABD$.

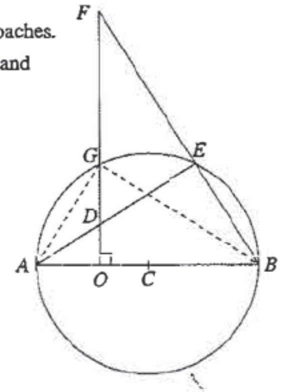


12A.21 <u>HKCEE MA 2002 – I – 16</u> (To continue as **16C.23**.)

In the figure, AB is a diameter of the circle $ABEG$ with centre C. The perpendicular from G to AB cuts AB at O. AE cuts OG at D. BE and OG are produced to meet at F.
Mary and John try to prove $OD \cdot OF = OG^2$ by using two different approaches.

(a) Mary tackles the problem by first proving that $\triangle AOD \sim \triangle FOB$ and $\triangle AOG \sim \triangle GOB$. Complete the following tasks for Mary.

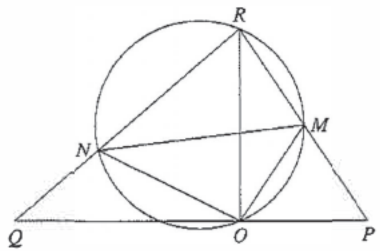
- (i) Prove that $\triangle AOD \sim \triangle FOB$.
- (ii) Prove that $\triangle AOG \sim \triangle GOB$.
- (iii) Using (a)(i) and (a)(ii), prove that $OD \cdot OF = OG^2$.



12A.22 <u>HKCEE MA 2005 – I – 17</u> (To continue as **16C.26**.)

(a) In the figure, MN is a diameter of the circle $MONR$. The chord RO is perpendicular to the straight line POQ. RNQ and RMP are straight lines.

- (i) By considering triangles OQR and ORP, prove that $OR^2 = OP \cdot OQ$.
- (ii) Prove that $\triangle MON \sim \triangle POR$.

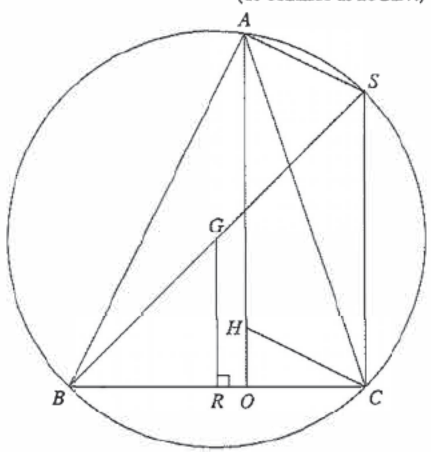


12A.23 <u>HKCEE MA 2006 – I – 16</u> (To continue as **16C.27**.)

In the figure, G and H are the circumcentre and the orthocentre of $\triangle ABC$ respectively. AH produced meets BC at O. The perpendicular from G to BC meets BC at R. BS is a diameter of the circle which passes through A, B and C.

(a) Prove that

- (i) $AHCS$ is a parallelogram,
- (ii) $AH = 2GR$.

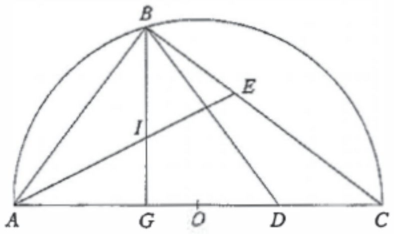


12A.24 HKCEE MA 2007 – I – 17 (To continue as **16C.28.**)

(a) In the figure, AC is the diameter of the semi circle ABC with centre O. D is a point lying on AC such that $AB = BD$. I is the in-centre of $\triangle ABD$. AI is produced to meet BC at E. BI is produced to meet AC at G.

(i) Prove that $\triangle ABG \cong \triangle DBG$.

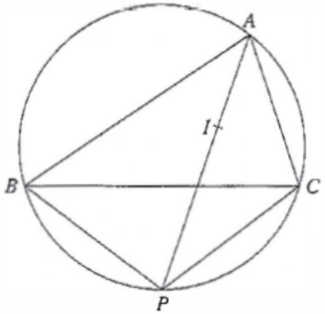
(ii) By considering the triangles AGI and ABE, prove that $\frac{GI}{AG} = \frac{BE}{AB}$.



12A.25 HKCEE MA 2008 – I – 17 (To continue as **16C.29.**)

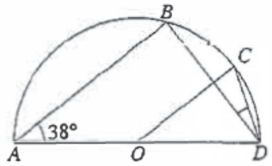
The figure shows a circle passing through A, B and C. I is the in centre of $\triangle ABC$ and AI produced meets the circle at P.

(a) Prove that $BP = CP = IP$.



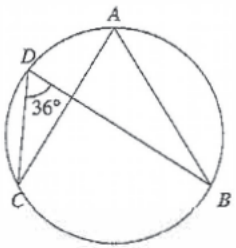
12A.26 HKDSE MA SP – I – 7

In the figure, O is the centre of the semicircle $ABCD$.
If $AB // OC$ and $\angle BAD = 38°$, find $\angle BDC$.



12A.27 HKDSE MA PP – I – 7

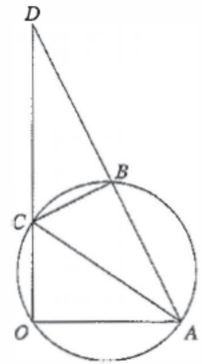
In the figure, BD is a diameter of the circle $ABCD$.
If $AB = AC$ and $\angle BDC = 36°$, find $\angle ABD$.



12A.28 HKDSE MA PP – I – 14 (To continue as **16C.51.**)

In the figure, $OABC$ is a circle. It is given that AB produced and OC produced meet at D.

(a) Write down a pair of similar triangles in the figure.

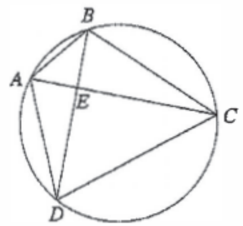


12A.29 HKDSE MA 2012 – I – 8

In the figure, AB, BC, CD and AD are chords of the circle. AC and BD intersect at E. It is given that $BE = 8$ cm, $CE = 20$ cm and $DE = 15$ cm.

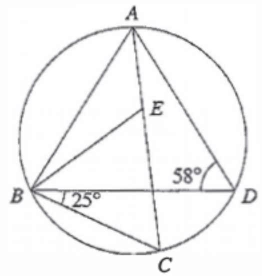
(a) Write down a pair of similar triangles in the figure. Also find AE.

(b) Suppose that $AB = 10$ cm. Are AC and BD perpendicular to each other? Explain your answer.



12A.30 HKDSE MA 2015 – I – 8

In the figure, $ABCD$ is a circle. E is a point lying on AC such that $BC = CE$. It is given that $AB = AD$, $\angle ADB = 58°$ and $\angle CBD = 25°$.
Find $\angle BDC$ and $\angle ABE$.

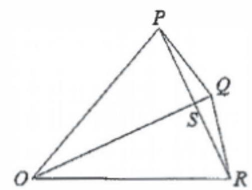


12A.31 HKDSE MA 2017 – I – 10 (Continued from **11B.11.**)

In the figure, $OPQR$ is a quadrilateral such that $OP = OQ = OR$. OQ and PR intersect at the point S. S is the mid-point of PR.

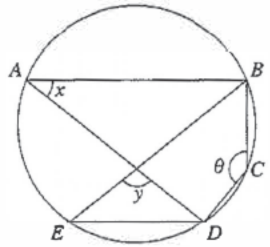
(a) Prove that $\triangle OPS \cong \triangle ORS$.

(b) It is given that O is the centre of the circle which passes through P, Q and R. If $OQ = 6$ cm and $\angle PRQ = 10°$, find the area of the sector $OPQR$ in terms of π.



12A.32 HKDSE MA 2018 – I – 8

In the figure, $ABCDE$ is a circle. It is given that $AB//ED$. AD and BE intersect at the point F.
Express x and y in terms of θ.

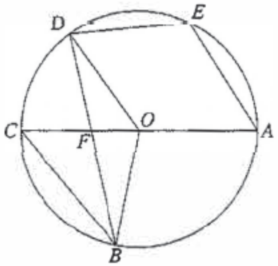


12A.33 HKDSE MA 2019 – I – 13

In the figure, O is the centre of circle $ABCDE$. AC is a diameter of the circle. BD and OC intersect at the point F. It is given that $\angle AED = 115°$.

(a) Find $\angle CBF$.

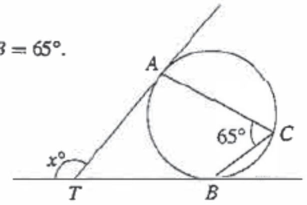
(b) Suppose that $BC//OD$ and $OB = 18$ cm. Is the perimeter of the sector OBC less than 60 cm? Explain your answer.



12B Tangents of circles

12B.1 HKCEE MA 1980(1*) – I 8

In the figure, TA and TB touch the circle at A and B respectively. $\angle ACB = 65°$.
Find the value of x.



12B.2 HKCEE MA 1981(2) I – 13

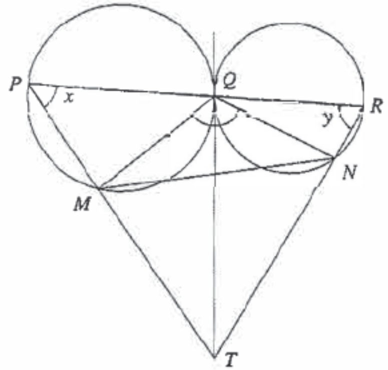
In the figure, circles PMQ and QNR touch each other at Q. QT is a common tangent. PQR is a straight line. TP and TR cut the circles at M and N respectively.

(a) If $\angle P = x$ and $\angle R = y$, express $\angle MQN$ in terms of x and y.

(b) Prove that Q, M, T and N are concyclic.

(c) Prove that P, M, N and R are concyclic.

(d) There are several pairs of similar triangles in the figure. Name any two pairs (no proof is required).



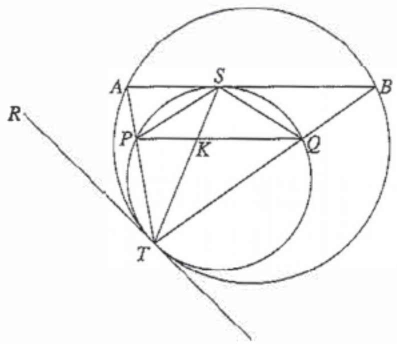
12B.3 HKCEE MA 1982(2) I – 14

In the figure, two circles touch internally at T. TR is their common tangent. AB touches the smaller circle at S. AT and BT cut the smaller circle at P and Q respectively. PQ and ST intersect at K.

(a) Prove that $PQ//AB$.

(b) Prove that ST bisects $\angle ATB$.

(c) $\triangle STQ$ is similar to four other triangles in the figure. Write down any three of them.
(No proof is required.)

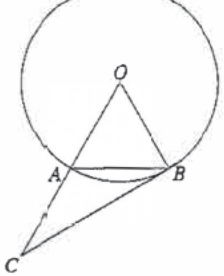


12B.4 HKCEE MA 1983(A/B) – I 2

In the figure, O is the centre of the circle. A and B are two points on the circle such that OAB is an equilateral triangle. OA is produced to C such that $OA = AC$.

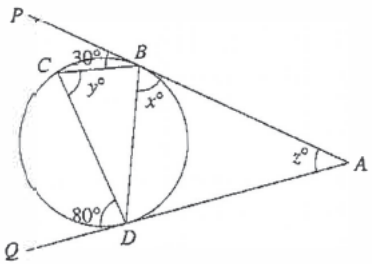
(a) Find $\angle ABC$.

(b) Is CB a tangent to the circle at B? Give a reason for your answer.



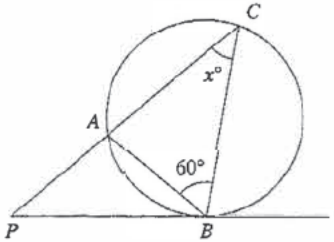
12B.5 HKCEE MA 1984(A/B) I – 5

In the figure, AP and AQ touch the circle BCD at B and D respectively. $\angle PBC = 30°$ and $\angle CDQ = 80°$. Find the values of x, y and z.



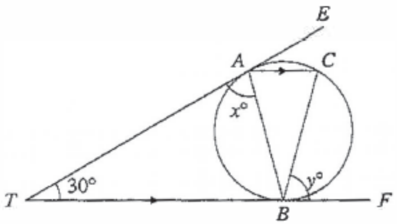
12B.6 HKCEE MA 1985(A/B) – I – 2

In the figure, PB touches the circle ABC at B. PAC is a straight line. $\angle ABC = 60°$. $AP = AB$. Find the value of x.



12B.7 HKCEE MA 1986(A/B) I – 2

In the figure, TAE and TBF are tangents to the circle ABC. If $\angle ATB = 30°$ and $AC // TF$, find x and y.

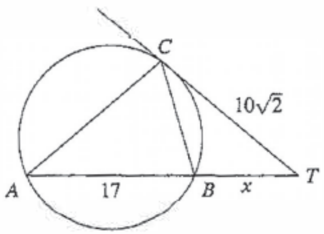


12B.8 HKCEE MA 1986(A/B) – I – 6

In the figure, A, B and C are three points on the circle. CT is a tangent and ABT is a straight line.

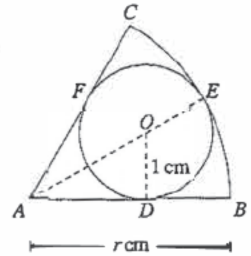
(a) Name a triangle which is similar to $\triangle BCT$.

(b) Let $BT = x$, $AB = 17$ and $CT = 10\sqrt{2}$. Find x.



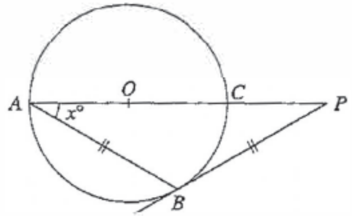
12B.9 HKCEE MA 1987(A/B) I 6

The figure shows a circle, centre O, inscribed in a sector ABC. D, E and F are points of contact. $OD = 1$ cm, $AB = r$ cm and $\angle BAC = 60°$. Find r.



12B.10 HKCEE MA 1987(A/B) I – 7

In the figure, O is the centre of the circle. $AOCP$ is a straight line, PB touches the circle at B, $BA = BP$ and $\angle PAB = x°$. Find x.

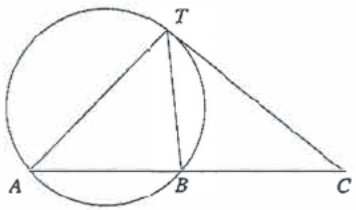


12B.11 HKCEE MA 1988 I – 8(b)

In the figure, CT is tangent to the circle ABT.

(i) Find a triangle similar to $\triangle ACT$ and give reasons.

(ii) If $CT = 6$ and $BC = 5$, find AB.



12B.12 HKCEE MA 1991 I – 13

In the figure, A, B are the centres of the circles DEC and DFC respectively. ECF is a straight line.

(a) Prove that triangles ABC and ABD are congruent.

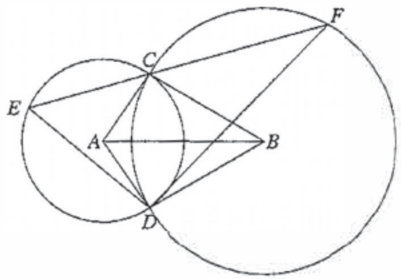
(b) Let $\angle FED = 55°$, $\angle ACB = 95°$.

 (i) Find $\angle CAB$ and $\angle EFD$.

 (ii) A circle S is drawn through D to touch the line CF at F.

  (1) Draw a labelled rough diagram to represent the above information.

  (2) Show that the diameter of the circle S is $2DF$.



12B.13 HKCEE MA 1995 – I – 14

In Figure (1), AP and AQ are tangents to the circle at P and Q. A line through A cuts the circle at B and C and a line through Q parallel to AC cuts the circle at R. PR cuts BC at M.

(a) Prove that

(i) M, P, A and Q are concyclic;

(ii) $MR = MQ$.

(b) If $\angle PAC = 20°$ and $\angle QAC = 50°$, find $\angle QPR$ and $\angle PQR$. (You are not required to give reasons.)

(c) The perpendicular from M to RQ meets RQ at H (see Figure (2)).

(i) Explain briefly why MH bisects RQ.

(ii) Explain briefly why the centre of the circle lies on the line through M and H.

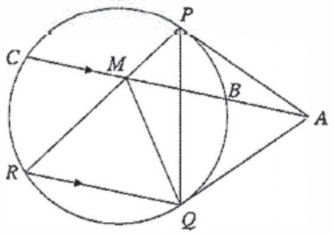


Figure (1)

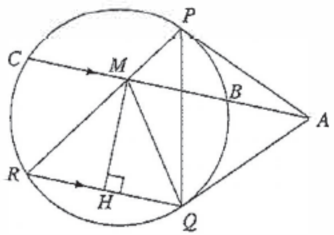


Figure (2)

12B.14 HKCEE MA 1997 – I – 16

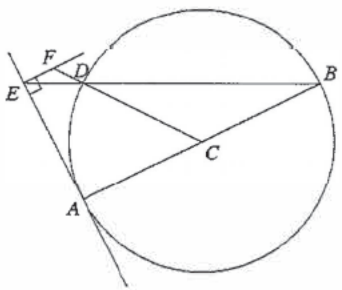
(To continue as **16C.18**.)

(a) In the figure, D is a point on the circle with AB as diameter and C as the centre. The tangent to the circle at A meets BD produced at E. The perpendicular to this tangent through E meets CD produced at F.

(i) Prove that $AB // EF$.

(ii) Prove that $FD = FE$.

(iii) Explain why F is the centre of the circle passing through D and touching AE at E.



12B.15 HKCEE MA 2000 – I – 16

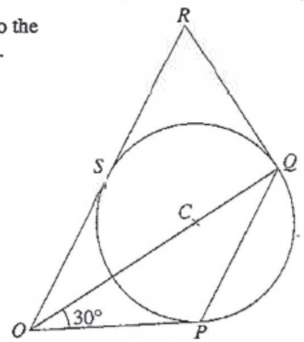
(To continue as **16C.21**.)

In the figure, C is the centre of the circle PQS. OR and OP are tangent to the circle at S and P respectively. OCQ is a straight line and $\angle QOP = 30°$.

(a) Show that $\angle PQO = 30°$.

(b) Suppose $OPQR$ is a cyclic quadrilateral.

(i) Show that RQ is tangent to circle PQS at Q.



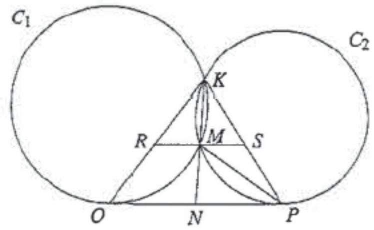
12B.16 HKCEE MA 2003 – I – 17

(To continue as **16C.24**.)

(a) In the figure, OP is a common tangent to the circles C_1 and C_2 at the points O and P respectively. The common chord KM when produced intersects OP at N. R and S are points on KO and KP respectively such that the straight line RMS is parallel to OP.

(i) By considering triangles NPM and NKP, prove that $NP^2 = NK \cdot NM$.

(ii) Prove that $RM = MS$.



12B.17 HKCEE MA 2004 I 16(a),(b),(c)(i)

(To continue as **16C.25**.)

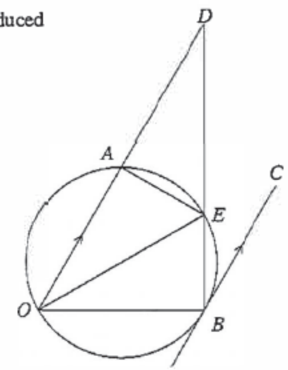
In the figure, BC is a tangent to the circle OAB with $BC // OA$. OA is produced to D such that $AD = OB$. BD cuts the circle at E.

(a) Prove that $\triangle ADE \cong \triangle BOE$.

(b) Prove that $\angle BEO = 2\angle BOE$.

(c) Suppose OE is a diameter of the circle $OAEB$.

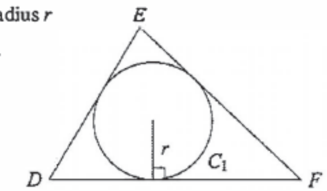
(i) Find $\angle BOE$.



12B.18 HKCEE AM 2002 – 15

(To continue as **16C.45**.)

(a) DEF is a triangle with perimeter p and area A. A circle C_1 of radius r is inscribed in the triangle (see the figure). Show that $A = \frac{1}{2}pr$.



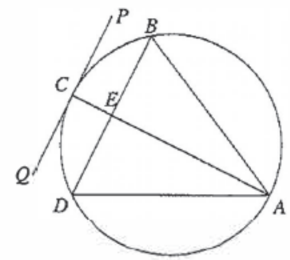
12B.19 HKDSE MA SP – I – 19

(To continue as **16C.50**.)

In the figure, the circle passes through four points A, B, C and D. PQ is the tangent to the circle at D and is parallel to BD. AC and BD intersect at E. It is given that $AB = AD$.

(a) (i) Prove that $\triangle ABE \cong \triangle ADE$.

(ii) Are the in centre, the orthocentre, the centroid and the circum centre of $\triangle ABD$ collinear? Explain your answer.



12B.20 HKDSE MA 2016 – I – 20 (To continue as **16C.54.**)

$\triangle OPQ$ is an obtuse-angled triangle. Denote the in-centre and the circumcentre of $\triangle OPQ$ by I and J respectively. It is given that P, I and J are collinear.

(a) Prove that $OP = PQ$.

12B.21 HKDSE MA 2019 I 17 (To continue as **16D.14.**)

(a) Let a and p be the area and perimeter of $\triangle CDE$ respectively. Denote the radius of the inscribed circle of $\triangle CDE$ by r. Prove that $pr = 2a$.

12 Geometry of Circles

12A Angles and chords in circles

12A.1 HKCEE MA 1980(1/1*/3) – I – 10

(a) $\angle PAX = 2\theta$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

Similarly, $\angle QBX = \angle RCX = 2\theta$

(b) Areas of sector $OAP : OBQ : OCR = (OA : OB : OC)^2$

$= 4 : 9 : 16$

(c) $\cos\angle RCX = \frac{CD}{CR} = \frac{2}{4} = \frac{1}{2} \Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ$

12A.2 HKCEE MA 1980(1*) – I – 14

(a) $\angle CAD = \angle CAD$ (common)

$x + \angle CAD = \angle CAD + y$ (given)

$\Rightarrow \angle BAD = \angle CAE$

In $\triangle ABD$ and $\triangle ACE$,

$AB = AC$ (given)

$\angle BAD = \angle CAE$ (proved)

$AD = AE$ (given)

$\therefore \triangle ABD \cong \triangle ACE$ (SAS)

(b) $\because \angle ABK = \angle ACK$ (corr. $\angle$s, $\cong\triangle$s)

$\therefore ABCK$ is cyclic. (converse of $\angle$s in the same segment)

(c) $AEDK$

12A.3 HKCEE MA 1981(2) – I – 7

$\angle OBA = 40^\circ$ (base $\angle$s, isos. $\triangle$)

$\angle BOA = 180^\circ - 40^\circ - 40^\circ = 100^\circ$ ($\angle$ sum of $\triangle$)

$\angle BCA = 100^\circ \div 2 = 50^\circ$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

12A.4 HKCEE MA 1982(2) – I – 6

$x = 50^\circ - 20^\circ = 30^\circ$ (ext. $\angle$ of $\triangle$)

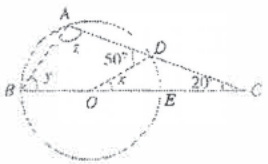
Let OC meet the circle at E. Then

$\angle BOD = 180^\circ - x = 150^\circ$ (adj. $\angle$s on st. line)

$\Rightarrow \angle BED = 150^\circ \div 2 = 75^\circ$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$\therefore z = 180^\circ - \angle BED = 105^\circ$ (opp. $\angle$s, cyclic quad.)

$\Rightarrow y = 180^\circ - 20^\circ - z = 55^\circ$ ($\angle$ sum of $\triangle$)



12A.5 HKCEE MA 1982(2) – I – 13

(a) $\angle DAB = \angle EAC = 60^\circ$ (property of equil. $\triangle$)

$\angle DAB + \angle BAC = \angle EAC + \angle BAC$

$\angle DAC = \angle BAE$

In $\triangle ADC$ and $\triangle ABE$,

$DA = BA$ (property of equil. $\triangle$)

$\angle DAC = \angle BAE$ (proved)

$AC = AE$ (property of equil. $\triangle$)

$\therefore \triangle ADC \cong \triangle ABE$ (SAS)

$\therefore DC = BE$ (corr. sides, $\cong\triangle$s)

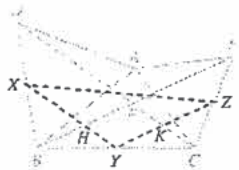
(b) (i) $\because \angle ADC = \angle ABF$ (corr. $\angle$s, $\cong\triangle$s)

$\therefore A, D, B$ and F are concyclic.

(converse of $\angle$s in the same segment)

(ii) $\angle BFD = \angle BAD = 60^\circ$ ($\angle$s in the same segment)

(c)



$\because BX = XD$ and $BY = YC$ (given)

$\therefore XY = \frac{1}{2}DC$ and $XY // DC$ (mid-pt thm)

Similarly, $YZ = \frac{1}{2}BE$ and $YZ // BE$ (mid-pt thm)

$\because DC = BE$ (proved) $\therefore XY = YZ$

$\because \angle BFD = 60^\circ$ (proved)

$\therefore \angle BFC = 180^\circ - 60^\circ = 120^\circ$ (adj. $\angle$s on st. line)

and $\angle CFE = 60^\circ$ (vert. opp. $\angle$s)

Suppose XY meets BE at H and YZ meets DC at K. Then

$\angle YHF = \angle CFE = 60^\circ$ (corr. $\angle$s, $XY // DC$)

$\angle YKF = \angle BFD = 60^\circ$ (corr. $\angle$s, $YZ // BE$)

Hence,

$\angle XYZ = 360^\circ - \angle YHF - \angle YKF - \angle BFC = 120^\circ$

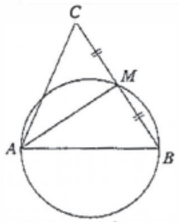
($\angle$ sum of polygon)

$\angle XZY = \angle ZXY$ (base $\angle$s, isos. $\triangle$)

$= (180^\circ - 120^\circ) \div 2 = 30^\circ$ ($\angle$ sum of $\triangle$)

12A.6 HKCEE MA 1989 – I – 4

(a)



(b) In $\triangle ABM$ and $\triangle ACM$,

$AM = AM$ (common)

$MB = MC$ (given)

$\angle AMB = \angle AMC = 90^\circ$ ($\angle$ in semi-circle)

$\therefore \triangle ABM \cong \triangle ACM$ (SAS)

$\therefore \angle BAM = \angle CAM$ (corr. $\angle$s, $\cong\triangle$s)

i.e. AM bisects $\angle BAC$.

12A.7 HKCEE MA 1989 – I – 6

(a) $\angle ABD = \angle ACD = 60^\circ$ ($\angle$s in the same segment)

$\angle BAD = 180^\circ - (60^\circ + 40^\circ)$ (opp. $\angle$s, cyclic quad.)

$= 80^\circ$

12A.8 HKCEE MA 1990 – I – 9

(a) In $\triangle ABD$ and $\triangle ACD$,

$\angle ADB = \angle ADC = 90^\circ$ ($\angle$ in semi-circle)

$AB = AC$ (given)

$AD = AD$ (common)

$\therefore \triangle ABD \cong \triangle ACD$ (RHS)

(b) In $\triangle ABD$ and $\triangle ADE$,

$\angle ABD = \angle ADE$ ($\angle$ in alt. segment)

$\angle BAD = \angle DAE$ (corr. $\angle$s, $\cong\triangle$s)

$\angle ADB = \angle AED$ ($\angle$ sum of $\triangle$)

$\therefore \triangle ABD \sim \triangle ADE$ (AAA)

(c) (i) $AD = \sqrt{AB^2 - BD^2} = 3$ (Pyth. thm)

$\frac{AB}{BD} = \frac{AD}{DE}$ (corr. sides, $\cong\triangle$s)

$\frac{5}{4} = \frac{3}{DE}$

$DE = 2.4$

(ii) $\angle AED = \angle ADB = 90^\circ$ (corr. $\angle$s, $\sim\triangle$s)

$\angle CFB = 90^\circ$ ($\angle$ in semi-circle)

In $\triangle CFB$ and $\triangle CDA$,

$\angle CFB = \angle CDA = 90^\circ$ (proved)

$\angle C = \angle C$ (common)

$\angle CBF = \angle CAD$ ($\angle$ sum of $\triangle$)

$\therefore \triangle CFB \sim \triangle CDA$ (AAA)

$\therefore \frac{CF}{CB} = \frac{CD}{CA}$ (corr. sides, $\cong\triangle$s)

$\frac{AC + AF}{CD + DB} = \frac{CD}{CA}$

$\frac{5 + AF}{4 + 4} = \frac{4}{5} \Rightarrow AF = 1.4$

12A.9 HKCEE MA 1992 – I – 11

(a) $e_3 = d$ (corr. $\angle$s, $FE // AD$)

$b = d$ ($\angle$s in the same segment)

$d = f_1$ (ext. $\angle$, cyclic quad.)

$\therefore e_3 = f_1$

i.e. $\triangle EFY$ is isosceles. (sides opp. equal $\angle$s)

(b) $\because \widehat{BCD} = \widehat{AFE}$ (given)

$\therefore e_1 = b$ (equal arcs, equal $\angle$s)

$\therefore BA // DE$ (alt. $\angle$s equal)

(c) $f_1 = b$ (ext. $\angle$, cyclic quad.)

$= e_1$ (proved)

$e_3 = d$ (proved)

$\therefore f_1 + e_3 + y = 180^\circ$ ($\angle$ sum of $\triangle$)

$\Rightarrow (e_1) + (d) + y = 180^\circ$

$x + y = 180^\circ$ (ext. $\angle$ of $\triangle$)

$\therefore A, X, E$ and Y are concyclic. (opp. $\angle$s supp.)

(d) $f_1 = b = 47^\circ$ (proved)

$e_3 = f_1 = 47^\circ$ (proved)

$\therefore y = 180^\circ - f_1 - e_3 = 86^\circ$ ($\angle$ sum of $\triangle$)

$x = 180^\circ - y = 94^\circ$ (opp. $\angle$s, cyclic quad.)

12A.10 HKCEE MA 1993 – I – 11

(a) $\angle ABP = 90^\circ$ ($\angle$ in semi-circle)

$\angle PQD = 90^\circ$ (given)

$\because \angle ABP = \angle PQD$

$\therefore A, Q, P$ and B are concyclic. (ext. $\angle$ = int. opp. $\angle$)

(b) (i) $\angle BAC = \angle BQP = \theta$ ($\angle$s in the same segment)

$\Rightarrow \angle BDC = \theta$ ($\angle$s in the same segment)

Similar to (a), we get D, Q, P and C are concyclic.

$\Rightarrow \angle PQC = \angle BDC = \theta$ ($\angle$s in the same segment)

$\therefore \angle BQC = \angle BQP + \angle PQC = 2\theta$

(ii) $\angle BOC = 2\angle BAC = 2\theta$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

(c) $\because \angle BQC = \angle BOC = 2\theta$ (proved)

$\therefore BOQC$ is cyclic. (converse of $\angle$s in the same segment)

$\therefore \angle CBQ = \angle COQ$ ($\angle$s in the same segment)

$2\angle CAD = 2\phi$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

12A.11 HKCEE MA 1994 – I – 13

(a) $d = b$ (ext. $\angle$, cyclic quad.)

$\therefore g = 180^\circ - d - \angle DEG$ ($\angle$ sum of $\triangle$)

$= 180^\circ - d - e$

$k_2 = k_1$ (vert. opp. $\angle$s)

$= 180^\circ - b - \angle AEG$ ($\angle$ sum of $\triangle$)

$= 180^\circ - d - e = g$ (proved)

$\therefore \angle FGH = \angle FKH$

(b) $h_2 = g + \angle GFH = g + f$ (ext. $\angle$ of $\triangle$)

$h_1 = k_2 + \angle KFH = k_2 + f$ (ext. $\angle$ of $\triangle$)

$= g + f = h_2$ (proved)

$\therefore h_1 = h_2 = 180^\circ \div 2 = 90^\circ$ (adj. $\angle$s on st. line)

i.e. $FH \perp GK$

(c) (i) $d = 180^\circ - a - 2e$ ($\angle$ sum of $\triangle$)

$= 180^\circ - a \quad 2f$ (given)

$= \angle ABF$ ($\angle$ sum of $\triangle$)

$\because d + \angle ABF = 180^\circ$ (opp. $\angle$s, cyclic quad.)

$\therefore d = 180^\circ \div 2 = 90^\circ$

Hence, $d = h_2 = 90^\circ$ (proved)

$\Rightarrow D, J, H$ and G are concyclic. (ext. $\angle$ = int. opp. $\angle$)

(ii) $d = 180^\circ \quad 28^\circ - a = 152^\circ - a$ ($\angle$ sum of $\triangle$)

$b = a + 46^\circ$ (ext. $\angle$ of $\triangle$)

$152^\circ \quad a = a + 46^\circ$ (ext. $\angle$, cyclic quad.)

$a = 53^\circ$

$\therefore \angle BCD = 180^\circ \quad 53^\circ$ (opp. $\angle$s, cyclic quad.)

$= 127^\circ$

12A.12 HKCEE MA 1996 – I – 6

$\angle BAP = \angle DCP$ (ext. $\angle$, cyclic quad.)

$= \angle ABP$ (corr. $\angle$s, $AB // DC$)

$\therefore AP = BP$ (sides opp. equal $\angle$s)

12A.13 HKCEE MA 1997 – I – 9

(a) $\angle BDC = \angle BAC = 30^\circ$ ($\angle$s in the same segment)

$\angle ADB = 90^\circ - \angle BDC = 60^\circ$ ($\angle$ in semi-circle)

(b) $\widehat{AB} : \widehat{BC} = \angle ADB : \angle BDC = 2 : 1$ (arcs prop. to $\angle$s at $\odot^{ce}$)

(c) $\angle ABC = 90^\circ$ ($\angle$ in semi-circle)

$\Rightarrow AB = 4\cos 30^\circ = 2\sqrt{3}$, $BC = 4\sin 30^\circ = 2$

$\therefore AB : BC = \sqrt{3} : 1$

12A.14 HKCEE MA 1998 – I – 6

(a) $\triangle EBA$

(b) $\frac{y}{3} = \frac{6}{4} \Rightarrow y = \frac{9}{2}$ (corr. sides, $\sim\triangle$s)

12A.15 HKCEE MA 1998 – I – 14

$\because OB = OD$ (radii)

$\therefore \angle ODB = \angle OBD$ (base $\angle$s, isos. $\triangle$)

$\because CB = BA$ (given)

$\therefore \angle CDB = \angle BDA$ (equal chords, equal $\angle$s)

$= \angle OBD$

$\therefore BO // CD$ (alt. $\angle$s equal)

12A.16 HKCEE MA 1999 – I – 5

$\angle ADC = 90°$ ($\angle$ in semi-circle)
$\angle ADB = 50°$ ($\angle$s in the same segment)
$\therefore\ y = 90 - 50 = 40$
$x = 180 - 20 - 90 = 70$ ($\angle$ sum of $\triangle$)

12A.17 HKCEE MA 1999 – I – 16

(a) (i) $\angle BFE = \angle BDE$ ($\angle$s in the same segment)
$= \angle BAC$ (corr. $\angle$s, $AC // DE$)
$\therefore$ A, F, B and C are concyclic.
(converse of $\angle$s in the same segment)
(ii) $\because\ \angle ABC = 90°$ (given)
$\therefore$ AC is a diameter of circle $AFBC$.
(converse of $\angle$ in sem-circle)
$\Rightarrow$ M is the centre of circle $AFBC$ $\Rightarrow$ $MB = MF$

12A.18 HKCEE MA 2000 – I – 7

$x = 25$ ($\angle$ in alt. segment)AD//BC
$\angle DBC = \angle DAC = 25°$ ($\angle$s in the same segment)
$\angle DAB + \angle ABC = 180°$ (int. $\angle$s, $AD // BC$)
$\therefore\ y = 180 - 25 - 56 - 25 = 74$

12A.19 HKCEE MA 2001 – I – 5

$\angle ADC = 90°$ ($\angle$ in semi-circle)
$\angle ACD = 30°$ ($\angle$s in the same segment)
$\therefore\ \angle DAC = 180° - 90° - 30° = 60°$ ($\angle$ sum of $\triangle$)

12A.20 HKCEE MA 2002 – I – 9

$\angle BCD = 90°$ ($\angle$ in semi-circle)
$\angle DBC = 180° - 90° - 40° = 50°$ ($\angle$ sum of $\triangle$)
$\angle BAC = 40°$ ($\angle$s in the same segment)
$\angle ABC = \angle ACB$ (base $\angle$s, isos. $\triangle$)
$= (180° - 40°) \div 2 = 70°$ ($\angle$ sum of $\triangle$)
$\therefore\ \angle ABD = 70° - 50° = 20°$

12A.21 HKCEE MA 2002 – I – 16

(a) (i) $\angle AEB = 90°$ ($\angle$ in semi-circle)
$\angle DAO = 180° - \angle AEB - \angle ABE$ ($\angle$ sum of $\triangle$)
$= 90° - \angle ABE$
$\angle BFO = 180° - \angle FOB - \angle ABE$ ($\angle$ sum of $\triangle$)
$= 90° - \angle ABE$
$\therefore\ \angle DAO = \angle BFO$
In $\triangle AOD$ and $\triangle FOB$,
$\angle DAO = \angle BFO$ (proved)
$\angle AOD = \angle FOB = 90°$ (given)
$\angle ADO = \angle FBO$ ($\angle$ sum of $\triangle$)
$\therefore\ \triangle AOD \sim \triangle FOB$ (AAA)
(ii) $\angle AGB = 90°$ ($\angle$ in semi-circle)
$\angle GAO = 180° - \angle AGO - \angle AOG$ ($\angle$ sum of $\triangle$)
$= 90° - \angle AGO = \angle BGO$
In $\triangle AOG$ and $\triangle GOB$,
$\angle GAO = \angle BGO$ (proved)
$\angle AOG = \angle GOB = 90°$ (given)
$\angle OGA = \angle OBG$ ($\angle$ sum of $\triangle$)
$\therefore\ \triangle AOG \sim \triangle GOB$ (AAA)
(iii) From (i), $\dfrac{AO}{OD} = \dfrac{FO}{OB}$ (corr. sides, $\sim\triangle$s)
$AO \cdot OB = OD \cdot OF$
From (ii), $\dfrac{AO}{OG} = \dfrac{GO}{OB}$ (corr. sides, $\sim\triangle$s)
$AO \cdot OB = OG^2$
$\therefore\ OD \cdot OF = OG^2$

12A.22 HKCEE MA 2005 – I – 17

(a) (i) $\because$ MN is a diameter (given)
$\therefore\ \angle NOM = \angle QRP = 90°$ ($\angle$ in semi-circle)
In $\triangle OQR$ and $\triangle ORP$,
$\angle ROQ = \angle POR = 90°$ (given)
$\angle QRO = \angle QRP - \angle PRO$
$= 90° - \angle PRO$
$\angle POR = 180° - \angle ROP - \angle PRO$ ($\angle$ sum of $\triangle$)
$= 90° - \angle PRO$
$\Rightarrow\ \angle QPO = \angle PRO$
$\angle RQO = \angle PRO$ ($\angle$ sum of $\triangle$)
$\therefore\ \triangle OQR \sim \triangle ORP$ (AAA)
$\Rightarrow\ \dfrac{OR}{OQ} = \dfrac{OP}{OR}$ (corr. sides, $\sim\triangle$s)
$OR^2 = OP \cdot OQ$
(ii) In $\triangle MON$ and $\triangle POR$,
$\angle NMO = \angle QRO$ ($\angle$s in the same segment)
$= \angle RPO$ (proved)
$\angle MON = \angle POR$ (proved)
$\angle MNO = \angle RQO$ ($\angle$ sum of $\triangle$)
$\therefore\ \triangle MON \sim \triangle RQO$ (AAA)

12A.23 HKCEE MA 2006 – I – 16

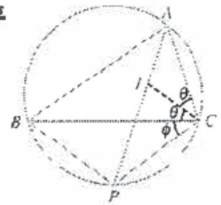
(a) (i) $\because$ G is the circumcentre (given)
$\therefore\ SC \perp BC$ and $SA \perp AB$ ($\angle$ in semi-circle)
$\because$ H is the orthocentre (given)
$\therefore\ AH \perp BC$ and $CH \perp AB$
Thus, $SC // AH$ and $SA // CH$ $\Rightarrow$ $AHCS$ is a //gram.
(ii) *Method 1*
$\because\ \angle GRB = \angle SCB = 90°$ (proved)
$\therefore\ GR // SC$ (corr. $\angle$s equal)
$\because\ BG = GS =$ radius
$\therefore\ BR = RC$ (intercept thm)
$\Rightarrow\ SC = 2GR$ (mid-pt thm)
Hence, $AH = SC = 2GR$ (property of //gram)
Method 2
$\because\ BG = GS =$ radius
and $BR = RC$ ($\perp$ from centre to chord bisects chord)
$\Rightarrow\ SC = 2GR$ (mid-pt thm)
Hence, $AH = SC = 2GR$ (property of //gram)

12A.24 HKCEE MA 2007 – I – 17

(a) (i) $\because$ I is the incentre of $\triangle ABD$ (given)
$\therefore\ \angle ABG = \angle DBG$ and $\angle BAE = \angle CAE$
In $\triangle ABG$ and $\triangle DBG$,
$\angle ABG = \angle DBG$ (proved)
$AB = DB$ (given)
$BG = BG$ (common)
$\therefore\ \triangle ABG \cong \triangle DBG$ (SAS)
(ii) $\because$ $\triangle ABD$ is isosceles and $\angle ABG = \angle DBG$
$\therefore\ \angle BGA = 90°$ (property of isos. $\triangle$)
In $\triangle AGI$ and $\triangle ABE$,
$\angle AGI = 90° = \angle ABE$ ($\angle$ in semi-circle)
$\angle IAG = \angle EAB$ (proved)
$\angle AIG = \angle AEB$ ($\angle$ sum of $\triangle$)
$\therefore\ \triangle AGI \sim \triangle ABE$ (AAA)
$\Rightarrow\ \dfrac{GI}{AG} = \dfrac{BE}{AB}$ (corr. sides, $\sim\triangle$s)

12A.25 HKCEE MA 2008 – I – 17

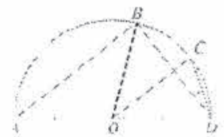
(a) *Method 1*
$\because$ I is the incentre of $\triangle ABC$ (given)
$\therefore\ \angle BAP = \angle CAP$
$\therefore\ BP = CP$ (equal $\angle$s, equal chords)
Method 2
$\because$ I is the incentre of $\triangle ABC$ (given)
$\therefore\ \angle BAP = \angle CAP$
$\angle BCP = \angle BAP$ ($\angle$s in the same segment)
$= \angle CAP$ (proved)
$= \angle CBP$ ($\angle$s in the same segment)
$\Rightarrow\ BP = CP$ (sides opp. equal $\angle$s)
Both methods



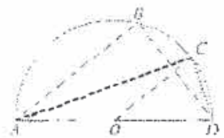
Join CI. Let $\angle ACI = \angle BCI = \theta$ and $\angle BCP = \phi$.
$\angle PAC = \phi$ (equal chords, equal $\angle$s)
$\Rightarrow\ \angle PIC = \angle PAC + \angle ACI = \theta + \phi$ (ext. $\angle$ of $\triangle$)
$= \angle PCI$
$\therefore\ IP = CP$ (sides opp. equal $\angle$s)
i.e. $BP = CP = IP$

12A.26 HKDSE MA SP – I – 7

Method 1
$\angle ABD = 90°$ ($\angle$ in semi-circle)
$\angle BDA = 180° - 90° - 38° = 52°$ ($\angle$ sum of $\triangle$)
$\angle COD = 38°$ (corr. $\angle$s, $AB // OC$)
$\because\ OC = OD$ (radii)
$\therefore\ \angle ODC = \angle OCD$ (base $\angle$s, isos. $\triangle$)
$= (180° - 38°) \div 2 = 71°$ ($\angle$ sum of $\triangle$)
Hence, $\angle BDC = 71° - 52° = 19°$
Method 2



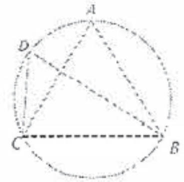
$\angle BOD = 2(38°) = 76°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
$\angle COD = 38°$ (corr. $\angle$s, $AB // OC$)
$\Rightarrow\ \angle BOC = 76° - 38° = 38°$
$\therefore\ \angle BDC = 38° \div 2 = 19°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
Method 3



$\angle COD = 38°$ (corr. $\angle$s, $AB // OC$)
$OA = OC$ (radii)
$\Rightarrow\ \angle OAC = \angle OCA$ (base $\angle$s, isos. $\triangle$)
$= \angle COD \div 2 = 19°$ (ext. $\angle$ of $\triangle$)
$\therefore\ \angle BAC = 38° - 19° = 19°$
$\Rightarrow\ \angle BDC = \angle BAC = 19°$ ($\angle$s in the same segment)

12A.27 HKDSE MA PP – I – 7

$\angle DCB = 90°$ ($\angle$ in semi-circle)
$\Rightarrow\ \angle DBC = 180° - 90° - 36° = 54°$ ($\angle$ sum of $\triangle$)
$\angle CAB = 36°$ ($\angle$s in the same segment)
$\angle ABC = \angle ACB$ (base $\angle$s, isos. $\triangle$) / (equal chords, equal $\angle$s)
$= (180° - \angle CAB) \div 2 = 72°$ ($\angle$ sum of $\triangle$)
$\therefore\ \angle ABD = 72° - 54° = 18°$



12A.28 HKDSE MA PP – I – 14

(a) $\triangle AOD \sim \triangle CBD$

12A.29 HKDSE MA 2012 – I – 8

(a) $\triangle AED \sim \triangle BEC$
$\therefore\ \dfrac{AE}{DE} = \dfrac{BE}{CE}$ (corr. sides, $\sim\triangle$s)
$\Rightarrow\ AE = \dfrac{8}{20} \times 15 = 6$ (cm)
(b) $AB^2 = 10^2 = 100$
$AE^2 + EB^2 = 6^2 + 8^2 = 100 = AB^2$
$\therefore\ AC \perp BD$ (converse of Pyth. thm)

12A.30 HKDSE MA 2015 – I – 8

Method 1
$\angle ACB = \angle ADB = 58°$ ($\angle$s in the same segment)
$\angle ABD = \angle ADB$ (base $\angle$s, isos. $\triangle$) / (equal chords, equal $\angle$s)
$= 58°$
$\angle BDC = \angle BAC$ ($\angle$s in the same segment)
$= 180° - \angle ABC - \angle ACB$ ($\angle$ sum of $\triangle$)
$= 180° - (58° + 25°) - 58° = 39°$
Method 2
$\angle ABD = \angle ADB$ (base $\angle$s, isos. $\triangle$) / (equal chords, equal $\angle$s)
$= 58°$
$\angle ADC + \angle ABC = 180°$ (opp. $\angle$s, cyclic quad.)
$58° + \angle BDC + (58° + 25°) = 180°$
$\angle BDC = 39°$
Both methods
$\angle BAC = \angle BDC = 39°$ ($\angle$s in the same segment)
In $\triangle BCE$, $\angle BEC = \angle EBC$ (base $\angle$s, isos. $\triangle$)
$= (180° - \angle BCA) \div 2$ ($\angle$ sum of $\triangle$)
$= 61°$
$\therefore\ \angle ABE = \angle BEC - \angle BAC = 22°$ (ext. $\angle$ of $\triangle$)

12A.31 HKDSE MA 2017 – I – 10

(a) In $\triangle OPS$ and $\triangle ORS$,
$OP = OR$ (given)
$OS = OS$ (common)
$PS = RS$ (given)
$\therefore\ \triangle OPS \cong \triangle ORS$ (SSS)
(b) $\angle ROQ = \angle POQ$ (corr. $\angle$s, $\cong\triangle$s)
$= 2\angle PRQ = 20°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
$\therefore$ Area of sector $= \dfrac{2(20°)}{360°} \times \pi(6)^2 = 4\pi$ (cm²)

12A.32 HKDSE MA 2018 – I – 8

$x = 180° - \theta$ (opp. ∠s, cyclic quad.)
$\angle BED = \angle BAD = x$ (∠s in the same segment)
$= \angle ADE$ (alt. ∠s, $AB//ED$)
∴ $y = 180° - \angle BED - \angle ADE$ (∠ sum of △)
$= 180° - 2(180° - \theta) = 2\theta - 180°$

12A.33 HKDSE MA 2019 – I – 13

(a) *Method 1*
Reflex $\angle DOA = 2\angle DEA$ (∠ at centre twice ∠ at ⊙^{ce})
$= 230°$
⇒ $\angle DOC = 230° - 180° = 50°$
∴ $\angle CBF = \angle DOC \div 2 = 25°$ (∠ at centre twice ∠ at ⊙^{ce})

Method 2
$\angle ABD = 180° - \angle AED = 65°$ (opp. ∠s, cyclic quad.)
$\angle ABC = 90°$ (∠ in semi-circle)
∴ $\angle CBF = 90° - 65° = 25°$

(b) $\angle OCB = \angle DOC = 50°$ (alt. ∠s, $BC//OD$)
⇒ $\angle BOC = 180° - 2\angle OCB = 80°$
∴ Perimeter of sector $OBC = 2 \times 18 + \widehat{BC}$
$= 36 + \frac{80°}{360°} \times 2\pi(18)$
$= 61.13 > 60$ (cm)

∴ NO

12B Tangents of circles

12B.1 HKCEE MA 1980(1*) – I – 8

$\angle TAB = \angle TBA = 65°$ (∠ in alt. segment)
∴ $x = \angle TAB + \angle TBA = 130°$ (ext. ∠ of △)

12B.2 HKCEE MA 1981(2) – I – 13

(a) $\angle MQT = x$ (∠ in alt. segment)
$\angle NQT = y$ (∠ in alt. segment)
∴ $\angle MQN = x + y$

(b) $\angle PTR = 180° - \angle TPR - \angle PRT$ (∠ sum of ▲)
$= 180° - x - y$
∵ $\angle MQN + \angle MTN = (x+y) + (180° - x - y) = 180°$
∴ Q, M, T and N are concyclic. (opp. ∠s supp.)

(c) ∵ $QMTN$ is cyclic, (proved)
∴ $\angle NMT = \angle NQT = y$ (∠s in the same segment)
∵ $\angle NMT = \angle PRN = y$ (proved)
∴ P, M, N and R are concyclic. (ext. ∠ = int. opp. ∠)

(d) $\triangle MNT \sim \triangle RPT$, $\triangle MQT \sim \triangle QPT$, $\triangle NQT \sim \triangle QRT$

12B.3 HKCEE MA 1982(2) – I – 14

(a) $\angle ABT = \angle ATR$ (∠ in alt. segment)(large circle)
$= \angle PQT$ (∠ in alt. segment)(small circle)
∴ $AB//PQ$ (corr. ∠s equal)

(b) Consider the small circle.
$\angle QTS = \angle BSQ$ (∠ in alt. segment)
$= \angle SQP$ (alt. ∠s, $AB//PQ$)
$= \angle STP$ (∠s in the same segment)
i.e. ST bisects $\angle ATB$.

(c) ▲PTK, △ATS, △ASP, △SQK

12B.4 HKCEE MA 1983(A/B) – I – 2

(a) $\angle OAB = \angle OBA = 60°$ (property of equil △)
$AC = OA = AB$ (given)
∴ $\angle ABC = \angle ACB$ (base ∠s, isos. △)
$= \angle OAB \div 2 = 30°$ (ext. ∠ of △)

(b) ∵ $\angle OBC = 60° + 30° = 90°$
∴ CB is tangent to the circle at B.
(converse of tangent ⊥ radius)

12B.5 HKCEE MA 1984(A/B) – I – 5

$\angle CBD = 80°$ (∠ in alt. segment)
$x = 180 - 30 - 80 = 70$ (adj. ∠s on st. line)
$y = x = 70$ (∠ in alt. segment)
$AB = AD$ (tangent properties)
⇒ $\angle BDA = x°$ (base ∠s, isos. △)
∴ $z = 180 - x - x = 40$ (∠ sum of △)

12B.6 HKCEE MA 1985(A/B) – I – 2

$\angle APB = \angle ABP$ (base ∠s, isos. ▲)
$= x°$ (∠ in alt. segment)
∴ In $\triangle BCP$, $x° + x° + (x° + 60°) = 180°$ (∠ sum of △)
$x = 40$

12B.7 HKCEE MA 1986(A/B) – I – 2

$TA = TB$ (tangent properties)
$\angle ABT = x°$ (base ∠s, isos. △)
$= (180° - 30°) \div 2$ (∠ sum of △) ⇒ $x = 75$
$y° = \angle ACB$ (alt. ∠s, $AC//TF$)
$= \angle ABT = x°$ (∠ in alt. segment) ⇒ $y = 75$

12B.8 HKCEE MA 1986(A/B) – I – 6

(a) △CAT

(b) ∵ $\triangle BCT \sim \triangle CAT$
∴ $\frac{BT}{CT} = \frac{CT}{AT}$ (corr. sides, ~ △s)
$\frac{x}{10\sqrt{2}} = \frac{10\sqrt{2}}{17+x}$
$17x + x^2 = 200$ ⇒ $x = 8$ or -25 (rejected)

12B.9 HKCEE MA 1987(A/B) – I – 6

$\angle ODA = 90°$ (tangent ⊥ radius)
$\angle OAD = 60° \div 2 = 30°$ (tangent properties)
∴ $AO = \frac{1}{\sin 30°} = 2$ (cm)
∴ $r = AE = 2 + 1 = 3$

12B.10 HKCEE MA 1987(A/B) – I – 7

$\angle ABC = 90°$ (∠ in semi-circle)
$\angle APB = \angle PAB = x°$ (base ∠s, isos. △)
$= \angle CBP$ (∠ in alt. segment)
∴ In $\triangle ABP$, $x° + x° + (90° + x°) = 180°$ (∠ sum of △)
$x = 30$

12B.11 HKCEE MA 1988 – I – 8(b)

(i) In $\triangle ACT$ and $\triangle TCB$,
$\angle TCA = \angle BCT$ (common)
$\angle TAC = \angle BTC$ (∠ in alt. segment)
$\angle CTA = \angle CBT$ (∠ sum of △)
∴ $\triangle ACT \sim \triangle TCB$ (AAA)

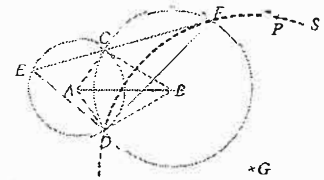
(ii) $\frac{AC}{CT} = \frac{TC}{CB}$ (corr. sides, ~ △s)
$\frac{AB+5}{6} = \frac{6}{5}$ ⇒ $AB = \frac{11}{5}$

12B.12 HKCEE MA 1991 – I – 13

(a) In $\triangle ABC$ and $\triangle ABD$,
$AC = AD$ (radii)
$BC = BD$ (radii)
$AB = AB$ (common)
∴ $\triangle ABC \cong \triangle ABD$ (SSS)

(b) (i) ∵ $\angle CAD = 2(55°)$ (∠ at centre twice ∠ at ⊙^{ce})
$= 110°$
and $\angle CAB = \angle DAB$ (corr. ∠s, ≅ △s)
∴ $\angle CAB = 110 \div 2 = 55°$
$\angle DBA = \angle CBA$ (corr. ∠s, ≅ △s)
$= 180° - \angle ACB - \angle CAB$ (∠ sum of △)
$= 30°$
⇒ $\angle CBD = 30° + 30° = 60°$
∴ $\angle EFD = \frac{1}{2}\angle CBD$ (∠ at centre twice ∠ at ⊙^{ce})
$= \frac{1}{2}(60°) = 30°$

(ii) (1)



(The centre of S lies on the intersection of the perpendicular bisector of DF and the line at F perpendicular to CF.)

(2) Let P be a point on major $\widehat{DF}$ and G be the centre of S.
$\angle CFD = \angle FPD = 30°$ (∠ in alt. segment)
$\angle FGD = 2 \times 30°$ (∠ at centre twice ∠ at ⊙^{ce})
$= 60°$
Hence, $\triangle FGD$ is equilateral.
⇒ Diameter $= 2GF = 2DF$

12B.13 HKCEE MA 1995 – I – 14

(a) (i) ∵ $\angle PQA = \angle PRQ$ (∠ in alt. segment)
$= \angle PMA$ (corr. ∠s, $AC//QR$)
∴ M, P, A and Q are concyclic.
(converse of ∠s in the same segment)

(ii) $\angle MQR = \angle AMQ$ (alt. ∠s, $AC//QR$)
$= \angle APQ$ (∠s in the same segment)
$= \angle MRQ$ (∠ in alt. segment)
∴ $MR = MQ$ (sides opp. equal ∠s)

(b) $\angle QPR = \angle QAC = 50°$ (∠s in the same segment)
$\angle RMQ = \angle PAQ = 70°$ (opp. ∠s, cyclic quad.)
$\angle MQR = (180° - 70°) \div 2 = 55°$ (∠ sum of △)
$\angle MQP = \angle PAC = 20°$ (∠s in the same segment)
∴ $\angle PQR = \angle MQR + \angle\angle MQP = 75°$

(c) (i) Property of isos. △
(ii) ⊥ bisector of chord passes through centre

12B.14 HKCEE MA 1997 – I – 16

(a) (i) $\angle EAB = 90°$ (tangent ⊥ radius)
∵ $\angle FEA + \angle EAB = 90° + 90° = 180°$
∴ $AB//EF$ (int. ∠s supp.)

(ii) $\angle FDE = \angle BDC$ (vert. opp. ∠s)
$= \angle DBC$ (base ∠s, isos. △)
$= \angle FED$ (alt. ∠s, $AB//EF$)
∴ $FD = FE$ (sides opp. equal ∠s)

(iii) If the circle touches AE at E, its centre lies on EF.
If ED is a chord, the centre lies on the ⊥ bisector of ED.
∴ The intersection of these two lines, F, is the centre of the circle described.

12B.15 HKCEE MA 2000 – I – 16

(a) In $\triangle OCP$, $\angle CPO = 90°$ (tangent ⊥ radius)
$\angle PCO = 180° - 30° - 90°$ (∠ sum of △)
∴ $\angle PQO = 60° \div 2 = 30°$ (∠ at centre twice ∠ at ⊙^{ce})

(b) (i) $\angle SOC = \angle POC = 30°$ (tangent properties)
$\angle PQR = 180° - \angle POS$ (opp. ∠s, cyclic quad.)
$= 120°$
⇒ $\angle RQO = 120° - 30° = 90°$
∴ RQ is tangent to the circle at Q.
(converse of tangent ⊥ radius)

12B.16 <u>HKCEE MA 2003 – I – 17</u>

(a) (i) In $\triangle NPM$ and $\triangle NKP$,

$\angle PNM = \angle KNP$ (common)

$\angle NPM = \angle NKP$ ($\angle$ in alt. segment)

$\angle PMN = \angle KPN$ ($\angle$ sum of $\triangle$)

$\therefore \triangle NPM \sim \triangle NKP$ (AAA)

$\Rightarrow \dfrac{NP}{NM} = \dfrac{NK}{NP}$ (corr. sides, $\sim \triangle$s)

$NP^2 = NK \cdot NM$

(ii) $\because RS // OP$ (given)

$\therefore \triangle KRM \sim \triangle KON$ and $\triangle KSM \sim \triangle KPN$

$\Rightarrow \dfrac{RM}{ON} = \dfrac{KM}{KN}$ and $\dfrac{SM}{PN} = \dfrac{KM}{KN}$

$\Rightarrow \dfrac{RM}{ON} = \dfrac{SM}{PN}$

Similar to (a), $NO^2 = NK \ \ NM \Rightarrow NP = NO$

Hence, $RM = MS$.

12B.17 <u>HKCEE MA 2004 – I – 16</u>

(a) In $\triangle ADE$ and $\triangle BOE$,

$\angle ADE = \angle EBC$ (alt. $\angle$s, $OD // BC$)

$= \angle BOE$ ($\angle$ in alt. segment)

$\angle DAE = \angle OBE$ (ext. $\angle$, cyclic quad.)

$AD = BO$ (given)

$\therefore \triangle ADE \cong \triangle BOE$ (ASA)

(b) $DE = OE$ (corr. sides, $\cong \triangle$s)

$\angle BOE = \angle ADE$ (proved)

$= \angle AOE$ (base $\angle$s, isos. $\triangle$)

i.e. $\angle AOB = 2\angle BOE$

$\therefore \angle BEO = \angle AED$ (corr. $\angle$s, $\cong \triangle$s)

$= \angle AOB$ (ext. $\angle$, cyclic quad.)

$= 2\angle BOE$ (proved)

(c) Suppose OE is a diameter of the circle $OAEB$.

(i) $\angle OBE = 90^\circ$ ($\angle$ in semi-circle)

In $\triangle OBE$, $\angle BOE = 180^\circ - 90^\circ - (2\angle BOE)$

($\angle$ sum of $\triangle$)

$3\angle BOE = 90^\circ \Rightarrow \angle BOE = 30^\circ$

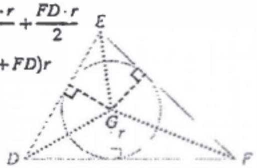
12B.18 <u>HKCEE AM 2002 – 15</u>

(a) Cut the triangle into $\triangle ODE$, $\triangle OEF$ and $\triangle OFD$. Then the radii are the heights of the triangles. (tangent $\perp$ radius)

$\therefore A = \dfrac{DE \cdot r}{2} + \dfrac{EF \cdot r}{2} + \dfrac{FD \cdot r}{2}$

$= \dfrac{1}{2}(DE + EF + FD)r$

$= \dfrac{1}{2}pr$



12B.19 <u>HKDSE MA SP – I – 19</u>

(a) (i) In $\triangle ABE$ and $\triangle ADE$,

$AB = AD$ (given)

$AE = DE$ (common)

$\angle BAE = \angle BCP$ ($\angle$ in alt. segment)

$= \angle EBC$ (alt. $\angle$s, $BD // PQ$)

$= \angle DAE$ ($\angle$s in the same segment)

$\therefore \triangle ABE \cong \triangle ADE$ (SAS)

(ii) $\because AB = AD$ (given)

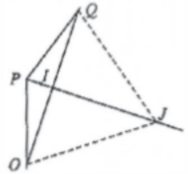
and AE is an $\angle$ bisector of $\triangle ADE$ (proved)

$\therefore AE$ is an altitude, a median and $\perp$ bisector of $\triangle ADE$. (property of isos. $\triangle$)

i.e. The in-centre, orthocentre, centroid and circumcentre of $\triangle ABD$ all lie on AE, and are thus collinear.

12B.20 <u>HKDSE MA 2016 – I – 20</u>

(a) *Method 1*



Let $\angle OPJ = \angle QPJ = \theta$. (in-centre)

$OJ = PJ = QJ$ (radii)

In $\triangle POJ$, $\angle POJ = \angle OPJ = \theta$ (base $\angle$s, isos. $\triangle$)

In $\triangle PQJ$, $\angle PQJ = \angle QPJ = \theta$ (base $\angle$s, isos. $\triangle$)

In $\triangle POJ$ and $\triangle PQJ$,

$\angle OPJ = \angle QPJ = \theta$ (in-centre)

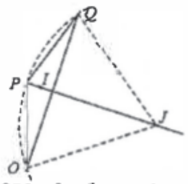
$\angle POJ = \angle PQJ = \theta$ (proved)

$PJ = PJ$ (common)

$\therefore \triangle POJ \cong \triangle PQJ$ (AAS)

$\therefore PO = PQ$ (corr. sides, $\cong \triangle$s)

Method 2



Let $\angle OPJ = \angle QPJ = \theta$. (in-centre)

$OJ = PJ = QJ$ (radii)

In $\triangle POJ$, $\angle POJ = \angle OPJ = \theta$ (base $\angle$s, isos. $\triangle$)

$\Rightarrow \angle PJO = 180^\circ \ \ 2\theta$ ($\angle$ sum of $\triangle$)

$\Rightarrow \angle PQO = (180^\circ - 2\theta) \div 2 = 90^\circ - \theta$

($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

In $\triangle PQJ$, $\angle PQJ = \angle QPJ = \theta$ (base $\angle$s, isos. $\triangle$)

$\Rightarrow \angle PJQ = 180^\circ - 2\theta$ ($\angle$ sum of $\triangle$)

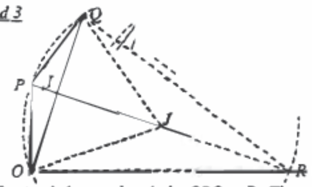
$\Rightarrow \angle POQ = (180^\circ - 2\theta) \div 2 = 90^\circ - \theta$

($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$\because \angle PQO = \angle POQ = 90^\circ - \theta$ (proved)

$\therefore PO = PQ$ (sides opp. equal $\angle$s)

Method 3



Let PJ extended meet the circle OPQ at R. Then PR is a diameter of the circle.

$\therefore \angle POR = \angle PQR = 90^\circ$ ($\angle$ in semi-circle)

Let $\angle OPR = \angle QPR = \theta$. (in-centre)

In $\triangle OPR$, $PO = PR\cos\theta$

In $\triangle QPR$, $PQ = PR\cos\theta$

$\therefore PO = PQ$

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(a) Let I be the in-centre of $\triangle CDE$. Then the perpendiculars from I to CD, DE and EC are all r.

$a = \dfrac{r \cdot CD}{2} + \dfrac{r \cdot DE}{2} + \dfrac{r \ EC}{2}$

$= \dfrac{r(CD + DE + EC)}{2} = \dfrac{r(p)}{2} \Rightarrow pr = 2a$